

Probabilistic safety analysis in three dimensional aircraft flight[†]

Jianghai Hu* Maria Prandini** Shankar Sastry*

*Dept. of Electrical Engineering and Computer Sciences
University of California at Berkeley - USA
E-mail: {jianghai,sastry}@eecs.berkeley.edu

**Dipartimento di Elettronica e Informazione
Politecnico di Milano - Italy
E-mail: prandini@elet.polimi.it

Abstract

In this paper we study the problem of evaluating if the flight plan assigned to an aircraft is safe. A flight plan is said to be safe if it guarantees that the aircraft will not enter any forbidden region of the airspace (aircraft-to-airspace problem) and will not come closer than a minimum prescribed distance to another aircraft (aircraft-to-aircraft problem). We address the general case when the flight plan possibly involves altitude changes.

The aircraft future position during the flight time is predicted based on a stochastic model that incorporates the information on the aircraft flight plan, and takes into account the presence of wind as main source of uncertainty on the aircraft actual motion. Based on the aircraft predicted motion, we estimate the probability that an aircraft-to-airspace or aircraft-to-aircraft problem will occur during the aircraft flight by a stochastic approximation method.

Some examples are presented where the proposed methodology for safety analysis is applied to predict unsafe situations in aircraft flight. As expected, the examples show that the wind spatial correlation has to be taken into account when analyzing aircraft-to-aircraft problems.

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1 Introduction

In many control applications, the dynamics of the system under study is subject to the perturbation of random noises that are either inherent or present in the environment. Typically, a certain part of the state space is “unsafe” and the control input to the system has to be chosen so as to keep the state away from it, despite of the presence of the random noises. This is the case, for example, in many safety-critical applications. In these applications it is then very important to have some measure of criticality for evaluating whether the selected control input is appropriate or a corrective action should be taken to timely steer the system out of the unsafe set. A natural choice for the measure of criticality is the probability of intrusion into the unsafe set within a finite/infinite time horizon.

In this paper we study one such a safety-critical application, developing a methodology for aircraft flight safety analysis in three dimensional airspace.

Consider an aircraft flying in some region of the airspace and suppose that the aircraft is given a certain flight plan to follow during some look-ahead time horizon. Our objective is evaluating if the flight plan assigned to the aircraft is safe. A flight plan is defined to be safe if the aircraft-to-aircraft and aircraft-to-airspace separation requirements are satisfied. An aircraft-to-aircraft conflict situation arises when the aircraft gets closer than a prescribed minimum distance to another aircraft, whereas an aircraft-to-airspace conflict situation arises when the aircraft enters a forbidden region of the airspace such as, for example, a Special Use Airspace (SUA) area, or an area with high congestion or severe weather conditions ([9]).

The procedure used to prevent the occurrence of a conflict typically consists of two phases, namely, aircraft conflict detection and aircraft conflict resolution. Automated tools are currently being studied to support the air traffic controllers (ATCs) in performing these tasks. A comprehensive overview of the methods proposed in the literature for aircraft-to-aircraft conflict detection can be found in [12]. In automated conflict detection, models for predicting the aircraft future position are introduced and the possibility that a conflict would happen within a certain time horizon is evaluated based on these models ([23, 18, 19, 4]). If a conflict is predicted, then the aircraft flight plan is modified in the conflict resolution phase so as to avoid the actual occurrence of the predicted conflict. The cost of the resolution action in terms of, for example, delay, fuel consumption, deviation from originally planned itinerary, is usually taken into account when selecting a new flight plan ([7, 22, 16, 6, 11, 17, 24, 10]).

In this paper we focus on the conflict detection issue and address it according to a probabilistic viewpoint. We describe the aircraft motion as the solution to a stochastic differential equation. In our probabilistic context, safety is then quantified in terms of the probability that the prescribed separation from another aircraft or a prohibited airspace area is violated: the lower the probability, the higher the safety level.

We address the general case when the aircraft might change altitude during its flight. Modeling altitude changes is important not only because the aircraft changes altitude when it is inside a Terminal Radar Approach Control (TRACON) area, but also because altitude changes can be used as resolution maneuvers to avoid, e.g., severe weather areas or conflict situations with other aircraft ([20],[14],[11]). Conflicts are typically resolved by resorting to the turn, climb/descend, and accelerate/decelerate actions, which affect the aircraft heading, altitude, and speed, respectively ([11]). It is found that climb/descend is the most efficient action for resolving short-term conflicts. Vertical maneuvers are in fact used to resolve imminent conflicts in the Traffic Alert and Collision Avoidance System (TCAS [20, 14]).

There are several factors that combined make the safety analysis problem in aircraft flight highly complicated, and as such impossible to solve analytically. First, aircraft flight plans can be, in principle, arbitrary motions in the three dimensional airspace, and they are generally more

complex than the simple planar linear motions assumed in [19, 5] when determining analytic expressions for the probability of an aircraft-to-aircraft conflict. Second, forbidden airspace areas may have an arbitrary shape, which can also change in time, as, for example, in the case of a storm that covers an area of irregular shape and evolves dynamically. Finally, and probably the most importantly, the random perturbation to the aircraft motion is spatially correlated. Wind is the main source of uncertainty on the aircraft position, and if we consider two aircraft, the closer the aircraft, the larger the correlation between the wind perturbations to their motions. Although this last factor is known to be critical, it is largely ignored in the current literature on aircraft safety study, probably because it is difficult to model and analyze. To our knowledge, the first attempt to model the wind perturbation to the aircraft motion for Air Traffic Management (ATM) applications was done in [15], which inspired this work.

One of the contributions of this paper is the introduction of a model of the aircraft motion in three dimensional airspace that explicitly takes into account the spatially correlated structure of the random perturbation to the aircraft position. Based on this model, we develop an algorithm to estimate the probability of conflict in both the aircraft-to-aircraft and aircraft-to-airspace problems. The algorithm is based on the approximation of the solution to stochastic differential equations by using Markov chains. For general references on stochastic approximation, see [2, 13]. The basic idea is to construct a Markov chain whose state space is obtained by discretizing the original space into grids. For properly chosen transition probabilities, the Markov chain converges weakly to the solution to the stochastic differential equation as the discretization step approaches zero. Therefore, an approximation of the probability of interest can be obtained by computing the corresponding quantity for the Markov chain.

The rest of the paper is organized as follows. In Section 2 we describe the model for predicting the aircraft position during the look-ahead time horizon based on the flight plan information. We then precisely formulate the safety analysis problem addressed in the paper (Section 3), and describe a Markov chain approximation scheme to estimate the safety level of the prescribed flight plan (Section 4). The iterative algorithm implementing the Markov chain approximation scheme in the finite horizon case is described in Section 5, and then illustrated through some simulation examples in Section 6. Finally, we draw some conclusions in Section 7.

2 Model of the aircraft motion

In this section we introduce a kinematic model of the aircraft motion to predict the aircraft future position during the time interval $T = [0, t_f]$, where 0 is the current time instant, and t_f is a positive real number (possibly infinity) representing the look-ahead time horizon. Based on this model, in Section 3 we shall propose a method for evaluating the safety level of the flight plan assigned to the aircraft.

