Graph Control Lyapunov Function for Stabilization of Discrete-Time Switched Linear Systems ⋆

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Abstract

The goal of this paper is to study stabilization problems for discrete-time switched linear systems (SLSs). To analyze the switching stabilizability, we introduce the notion of graph control Lyapunov functions (GCLFs). The GCLF is a set of Lyapunov functions which satisfy several Lyapunov inequalities associated with a weighted digraph. Each Lyapunov function represents each node in the digraph, and each Lyapunov inequality represents a subgraph consisting of the edges connecting a node and its out-neighbors (a directed rooted tree). The weight of each directed edge indicates the decay or growth rate of the Lyapunov functions from the tail to the head of the edge. It is proved that an SLS is switching stabilizable if and only if there exists a GCLF. The GCLF is an extension of the recently developed graph Lyapunov function for stability of SLSs under arbitrary switching to switching stabilization problems, and it is proved that the GCLF approach unifies several control Lyapunov functions and related stabilization theorems. Finally, computational methods are developed to evaluate the stabilizability and estimate exponential convergence rates of SLSs.

Key words: Switched linear systems; control Lyapunov function; switching stabilization; graph theory.

1 Introduction

Switched linear systems (SLSs) are a class of hybrid systems where the system dynamics matrix is switched among a finite set of indexed subsystem matrices, each of which is called a mode. The SLSs have received a great deal of attention during the past decades. A fundamental problem of the SLSs is to analyze their stability/stabilizability and design the stabilizing controls [1]. In the stability problem, it is assumed that the switchings among the modes are arbitrary, while in the stabilization problem, the mode is assumed to be controlled in the autonomous system case.

A predominant approach to tackle these problems is to construct a Lyapunov or Lyapunov-like function [2]. The simplest one is a common quadratic Lyapunov function [3–5], which however has inherent conservatism [6]. For instance, it was proved in [7] that, for stabilization, even the existence of a convex Lyapunov function is only sufficient but not necessary. For the stability problem, the existence of convex homogeneous Lyapunov functions is necessary and sufficient [8].

A natural way to reduce the conservatism is to search for more general Lyapunov functions, for instance, multiple Lyapunov functions [9, 10], piecewise quadratic Lyapunov functions (PWQLF) [11–16], polyhedral or polytopic Lyapunov functions [17], sum-of-squares polynomial Lyapunov functions [18, 19], convex hull Lyapunov functions [20, 21], and switched Lyapunov functions [22, 23]. Other approaches include the joint spectral radius (JSR) [24, 25] and the generating function method [26]. In particular, the existence of some classes of Lyapunov functions was proved to be necessary and sufficient for the stability/stabilizability of the SLSs, for example, the switched Lyapunov function [23], the polyhedral Lyapunov function [17], the sum-of-squares polynomial Lyapunov functions [19], and the PWQLF in [26] for the stability, and the PWQLF [16] for the stabilizability.

Another progress of the classical Lyapunov method is the so-called non-monotonic Lyapunov functions. The value of such functions may not necessarily decrease at each time step along the state trajectories as in the case of classical Lyapunov functions. To the authors' knowledge, the non-monotonic Lyapunov functions were first proposed in [27, 28] for nonlinear and switching systems, and recently generalized in [29] to the graph Lyapunov functions (GLFs), where a finite set of non-monotonic Lyapunov functions is used to certify the stability in

⋆ This paper was not presented at any IFAC meeting.

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a graph theoretic manner. A special class of the non-monotonic Lyapunov functions is the periodic or aperiodic Lyapunov functions (PLF or APLF) [30–32, 50] whose value decreases periodically or aperiodically in time. It was proved in [30–32, 50] that the existence of quadratic (PLF or APLF) functions is necessary and sufficient for the stabilizability of the SLSs. It should be pointed out that the PLF has been used earlier for the robust stability/stabilization of linear time-invariant (LTI) systems in [33,34], nonlinear systems [35], and periodic systems [36]. For continuous-time nonlinear systems, the notion of non-monotonic Lyapunov functions traces back to the use of the higher order derivatives of the Lyapunov functions developed in early work [37,38], which was further explored in the recent papers [39–41]. In spite of the extensive literature in this area, to the authors' knowledge, the GLF method has not been applied to the stabilization problem so far.