Since we address the general case when the aircraft might change altitude during the time horizon T , we represent the airspace and the aircraft position at time $t \in T$ by \mathbb{R}^3 and $X(t) \in \mathbb{R}^3$, respectively.

We assume that the flight plan assigned to the aircraft is specified in terms of a velocity profile $u : T \rightarrow \mathbb{R}^3$, meaning that at time $t \in T$ the aircraft plans to fly at velocity $u(t)$. According to the common practice in the current ATM systems, where aircraft are advised to travel at constant speed piecewise linear motions specified by a series of way-points, the velocity profile u can be chosen to be a piecewise constant function. This choice is also suitable when the flight plan u represents a resolution maneuver to avoid conflict situations, since resolution maneuvers typically consist of lateral or altitude changes, and only rarely involve speed changes.

We suppose that the main source of uncertainty in the aircraft future position during the time interval T is the wind, which affects the aircraft motion by acting on the aircraft velocity.

The actual velocity of the aircraft at time $t \in T$ is then the sum of $u(t)$ and an additional term (the *wind speed*) representing the wind contribution. The velocity u is called *airspeed*, whereas the sum of u and the wind speed is the *ground speed*. Note that here we adopt the ATM terminology and use the word ‘speed’ for the velocity vector in \mathbb{R}^3 .

The wind speed can be further decomposed into two components: i) a deterministic term representing the nominal wind speed, which may depend on the aircraft location and time t , and is assumed to be known to the ATC through measurements or forecast; and ii) a stochastic term representing the effect of air turbulence and errors in the wind speed measurements and forecast.

As a result of the above discussion, the position X of the aircraft during the time horizon T is governed by the following stochastic differential equation:

$$dX(t) = u(t)dt + f(X, t)dt + \Sigma(X, t)dB(X, t), \quad (1)$$

initialized with the aircraft current position $X(0)$. We next explain the different terms appearing in equation (1).

First of all, $f : \mathbb{R}^3 \times T \rightarrow \mathbb{R}^3$ is a time-varying vector field on \mathbb{R}^3 : for a fixed $(x, t) \in \mathbb{R}^3 \times T$, $f(x, t)$ represents the nominal wind speed at position x and at time t . We call f the *wind field*.

$B(\cdot, \cdot)$ is a time-varying random field on \mathbb{R}^3 modeling (the integral of) air turbulence perturbations to aircraft velocity as well as wind speed measurement errors, and has the following properties:

- i) for each fixed $x \in \mathbb{R}^3$, $B(x, \cdot)$ is a standard three-dimensional Brownian motion. Hence $dB(x, t)/dt$ can be thought of as a three-dimensional white noise process;
- ii) $B(\cdot, \cdot)$ is time increment independent. This implies, in particular, that the collections of random variables $\{B(x, t_2) - B(x, t_1)\}_{x \in \mathbb{R}^3}$ and $\{B(x, t_4) - B(x, t_3)\}_{x \in \mathbb{R}^3}$ are independent for any $t_1, t_2, t_3, t_4 \in T$, with $t_1 \leq t_2 \leq t_3 \leq t_4$;
- iii) for any $t_1, t_2 \in T$ with $t_1 \leq t_2$, $\{B(x, t_2) - B(x, t_1)\}_{x \in \mathbb{R}^3}$ is an (uncountable) collection of Gaussian random variables with zero mean and covariance

$$E\{[B(x, t_2) - B(x, t_1)][B(y, t_2) - B(y, t_1)]^T\} = \rho(x - y)(t_2 - t_1)I_3, \quad \forall x, y \in \mathbb{R}^3,$$

where I_3 is the 3-by-3 identity matrix, and $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function with $\rho(0) = 1$ and $\rho(x)$ decreases to zero as $x \rightarrow \infty$. In addition, ρ has to be non-negative definite in the sense that the k -by- k matrix $[\rho(x_i - x_j)]_{i,j=1}^k$ is non-negative definite for arbitrary $x_1, \dots, x_k \in \mathbb{R}^3$ and positive integer k . See [1] for other equivalent conditions of this non-negative definite requirement.

Remark 1 Typically the wind field f is supposed to satisfy some continuity property. This condition, together with the monotonicity assumption on the spatial correlation function ρ , is introduced to model the fact that the closer are two points in space, the more similar are the wind speeds at those points, and, as the two points move farther away from each other, their wind speeds become more and more independent. This is a reasonable assumption in the context of aircraft flight.

In this paper we suppose that ρ is given by $\rho(x) = \exp(-\beta_h \|x\|_h - \beta_v \|x\|_v)$ for some $\beta_v \geq \beta_h > 0$, where the subscripts h and v stand for “horizontal” and “vertical”, and $\|(x_1, x_2, x_3)\|_h \triangleq \sqrt{x_1^2 + x_2^2}$ and $\|(x_1, x_2, x_3)\|_v \triangleq |x_3|$ for any $(x_1, x_2, x_3) \in \mathbb{R}^3$. This is to model the fact that the wind correlation in space is weaker in the vertical direction.

As a random field, $B(\cdot, \cdot)$ is Gaussian, stationary (its finite dimensional distributions remain unchanged when the origin of $\mathbb{R}^3 \times T$ is shifted), and isotropic in the horizontal directions (its

finite dimensional distributions are invariant with respect to changes of orthonormal coordinates in the horizontal directions).

Finally, $\Sigma : \mathbb{R}^3 \times T \rightarrow \mathbb{R}^{3 \times 3}$ is introduced to modulate the variance of the random perturbation to the aircraft velocity. In this paper, we assume that $\Sigma(\cdot, \cdot)$ is a constant diagonal matrix and denote it by Σ . Specifically, $\Sigma(x, t) = \Sigma \triangleq \text{diag}(\sigma_h, \sigma_h, \sigma_v)$, $(x, t) \in \mathbb{R}^3 \times T$, for some constant $\sigma_h > \sigma_v > 0$. Note that after the modulation of Σ the random contribution of the wind to the aircraft velocity remains isotropic horizontally. However, its variance is smaller in the vertical direction than in the horizontal ones, which is a reasonable assumption in our context.

Equation (1) can then be rewritten as

$$dX(t) = u(t)dt + f(X, t)dt + \Sigma dB(X, t) \quad (2)$$

with initial condition $X(0)$. In the following sections, a methodology for aircraft safety analysis will be developed based on this model.

3 Formulation of the probabilistic safety analysis problem

In this section we describe two situations in aircraft flight where safety analysis can be useful, and precisely formulate the safety analysis problem that we address in this paper. The objective in both situations is to evaluate if the flight plan assigned to the aircraft is safe by determining the probability that the aircraft enters some unsafe region. Given that it is difficult to obtain an analytic expression for this probability, in Section 4 we shall pursue the more realistic goal of estimating it numerically by using a Markov chain approximation method.

3.1 Aircraft-to-aircraft conflict problem

Consider two aircraft, say “aircraft 1” and “aircraft 2”, flying in the same region of the airspace during the time interval $T = [0, t_f]$. According to the ATM definition, the two aircraft encounter is said to be safe if the two aircraft are either at a horizontal distance greater than r or at a vertical distance greater than H during the whole duration of the encounter, where r and H are prescribed quantities [20]. Currently, for en-route airspace the minimum horizontal separation is 5 nautical miles (nmi), while inside the TRACON area it is reduced to 3 nmi. The minimum vertical separation is 2000 feet (ft) above the altitude of 29,000 ft (FL290), and 1000 ft below FL290. If the two aircraft get closer than r horizontally and H vertically at some instant $t \in T$, then, an aircraft-to-aircraft conflict occurs.

Denote the position of aircraft 1 and aircraft 2 by X_1 and X_2 , respectively. Based on equation (2), the evolutions of $X_1(\cdot)$ and $X_2(\cdot)$ over the time interval T are governed by the following stochastic differential equations:

$$dX_1(t) = u_1(t)dt + f(X_1, t)dt + \Sigma dB(X_1, t), \quad (3)$$

$$dX_2(t) = u_2(t)dt + f(X_2, t)dt + \Sigma dB(X_2, t), \quad (4)$$

with initial condition $X_1(0)$ and $X_2(0)$, respectively.

Our objective is to estimate the probability that a conflict occurs during the encounter. The probability of conflict can be expressed in terms of the relative position of the two aircraft as follows

$$P\{X_1(t) - X_2(t) \in \mathcal{D} \text{ for some } t \in T\},$$

where $\mathcal{D} \in \mathbb{R}^3$ is the closed cylinder of radius r and height $2H$ centered at the origin modeling the protection zone surrounding each aircraft.

Here, we focus our attention on the case when the wind field $f(x, t)$ is affine in x , i.e., it can be expressed as

$$f(x, t) = R(t)x + d(t), \quad \forall x \in \mathbb{R}^3, t \in T,$$

where $R: T \rightarrow \mathbb{R}^{3 \times 3}$ and $d: T \rightarrow \mathbb{R}^3$ are continuous functions. We shall show that in this case there is a particularly simple way to approximate the probability of conflict.

The relative position Y and the relative airspeed v of aircraft 1 and aircraft 2 are respectively given by

$$Y \triangleq X_2 - X_1, \quad v \triangleq u_2 - u_1.$$

Since the positions of the two aircraft, X_1 and X_2 , are governed by equations (3) and (4), by subtracting (3) from (4), we have

$$dY(t) = v(t)dt + R(t)Y(t)dt + \Sigma d[B(X_2, t) - B(X_1, t)]. \quad (5)$$

Define $Z(t) \triangleq B(X_2, t) - B(X_1, t)$. For each fixed X_1 and X_2 , $Z(t)$ is a Gaussian process with zero mean and covariance

$$E\{[Z(t_2) - Z(t_1)][Z(t_2) - Z(t_1)]^T\} = 2[1 - \rho(X_2 - X_1)](t_2 - t_1)I_3, \quad \forall t_1 \leq t_2.$$

Note also that $Z(0) = 0$. Therefore, in terms of distribution we have for fixed X_1 and X_2

$$Z(t) \stackrel{d}{\simeq} \sqrt{2[1 - \rho(X_2 - X_1)]} W(t), \quad (6)$$

where $W(t)$ is a standard three-dimensional Brownian motion. In general, since X_1, X_2 are themselves time-varying stochastic processes whose outcomes depend on B , hence on Z , (6) is not valid. On the other hand, if the function ρ is relatively slow-varying outside the cylinder \mathcal{D} , then $\rho(X_2 - X_1)$ can be thought of as locally constant near each time epoch. As a result, (6) still holds approximately, and in turn $dZ(t)$ can be approximated by $\sqrt{2[1 - \rho(X_2 - X_1)]} dW(t) = \sqrt{2[1 - \rho(Y)]} dW(t)$. We can thus approximate equation (5) weakly by

$$dY(t) = v(t)dt + R(t)Y(t)dt + \sqrt{2[1 - \rho(y)]}\Sigma dW(t). \quad (7)$$

3.2 Aircraft-to-airspace conflict problem

Consider an aircraft flying in some region of the airspace, close to a certain prohibited area such as, e.g., a SUA area or a severe weather zone. An aircraft-to-airspace conflict occurs if the aircraft enters the prohibited area within the look-ahead time horizon T . Suppose that the aircraft is following a specified flight plan and we want to evaluate if its flight plan is conflict-free or it should be modified so as to timely steer the aircraft out of the forbidden airspace area. If this area can be described by a compact set $\mathcal{D} \subset \mathbb{R}^3$ and the aircraft position $X(\cdot)$ during the time horizon of interest T is governed by equation (2), then this problem can be solved by estimating the probability

$$P\{X(t) \in \mathcal{D} \text{ for some } t \in T\}. \quad (8)$$

For the purpose of computing this probability, we can replace $B(\cdot, \cdot)$ with a standard three dimensional Brownian motion $W(\cdot)$. This is because we are considering a single aircraft, and we assumed that, for each fixed $x \in \mathbb{R}^3$, $B(x, \cdot)$ is a standard three dimensional Brownian motion and $B(\cdot, \cdot)$ is time increment independent and stationary. We can then refer to the equation

$$dX(t) = u(t)dt + f(X, t)dt + \Sigma dW(t), \quad (9)$$

initialized with $X(0)$ to compute the probability in equation (8).

4 Stochastic approximation method

In this section we describe a method for estimating the probability that the solution over the time interval T to

$$dS(t) = a(S, t)dt + b(S)\Sigma dW(t) \quad (10)$$

with initial condition $S(0)$ enters a certain compact set \mathcal{D} , where $a : \mathbb{R}^3 \times T \rightarrow \mathbb{R}^3$ and $b : \mathbb{R}^3 \rightarrow R$ are piecewise continuous functions, $\Sigma = \text{diag}(\sigma_h, \sigma_h, \sigma_v)$, and $W(\cdot)$ is a standard three dimensional Brownian motion. Note that the aircraft-to-aircraft and aircraft-to-airspace safety analysis problems are particular cases of the problem addressed in this section. Indeed, if we set $a(s, t) = v(t) + R(t)s(t)$ and $b(s) = \sqrt{2[1 - \rho(s)]}$, $t \in T$ and $s \in \mathbb{R}^3$, equation (10) reduces to equation (7), whereas if we set $a(s, t) = v(t) + f(s, t)$ and $b(s) = 1$, it reduces to equation (9).

To evaluate the probability of interest

$$P\{S(t) \in \mathcal{D} \text{ for some } t \in T\}$$

numerically, we consider an open domain $\mathcal{U} \subset \mathbb{R}^3$ that contains \mathcal{D} and has a compact support. \mathcal{U} should be large enough so that the situation can be declared safe once S wanders outside \mathcal{U} . With reference to the domain \mathcal{U} , the probability of entering the unsafe set \mathcal{D} can be expressed as

$$P_c \triangleq P\{S \text{ hits } \mathcal{D} \text{ before hitting } \mathcal{U}^c \text{ within the time interval } T\}. \quad (11)$$

Implicit in the above definition is that if S hits neither \mathcal{D} nor \mathcal{U}^c during T , still no conflict occurs. For the purpose of computing (11), we can assume that in equation (10), S is defined on the open domain $\mathcal{U} \setminus \mathcal{D}$ with initial condition $S(0)$, and that it is stopped as soon as it hits the boundary $\partial\mathcal{U} \cup \partial\mathcal{D}$.