Motivated by this, the goal of this paper is to investigate switching stabilization of discrete-time autonomous SLSs. For this purpose, we introduce graph control Lyapunov functions (GCLFs), which are an extension of the GLF introduced in [29] for stability of SLSs. The GCLF is a set of Lyapunov functions that satisfy a set of Lyapunov inequalities associated with a weighted digraph (directed graph). Each Lyapunov function represents a node in the digraph, and each Lyapunov inequality corresponds to a subgraph consisting of the edges connecting a node and its out-neighbors (a directed rooted tree structure). Each edge represents state transitions, and the weight of each directed edge indicates the growth or decay rate of the Lyapunov functions along the state transitions corresponding to the edge. The state trajectories of SLSs correspond to paths in the digraph, along which the values of Lyapunov functions decrease to zero. Therefore, in the Lyapunov sense, GCLFs serve as certificates of stabilizability of SLSs.

The main contributions of this paper consist of the following: 1) The concept of GCLFs is introduced as an extension of the GLF [29]. Based on GCLFs, we derive a Lyapunov theorem, which provides a necessary and sufficient condition for stabilizability of SLSs. Compared to the GLF theorem in [29], our result does not require strict descent of the Lyapunov functions along each directed edge but along simple cycles in the digraph. 2) We develop two alternative Lyapunov theorems which are numerically more amenable but sufficient for stabilizability. To this end, we construct a positive SLSs associated with the digraph of the GCLF, and prove that the joint spectral radius (JSR) [24] of the positive SLSs provides a measure on how fast the exponential convergence of the original SLS is. 3) The proposed stabilizability conditions also provide explicit characterizations of the exponential convergence rate of state trajectories. 4) It is proved that the GCLF unifies several previous control Lyapunov theorems in the literature, hence providing connections among them. 5) Computational methods based on semidefinite programming (SDP) [42] are developed to evaluate stabilzability of SLSs. In addition, conservatism and convergence of the computational methods are studied. Lastly, a numerical example demonstrates that the GCLF potentially improve the existing stabilizability tests by yielding tighter approximations of the exponential convergence rate.

The paper is organized as follows. We begin in Section 2 by providing the notation, background on graph theory, and the problem formulation. Section 3 gives a formal definition of the GCLF, the graph control Lyapunov theorem, and two sufficient conditions for the existence of the GCLF. Section 5 contains examples of the GCLF and proves that the GCLF unifies various existing control Lyapunov functions. Section 6 suggests computational methods to compute the GCLF based on SDPs. Lastly, Section 7 concludes the paper.

2 Preliminaries

2.1 Notation

The adopted notation is as follows: $\mathbb{N}$ and $\mathbb{N}_+$: sets of nonnegative integers and positive integers, respectively; $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$: sets of real numbers, nonnegative real numbers, and positive real numbers, respectively; $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$: set of all $n \times m$ real matrices; A^T: transpose of matrix A; $A \succ 0$ ($A \prec 0$, $A \succeq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix A; I_n: $n \times n$ identity matrix; $||\cdot||$: Euclidean norm of a vector or spectral norm of a matrix; $\mathbb{S}^n$, $\mathbb{S}^n_+$, and $\mathbb{S}^n_{++}$: sets of symmetric, positive semi-definite, and positive definite $n\times n$ matrices, respectively; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$: minimum and maximum eigenvalues of symmetric matrix A, respectively; $e_j \in \mathbb{R}^n$, $j \in \{1, 2, \ldots, n\}$, is the j-th basis vector (all components are 0 except for the j-th component which is 1); for a matrix A, we write $[A]_{ij}$ to denote its entry on the i-th row and j-th column; $\mathrm{diag}(M_1, M_2, \ldots, M_n)$: the matrix with matrices $M_1, M_2, \ldots, M_n$ on the block-diagonal and zeros elsewhere; $\lceil x \rceil$: the minimum integer greater than $x \in \mathbb{R}$; $|\cdot|$: cardinality of a set and absolute value for real numbers; $\mathrm{trace}(A)$ and $\rho(A)$: the trace and the spectral radius of the matrix A; $\mathrm{conv}(\cdot)$: the convex hull.

2.2 Graph theory

A directed graph or digraph $G(\mathcal{V}, \mathcal{E})$ is defined by the set of the nodes $\mathcal{V} := \{1, 2, \ldots, m\}$ and the set of ordered note pairs $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ which represents the set of directed edges, where $(j, i) \in \mathcal{E}$ indicates the edge from node $j \in \mathcal{V}$ to node $i \in \mathcal{V}$. For a given node $j \in \mathcal{V}$, $\mathcal{N}_j^- := \{i \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ is called the set of its in-neighbors, and $\mathcal{N}_j^+ := \{i \in \mathcal{V} : (j, i) \in \mathcal{E}\}$