We now describe an approach to approximate the solution $S(\cdot)$ to equation (10) defined on $\mathcal{U} \setminus \mathcal{D}$. The idea is to discretize $\mathcal{U} \setminus \mathcal{D}$ into grid points that constitute the state space of a Markov chain. By carefully choosing the transition probabilities, the solution to the Markov chain will converge weakly to that of the stochastic differential equation (10) as the grid size approaches zero. Therefore, at a small grid size, a good estimate of P_c is provided by the corresponding quantity associated with the Markov chain, which is much easier to compute.

To define the Markov chain, we first introduce some notations. Fix a grid size $\delta > 0$. Denote by $\delta\mathbb{Z}^3$ the integer grids of \mathbb{R}^3 scaled properly, more precisely,

$$\delta\mathbb{Z}^3 = \{(m\delta, n\delta, l\kappa\delta) \mid m, n, l \in \mathbb{Z}\},$$

where κ is a constant defined as

$$\kappa \triangleq \sigma_v / \sigma_h.$$

Each grid point $q \in \delta\mathbb{Z}^3$ has six immediate neighbors:

$$\begin{aligned} q_w &= q + (-\delta, 0, 0), & q_e &= q + (\delta, 0, 0), \\ q_s &= q + (0, -\delta, 0), & q_n &= q + (0, \delta, 0), \\ q_d &= q + (0, 0, -\kappa\delta), & q_u &= q + (0, 0, \kappa\delta), \end{aligned} \quad (12)$$

four (q_w , q_e , q_s , and q_n) on the same horizontal plane at a distance δ along the x axis (q_w and q_e) and the y axis (q_s and q_n), and two (q_d and q_u) at a distance $\kappa\delta$ along the z axis.

Define $\mathcal{Q} = (\mathcal{U} \setminus \mathcal{D}) \cap \delta\mathbb{Z}^3$, which consists of all those grid points in $\delta\mathbb{Z}^3$ that lie inside \mathcal{U} but outside \mathcal{D} . The interior of \mathcal{Q} , denoted by \mathcal{Q}^0 , consists of all those points in \mathcal{Q} which have all their six neighbors in \mathcal{Q} . The boundary of \mathcal{Q} is defined to be $\partial\mathcal{Q} = \mathcal{Q} \setminus \mathcal{Q}^0$, and is the union of two disjoint sets: $\partial\mathcal{Q} = \partial\mathcal{Q}_\mathcal{U} \cup \partial\mathcal{Q}_\mathcal{D}$, where points in $\partial\mathcal{Q}_\mathcal{U}$ have at least one neighbor outside \mathcal{U} , and points in $\partial\mathcal{Q}_\mathcal{D}$ have at least one neighbor inside \mathcal{D} .

We now define a Markov chain $\{Q_k, k \geq 0\}$ on the state space \mathcal{Q} . $\{Q_k, k \geq 0\}$ is a time-inhomogeneous Markov chain such that:

1. each state in $\partial\mathcal{Q}$ is an absorbing state, i.e., the state of the chain remains unchanged once it has arrived at any of the states in $\partial\mathcal{Q}$;
2. starting from a state q in \mathcal{Q}^0 , the chain jumps to one of its six neighbors defined in (12) or stays at the same state according to transition probabilities determined by its current location q and the current time step k as follows:

$$P\{Q_{k+1} = q' | Q_k = q\} = \begin{cases} p_w^k(q) = \frac{\exp(-\delta\xi_q^k)}{C_q^k}, & q' = q_w \\ p_e^k(q) = \frac{\exp(\delta\xi_q^k)}{C_q^k}, & q' = q_e \\ p_s^k(q) = \frac{\exp(-\delta\eta_q^k)}{C_q^k}, & q' = q_s \\ p_n^k(q) = \frac{\exp(\delta\eta_q^k)}{C_q^k}, & q' = q_n \\ p_d^k(q) = \frac{\exp(-\delta\zeta_q^k)}{C_q^k}, & q' = q_d \\ p_u^k(q) = \frac{\exp(\delta\zeta_q^k)}{C_q^k}, & q' = q_u \\ p_o^k(q) = \frac{\chi_q^k}{C_q^k}, & q' = q. \end{cases} \quad (13)$$

The parameters in the above expression are given by

$$\begin{aligned} \xi_q^k &= \frac{[a(q, k\Delta t)]_1}{\sigma_h^2 b(q)^2}, \\ \eta_q^k &= \frac{[a(q, k\Delta t)]_2}{\sigma_h^2 b(q)^2}, \\ \zeta_q^k &= \frac{[a(q, k\Delta t)]_3}{\kappa \sigma_h^2 b(q)^2}, \\ \chi_q^k &= \frac{2}{\lambda \sigma_h^2 b(q)^2} - 6, \\ C_q^k &= 2 \cosh(\delta\xi_q^k) + 2 \cosh(\delta\eta_q^k) + 2 \cosh(\delta\zeta_q^k) + \chi_q^k, \end{aligned}$$

where for an arbitrary vector $z \in \mathbb{R}^3$, $[z]_i$ denotes its i -th component with $i = 1, 2, 3$. λ is a positive constant that has to be chosen small enough such that χ_q^k defined above is positive for all q and all k . In particular, this is guaranteed if $0 < \lambda \leq (3\sigma_h^2 \max_q b(q)^2)^{-1}$. $\Delta t > 0$ is the amount of time elapsed between any two successive discrete time steps k and $k+1$, $k \geq 0$. Here, we set $\Delta t = \lambda \delta^2$.

Suppose that at some time step k the chain is at state $q \in \mathcal{Q}^0$. Define

$$\begin{aligned} m_q^k &= \frac{1}{\Delta t} E\{Q_{k+1} - Q_k | Q_k = q\}, \\ V_q^k &= \frac{1}{\Delta t} E\{(Q_{k+1} - Q_k)(Q_{k+1} - Q_k)^T | Q_k = q\}. \end{aligned}$$

Direct computation shows that

$$m_q^k = \frac{2}{\lambda \delta C_q^k} [\text{sh}(\delta\xi_q^k) \quad \text{sh}(\delta\eta_q^k) \quad \kappa \text{sh}(\delta\zeta_q^k)]^T, \quad V_q^k = \frac{2}{\lambda C_q^k} \text{diag}(\cosh(\delta\xi_q^k), \cosh(\delta\eta_q^k), \kappa^2 \cosh(\delta\zeta_q^k)).$$

If for each $\delta > 0$ we choose q to be a point in \mathcal{Q}^0 closest to a fixed point $s \in \mathcal{U} \setminus \mathcal{D}$, then it can be easily verified that as $\delta \rightarrow 0$,

$$m_q^k \rightarrow a(s, k\Delta t), \quad V_q^k \rightarrow b(s)^2 \Sigma^2.$$

Suppose that the chain $\{Q_k, k \geq 0\}$ starts from a point $\bar{q} \in \mathcal{Q}$ closest to $S(0)$. Then, by Theorem 8.7.1 in [3] (see also [21]), we conclude that