is called the set of its out-neighbors. A node which has no in-neighbors is called a source, and a node which has no out-neighbors is called a sink. The adjacency matrix $E \in \mathbb{R}^{m\times m}$ of $G(\mathcal{V}, \mathcal{E})$ is defined as the matrix with $[E]_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $[E]_{ij} = 0$ otherwise. A (finite) walk in a digraph $G(\mathcal{V}, \mathcal{E})$ is a finite sequence of nodes $\mathcal{W} = (v_0, v_1, \dots, v_{k-1}) \in \mathcal{V}^k$ such that $(v_i, v_{i+1}) \in \mathcal{E}$, $i \in \{0, 1, \dots, k-2\}$. The length of the walk $\mathcal{W}$, denoted by $|\mathcal{W}|$, is the number of edges, i.e., $|\mathcal{W}| = k - 1$ (We note that it should not be confused with the cardinality of $\mathcal{W}$). An infinite walk will be denoted by $\mathcal{W}_\infty$, i.e., $\mathcal{W}_\infty = (v_0, v_1, \dots) \in \mathcal{V}^\infty$. A closed walk is a walk $\mathcal{W} := (v_0, v_1, \dots, v_{k-1}) \in \mathcal{V}^k$ such that $v_{k-1} = v_0$. A path $\mathcal{P} := (v_0, v_1, \dots, v_{k-1}) \in \mathcal{V}^k$ in the digraph $G(\mathcal{V}, \mathcal{E})$ is a walk such that $v_0, v_1, \dots, v_{k-1}$ are all distinct. A simple cycle $\mathcal{C} := (v_0, v_1, \dots, v_{k-1}) \in \mathcal{V}^k$ is a path with $k \geq 2$, and $v_{k-1} = v_0$. A (self) loop in $G(\mathcal{V}, \mathcal{E})$ is an edge $(v_1, v_2) \in \mathcal{E}$ such that $v_1 = v_2$, which is regarded as a simple cycle in this paper. The digraph $G(\mathcal{V}, \mathcal{E})$ is strongly connected if for every $v_1, v_2 \in \mathcal{V}$, there is a path starting at v_1 and ending at v_2. Given a digraph $G(\mathcal{V}, \mathcal{E})$, define a mapping $w : \mathcal{E} \to \mathbb{R}$, where $w(j, i)$, $(j, i) \in \mathcal{E}$, represents the weight of the edge $(j, i) \in \mathcal{E}$. The weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha_j \in \mathbb{R}_+$, $j \in \mathcal{V}$, is defined so that $w(j, i) = \alpha_i$ if $(j, i) \in \mathcal{E}$, and $w(j, i) = 0$ otherwise. Every notion for the digraph can be similarly applied to the weighted digraph. The adjacency matrix $E \in \mathbb{R}^{m\times m}$ of the weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ is defined as the matrix with $[E]_{ij} = \alpha_j$ if $(j, i) \in \mathcal{E}$ and $[E]_{ij} = 0$ otherwise. The gain $g(\mathcal{W})$ of the walk $\mathcal{W} = (v_0, v_1, \dots, v_{k-1}) \in \mathcal{V}^k$ in the weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ is defined by the product of weights of edges along the walk, i.e., $g(\mathcal{W}) := \prod_{t=0}^{k-2} w(v_t, v_{t+1})$. The cycle gain $g(\mathcal{C})$ of the simple cycle $\mathcal{C}$ is defined in a similar way.

2.3 Problem formulation

Consider the discrete-time (autonomous) SLS

$$x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = z \in \mathbb{R}^n, \quad k \in \mathbb{N}, \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state, $\sigma(k) \in \mathcal{M} := \{1, 2, \dots, M\}$ is called the mode, and A_μ, $\mu \in \mathcal{M}$, are the subsystem matrices. A switching sequence is denoted by $\sigma := (\sigma(0), \sigma(1), \dots) \in \mathcal{M}^\infty$. Starting from $x(0) = z \in \mathbb{R}^n$ and under a given switching sequence σ, the trajectory of the SLS (1) is denoted by $x(k; z, \sigma)$, $k \in \mathbb{N}$. In this paper, we assume that the switching sequence σ can be determined by the designer, i.e., σ is the control input. The following two notions of switching stabilizability can be defined.

Definition 1 *The SLS* (1) *is called (exponentially) switching stabilizable with the parameters K and ϕ if there exist $K \in [0, \infty)$ and $\phi \in [0, 1)$ such that starting from any initial state $x(0) = z \in \mathbb{R}^n$, there exists a switching sequence σ for which the trajectory $x(k; z, \sigma)$ satisfies*

$$\|x(k; z, \sigma)\| \leq K\phi^k \|z\|, \quad \forall k \in \mathbb{N}. \quad (2)$$

Any $\phi \in [0, 1)$ satisfying (2) will be called *an exponential convergence rate*. The problem addressed in this paper is stated as follows.