Proposition 1 *Fix $\delta > 0$ and consider the corresponding Markov chain $\{Q_k, k \geq 0\}$. Denote by $\{Q(t), t \geq 0\}$ the stochastic process that is equal to Q_k on the time interval $[k\Delta t, (k+1)\Delta t)$ for all k . Then as $\delta \rightarrow 0$, $\{Q(t), t \geq 0\}$ converges weakly to the solution $\{S(t), t \geq 0\}$ to equation (10) defined on $\mathcal{U} \setminus \mathcal{D}$ with absorption on the boundary $\partial\mathcal{U} \cup \partial\mathcal{D}$.*

Let $k_f \triangleq \lfloor \frac{t_f}{\Delta t} \rfloor$ be the largest integer not exceeding $t_f/\Delta t$ ($k_f = \infty$ if $t_f = \infty$). Because of the weak convergence of $Q(t)$ to $S(t)$, if we fix a small δ , then the probability P_c in (11) can be approximated by the corresponding probability

$$P_{c,\delta} \triangleq P\{Q_{k_f} \in \partial\mathcal{Q}_{\mathcal{D}}\} = P\{Q_k \text{ hits } \partial\mathcal{Q}_{\mathcal{D}} \text{ before hitting } \partial\mathcal{Q}_{\mathcal{U}} \text{ within } 0 \leq k \leq k_f\}, \quad (14)$$

with the chain $\{Q_k, k \geq 0\}$ starting from a point $\bar{q} \in \mathcal{Q}$ closest to $S(0)$.

Remark 2 *In a time interval of length $\Delta t = \lambda\delta^2$, the maximal distance that the Markov chain can travel is δ horizontally and $\kappa\delta$ vertically. Thus given \mathcal{U} and $a(s, t)$, $t \in T$, $s \in \mathcal{U} \setminus \mathcal{D}$, for $S(t)$ to be approximated by the Markov chain, the quantities $\|a(s, t)\|_h$ and $\|a(s, t)\|_v$, $t \in T$, $s \in \mathcal{U} \setminus \mathcal{D}$, have to be upper bounded roughly by $1/\lambda\delta$ and $\kappa/\lambda\delta$, respectively, where λ satisfies $0 < \lambda \leq (3\sigma_h^2 \max_q b(q)^2)^{-1}$. This condition translates into upper bounds on the admissible values for δ .*

5 Iterative Algorithm

We next describe an iterative procedure to compute $P_{c,\delta}$ defined in equation (14) in the finite horizon case ($t_f < \infty$). The infinite horizon case ($t_f = \infty$) can be addressed in a similar way as discussed in [8] for the two dimensional case.

Define the set of functions $P_{c,\delta}^{(k)} : \mathcal{Q} \rightarrow [0, 1]$, $k = 0, 1, \dots, k_f$, as follows:

$$P_{c,\delta}^{(k)}(q) \triangleq P\{Q_{k_f} \in \partial\mathcal{Q}_{\mathcal{D}} | Q_k = q\}, \quad q \in \mathcal{Q}. \quad (15)$$

Since the chain $\{Q_k, k \geq 0\}$ starts at \bar{q} at $k = 0$, the desired quantity $P_{c,\delta}$ can be expressed as $P_{c,\delta} = P_{c,\delta}^{(0)}(\bar{q})$. The procedure we propose below determines the whole set of functions $P_{c,\delta}^{(k)} : \mathcal{Q} \rightarrow \mathbb{R}$, $k = 0, 1, \dots, k_f$. This has the advantage that at any future time $t \in [0, t_f]$, should the flight plan on $[t, t_f]$ remain unchanged, an estimate of the probability of entering the unsafe region over the new time horizon $[t, t_f]$ from any $q \in \mathcal{Q}$ is readily available, eliminating the need for a re-computation.

Observe that for any k such that $0 \leq k < k_f$, $P_{c,\delta}^{(k)} : \mathcal{Q} \rightarrow [0, 1]$ satisfies the following recursive equation

$$P_{c,\delta}^{(k)}(q) = \begin{cases} p_o^k(q)P_{c,\delta}^{(k+1)}(q) + p_w^k(q)P_{c,\delta}^{(k+1)}(q_w) + p_e^k(q)P_{c,\delta}^{(k+1)}(q_e) \\ \quad + p_s^k(q)P_{c,\delta}^{(k+1)}(q_s) + p_n^k(q)P_{c,\delta}^{(k+1)}(q_n) \\ \quad + p_d^k(q)P_{c,\delta}^{(k+1)}(q_d) + p_u^k(q)P_{c,\delta}^{(k+1)}(q_u), & \text{if } q \in \mathcal{Q}^0 \\ 1, & \text{if } q \in \partial\mathcal{Q}_{\mathcal{D}} \\ 0, & \text{if } q \in \partial\mathcal{Q}_{\mathcal{U}}. \end{cases} \quad (16)$$

The probability $P_{c,\delta} = P_{c,\delta}^{(0)}(\bar{q})$ can then be computed in a recursive way by iterating equation (16) backward k_f times starting from $k = k_f - 1$ and using the initialization

$$P_{c,\delta}^{(k_f)}(q) = \begin{cases} 1, & \text{if } q \in \partial \mathcal{Q}_{\mathcal{D}} \\ 0, & \text{otherwise.} \end{cases} \quad (17)$$

The reason for the above initialization is obvious considering the definition (15) with $k = k_f$.

The following algorithm summarizes the procedure to compute an approximation of P_c in the finite horizon case:

Algorithm 1 Given $S(0)$, $a : \mathbb{R}^3 \times T \rightarrow \mathbb{R}^3$ and $b : \mathbb{R}^3 \rightarrow \mathbb{R}$, then

1. Fix $\delta > 0$ and $\lambda \in (0, \frac{1}{3\sigma_h^2 \max_s b(s)^2}]$. Set $\Delta t = \lambda \delta^2$. Define the Markov chain $\{Q_k, k \geq 0\}$ with state space $\mathcal{Q} = (\mathcal{U} \setminus \mathcal{D}) \cap \delta \mathbb{Z}^3$ and transition probabilities given by (13).
2. Set $k = k_f$ and initialize $P_{c,\delta}^{(k)}$ as in equation (17).
3. For $k = k_f - 1, \dots, 0$, compute $P_{c,\delta}^{(k)}$ from $P_{c,\delta}^{(k+1)}$ according to equation (16).
4. Choose a point \bar{q} in \mathcal{Q} closest to $S(0)$ and output $P_{c,\delta} = P_{c,\delta}^{(0)}(\bar{q})$.

Remark 3 For a fixed δ , the size of the state space \mathcal{Q} is of the order of $1/\delta^3$. Since the number of iterations is given by $k_f \simeq t_f/\Delta t = O(1/\delta^2)$, the running time of Algorithm 1 grows as $O(1/\delta^5)$ as $\delta \rightarrow 0$. Accuracy in estimating the probability P_c and computation time should be appropriately balanced when choosing the grid size δ .

Remark 4 Algorithm 1 can be extended to the case when the position $S(0)$ is not known precisely. Suppose that $S(0)$ is described as a random variable with distribution $\mu_S(s)$, $s \in \mathcal{U} \setminus \mathcal{D}$. Then, the probability of entering the unsafe set \mathcal{D} can be expressed as

$$P_c = \int_{\mathcal{U} \setminus \mathcal{D}} p_c(s) d\mu_S(s), \quad (18)$$

where $p_c : \mathcal{U} \setminus \mathcal{D} \rightarrow [0, 1]$ is defined by

$$p_c(s) \triangleq P\{S \text{ hits } \mathcal{D} \text{ before hitting } \mathcal{U}^c \text{ within the time interval } T \mid S(0) = s\}, \quad (19)$$

and assigns to each $s \in \mathcal{U} \setminus \mathcal{D}$ the probability of entering the unsafe set \mathcal{D} over the time horizon T when $S(0) = s$. The integral (18) reduces to a finite summation when approximating the map p_c with $P_{c,\delta}^{(0)}$.

6 Examples

6.1 Aircraft-to-aircraft conflict problem

We consider a two aircraft encounter and evaluate the probability that a conflict situation occurs within the time horizon $T = [0, 15]$.

The relative velocity of the two aircraft during T is given by

$$v(t) = \begin{cases} (2, 0, 0), & 0 \leq t < 5; \\ (0, 1, 0), & 5 \leq t < 10; \\ (2, 0, 0), & 10 \leq t \leq 15, \end{cases}$$

whereas the wind field f is suppose to be identically zero. We suppose that $r = 3$ and $H = 1$, and that the wind spatial correlation is specified by the function $\rho(x) = \exp(-\beta_h \|x\|_h - \beta_v \|x\|_v)$, $x \in \mathbb{R}^3$, with $\beta_h = 1/5$ and $\beta_v = 1/2$. Let $\sigma_h = 1$, $\sigma_v = 0.5$.

Based on the values taken by T , $v(\cdot)$, r and H , we choose the domain \mathcal{U} to be $\mathcal{U} = (-30, 6) \times (-15, 11) \times (-5, 5)$. Finally, we set the discretization step size $\delta = 1$, and $\lambda = (6\sigma_h^2)^{-1} = 1/6$. Thus $\Delta t = \lambda\delta^2 = 1/6$.

Figure 1 represents the estimated probability of conflict $P_{c,\delta}$ computed by Algorithm 1 as a function of the relative position of the two aircraft. The plots refer to the time instances $t = 0$, $t = 5$, and $t = 10$, shown column-wise from left to right. In each column, the first row represent the level curves of $P_{c,\delta}$ restricted to the horizontal plane at altitude 0, while the other rows represent the three dimensional isosurface $P_{c,\delta} = 0.2$ viewed from different angles. The relevance of isosurfaces is that, in practice, once the relative position of the two aircraft is within the isosurface at a prescribed threshold value, an alarm of corresponding severity should be issued to the pilots to warn them on the level of criticality of the situation ([23]).

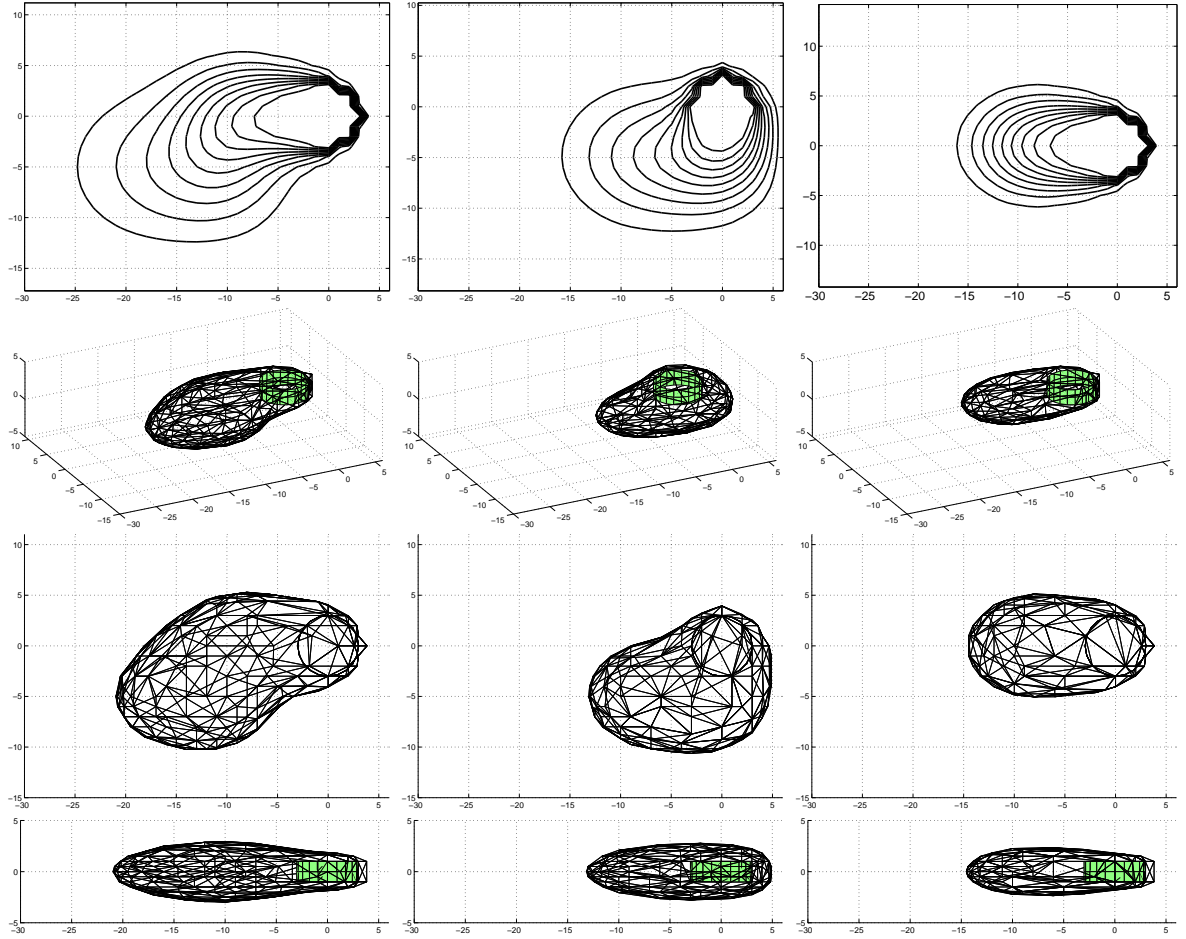


Figure 1: Estimated probability of conflict $P_{c,\delta}$ over the time horizon $[t, 15]$ as a function of the two aircraft relative position at time t . Left: $t = 0$; Center: $t = 5$; Right: $t = 10$. Top row: level curves of $P_{c,\delta}$ restricted to the plane at altitude 0; Second row: three dimensional plot of the isosurface $P_{c,\delta} = 0.2$; Third row: top view of the isosurface $P_{c,\delta} = 0.2$; Bottom row: side view of the isosurface $P_{c,\delta} = 0.2$ ($\beta_h = 1/5$; $\beta_v = 1/2$).

In Figure 2 we report similar plots for $P_{c,\delta}$ in the case when the parameters β_h and β_v of the spatial correlation function ρ are set equal to $\beta_h = \beta_v = 1/20$. Note that in this case the

wind spatial correlation is increased. As a consequence of this fact, the isosurface $P_{c,\delta} = 0.2$ concentrates more tightly along the deterministic path that leads to a conflict, and it extends longer as well.

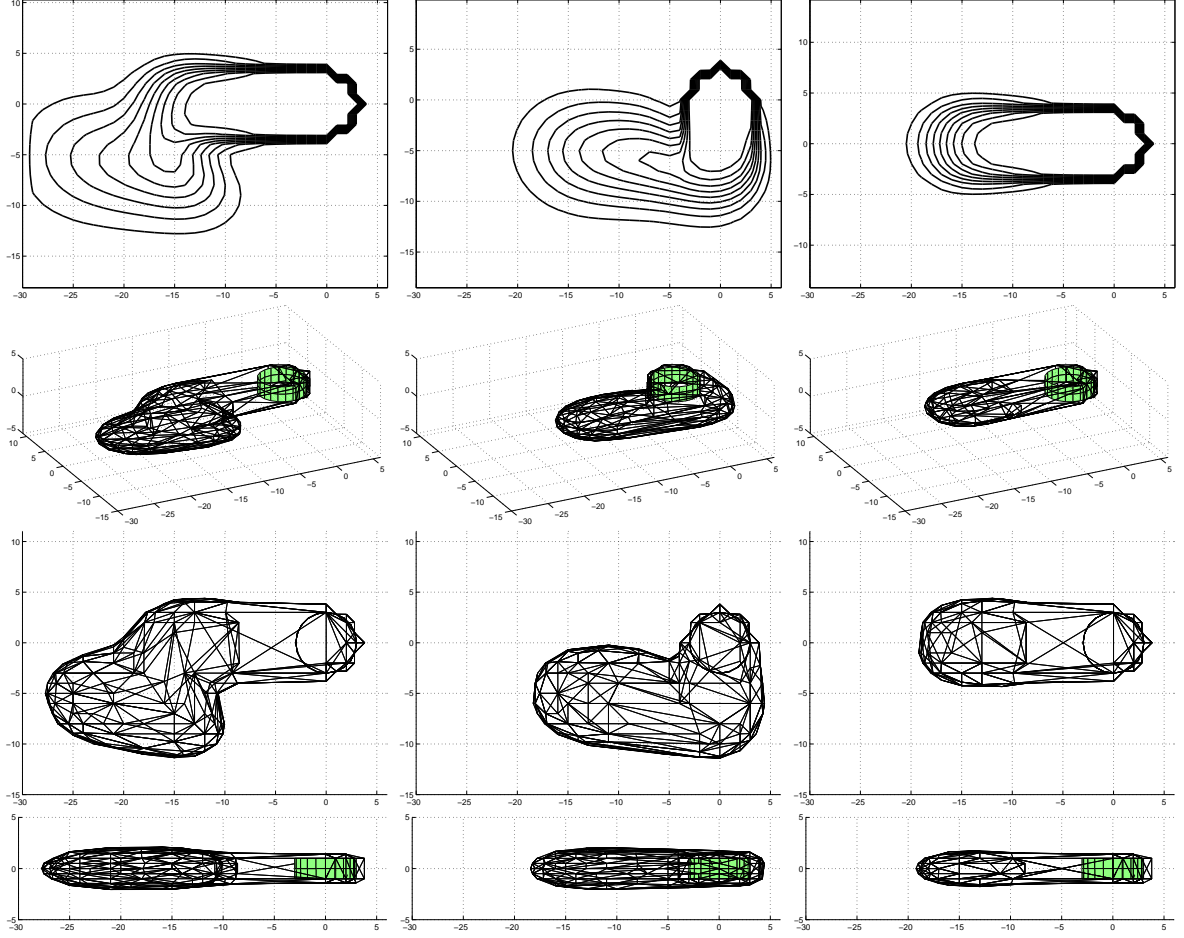


Figure 2: Estimated probability of conflict $P_{c,\delta}$ over the time horizon $[t, 15]$ as a function of the two aircraft relative position at time t . Left: $t = 0$; Center: $t = 5$; Right: $t = 10$. Top row: level curves of $P_{c,\delta}$ restricted to the plane at altitude 0; Second row: three dimensional plot of the isosurface $P_{c,\delta} = 0.2$; Third row: top view of the isosurface $P_{c,\delta} = 0.2$; Bottom row: side view of the isosurface $P_{c,\delta} = 0.2$ ($\beta_h = \beta_v = 1/20$).

6.2 Aircraft-to-airspace conflict problem

Consider a prohibited airspace area \mathcal{D} given by the union of two ellipsoids: the first one specified by

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2(x_1 + 4)^2 + (x_2 - 4)^2 + 10x_3^2 \leq 9\},$$

and the second one specified by

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + 2(x_2 + 5)^2 + 10x_3^2 \leq 16\},$$

in the (x_1, x_2, x_3) Cartesian coordinate system with x_3 representing the flight level. Suppose that the aircraft is flying along the x_1 -axis while climbing up at an accelerated rate according to the flight plan $u(t) = (3/2, 0, 2t/75)$, $t \in T = [0, 15]$. Let $\sigma_h = 1$, and $\sigma_v = 0.5$. Figure 3 shows the plots of the isosurface $P_{c,\delta} = 0.2$ at time $t = 0$, $t = 5$, and $t = 10$, viewed from three different angles. $P_{c,\delta}$ is computed according to Algorithm 1 with $\mathcal{U} = (-38, 6) \times (-15, 11) \times (-6, 3)$.

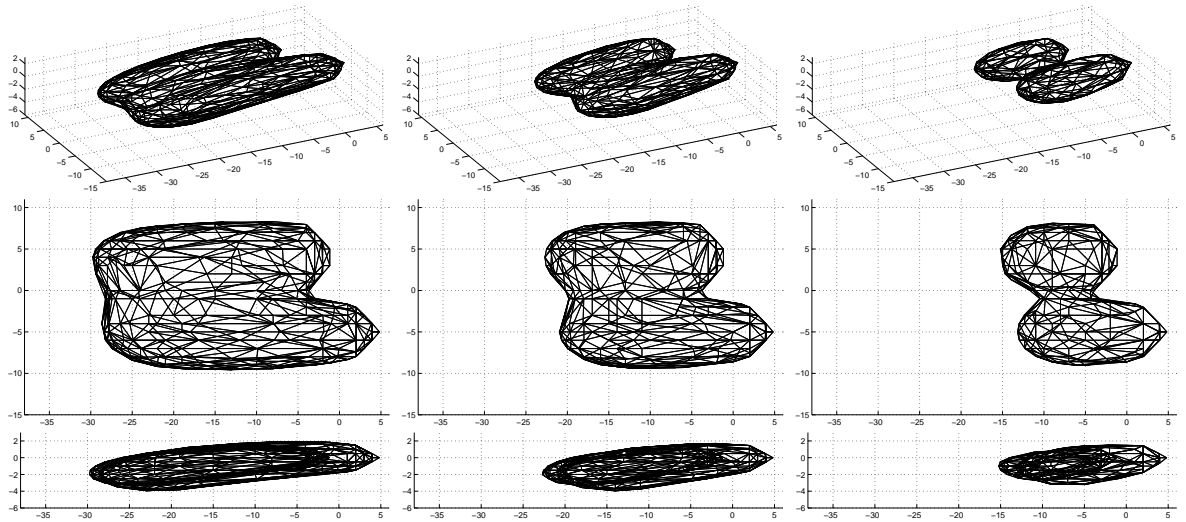


Figure 3: Estimated probability $P_{c,\delta}$ of entering the prohibited airspace area \mathcal{D} over the time horizon $[t, 15]$ as a function of the aircraft position at time t : isosurface corresponding to $P_{c,\delta} = 0.2$. Left: $t = 0$; Center: $t = 5$; Right: $t = 10$. First row: 3D plot; Second row: top view; Third row: side view.

7 Conclusions

In this paper a kinematic model of the aircraft motion in a three dimensional wind field with spatially correlated random perturbations was introduced. Based on this model, two safety problems arising in aircraft flight were studied: the conflict between two aircraft, and the intrusion into a forbidden airspace area by a single aircraft. An iterative algorithm was proposed to compute the probability of occurrence of these events given the aircraft flight plans, based on a Markov chain approximation scheme.

The method for safety analysis proposed in this paper can be extended in various ways to address more general cases. For example, the forbidden airspace area may evolve in time. Also, the model of the aircraft motion can be made more complicated, for example, including a dependence from the altitude of the random field variance, or considering a second order dynamics model with the wind affecting both first-order and second-order terms. In each of these cases, one can adopt the same approximation scheme. However, the dimension of the state space of the resulting Markov chain is generally larger than three, which is the state space dimension in the case addressed in this paper.

References

- [1] R.J. Adler. *The Geometry of Random Fields*. John Wiley & Sons, 1981.
- [2] H.-F. Chen. *Stochastic Approximation and Its Application*. Kluwer Academic Publishers, 2002.
- [3] R. Durrett. *Stochastic calculus: A practical introduction*. CRC Press, 1996.
- [4] H. Erzberger, R.A. Paielli, D.R. Isaacson, and M.M. Eshow. Conflict detection and resolution in the presence of prediction error. In *Proc. of the 1st USA/Europe Air Traffic Management R & D Seminar*, Saclay, France, June 1997.
- [5] J. Hu, J. Lygeros, M. Prandini, and S. Sastry. Aircraft conflict prediction and resolution using Brownian Motion. In *Proc. of the 38th Conf. on Decision and Control*, Phoenix, AZ, December 1999.

- [6] J. Hu, M. Prandini, and S. Sastry. Optimal maneuver for multiple aircraft conflict resolution: a braid point of view. In *Proc. of the 39th Conf. on Decision and Control*, pages 4164–4169, Sydney, Australia, December 2000.
- [7] J. Hu, M. Prandini, and S. Sastry. Optimal coordinated maneuvers for three dimensional aircraft conflict resolution. *Journal of Guidance, Control and Dynamics*, 25(5):888–900, 2002.
- [8] J. Hu, M. Prandini, and S. Sastry. Aircraft conflict detection in presence of spatially correlated wind perturbations. Technical report, University of Brescia, February 2003.
- [9] D.B. Kirk, K.C. Bowen, W.S. Heagy, N.E. Rozen, and K.J. Vietz. Development and assessment of problem resolution capabilities for the en route sector controller. In *Proc. of the AIAA Aircraft, Technology Integration, and Operations Forum, 1st, AIAA-2001-5255*, Los Angeles, CA, October 2001.
- [10] J. Kosecka, C. Tomlin, G.J. Pappas, and S. Sastry. Generation of Conflict Resolution Maneuvers For Air Traffic Management. In *Proc. of the IEEE Conference on Intelligent Robotics and System '97*, volume 3, pages 1598–1603, Grenoble, France, September 1997.
- [11] J. Krozel and M. Peters. Strategic conflict detection and resolution for free flight. In *Proc. of the 36th Conf. on Decision and Control*, volume 2, pages 1822–1828, San Diego, CA, December 1997.
- [12] J.K. Kuchar and L.C. Yang. A review of conflict detection and resolution modeling methods. *IEEE Transactions on Intelligent Transportation Systems, Special Issue on Air Traffic Control - Part I*, 1(4):179–189, 2000.
- [13] H. Kushner and G. Yin. *Stochastic Approximation Algorithms and Applications*. Springer-Verlag, 1997.
- [14] J. Lygeros and N. Lynch. On the formal verification of the TCAS conflict resolution algorithms. In *Proc. of the 36th Conf. on Decision and Control*, pages 1829–1834, San Diego, CA, December 1997.
- [15] J. Lygeros and M. Prandini. Aircraft and weather models for probabilistic conflict detection. In *Proc. of the 41st Conf. on Decision and Control*, Las Vegas, NV, December 2002.
- [16] F. Medioni, N. Durand, and J.M. Alliot. Air traffic conflict resolution by genetic algorithms. In *Proc. of the Artificial Evolution, European Conference (AE 95)*, pages 370–383, Brest, France, September 1995.
- [17] P.K. Menon, G.D. Sweriduk, and B. Sridhar. Optimal strategies for free-flight air traffic conflict resolution. *Journal of Guidance, Control, and Dynamics*, 22(2):202–211, 1999.
- [18] R.A. Paielli and H. Erzberger. Conflict probability estimation for free flight. *Journal of Guidance, Control, and Dynamics*, 20(3):588–596, 1997.
- [19] M. Prandini, J. Hu, J. Lygeros, and S. Sastry. A probabilistic approach to aircraft conflict detection. *IEEE Transactions on Intelligent Transportation Systems, Special Issue on Air Traffic Control - Part I*, 1(4):199–220, 2000.
- [20] Radio Technical Commission for Aeronautics. Minimum operational performance standards for traffic alert and collision avoidance system (TCAS) airborne equipment. Technical Report RTCA/DO-185, RTCA, September 1990. Consolidated Edition.
- [21] D.W. Stroock and S.R.S. Varadhan. *Multidimensional Diffusion Processes*. Springer-Verlag, 1979.
- [22] C. Tomlin, G.J. Pappas, and S. Sastry. Conflict resolution for air traffic management: A study in multi-agent hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):509–521, 1998.
- [23] L.C. Yang and J. Kuchar. Prototype conflict alerting system for free flight. In *Proc. of the AIAA 35th Aerospace Sciences Meeting, AIAA-97-0220*, Reno, NV, January 1997.
- [24] Y. Zhao and R. Schultze. Deterministic resolution of two aircraft conflict in free flight. In *Proc. of the AIAA Guidance, Navigation, and Control Conference, AIAA-97-3547*, New Orleans, LA, August 1997.