Problem 1 (Stabilizability problem) *Determine the stabilizability of the SLS* (1)*.*

As a byproduct of our development, we also solve the design problem.

Problem 2 *If the SLS* (1) *is stabilizable, then find a state-feedback switching policy under which the SLS* (1) *is stable.*

If one of the subsystem matrices is Schur stable, then the SLS (1) is trivially stabilizable. To avoid triviality, the following assumption is made in this paper.

Assumption 1 *Each subsystem matrix A_μ, $\mu \in \mathcal{M}$, is not Schur stable.*

As a result, we have

$$\tau := \max_{\mu \in \mathcal{M}} \|A_\mu\| \geq 1. \quad (3)$$

Lastly, some notions in [29] will be briefly reviewed. Hereafter, we will think of the set of subsystem matrices $\mathcal{A} := \{A_1, \dots, A_M\}$ as a finite *alphabet* and we will refer to a finite product of matrices from this set as a *word*. The set of all words $A_{\mu_{k-1}} \cdots A_{\mu_1} A_{\mu_0}$ of length $k \in \mathbb{N}$ is denoted by $\mathcal{A}^k := \{A_{\mu_{k-1}} \cdots A_{\mu_0}\}_{(\mu_0, \cdots, \mu_{k-1}) \in \mathcal{M}^k}$ with $\mathcal{A}^0 := \{I_n\}$; the set of all finite words is denoted by $\mathcal{A}^* := \bigcup_{h\in\mathbb{N}} \mathcal{A}^h$; and the set of all words with length from $k_1 \in \mathbb{N}$ to $k_2 \in \mathbb{N}$, $k_2 \geq k_1$, is denoted by $\mathcal{A}^{[k_1, k_2]} := \bigcup_{h\in\{k_1, k_1+1, \dots, k_2\}} \mathcal{A}^h$.

3 Graph control Lyapunov function

In this section, a formal definition of the graph control Lyapunov functions and the corresponding stabilization theorems are presented. The *graph control Lyapunov function (GCLF)* is defined as follows.

Definition 2 (GCLF) *Let a weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$, be given. A set of non-negative continuous functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, $i \in \mathcal{V}$, satisfying*

$$\underline{\kappa}_i \|z\|^2 \leq V_i(z) \leq \overline{\kappa}_i \|z\|^2, \quad \forall z \in \mathbb{R}^n, \quad (4)$$

for some positive constants $\underline{\kappa}_i$, $\overline{\kappa}_i \in \mathbb{R}_{++}$, $i \in \mathcal{V}$, will be called a graph control Lyapunov function (GCLF) associated with $G(\mathcal{V}, \mathcal{E}, \alpha)$ if

(1) there exist $\mathcal{A}_{j\to i} \subset \mathcal{A}^*$, $(j, i) \in \mathcal{E}$, such that the inequalities

$$\min_{i\in\mathcal{N}_j^+} \min_{A\in\mathcal{A}_{j\to i}} V_i(Az) \le \alpha_j V_j(z),$$
$$\forall z \in \mathbb{R}^n \backslash \{0_n\},\ j \in \mathcal{V} \tag{5}$$

associated with $G(\mathcal{V}, \mathcal{E}, \alpha)$ are satisfied;

(2) all the simple cycles in $G(\mathcal{V}, \mathcal{E}, \alpha)$ (including the loops) have the cycle gains strictly less than 1;

(3) $G(\mathcal{V}, \mathcal{E}, \alpha)$ has no sink.

When the GCLF $\{V_i\}_{i\in\mathcal{V}}$ consists of quadratic functions, then it will be called a quadratic GCLF (QGCLF).

Proposition 1 *Let a weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha_i \in [0, 1)$, $i \in \mathcal{V}$, be given. The set of functions $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with $G(\mathcal{V}, \mathcal{E}, \alpha)$ if all the conditions of Definition 2 except for the part 3) hold.*

PROOF. Straightforward. □

Example 1 *Consider the SLS* (1)*, and suppose that there exist nonnegative continuous functions V_1, V_2, V_3, V_4, satisfying* (4) *in Definition 2, and the words $\mathcal{A}_{1\to 2}$, $\mathcal{A}_{2\to 3}$, $\mathcal{A}_{2\to 4}$, $\mathcal{A}_{3\to 4}$, $\mathcal{A}_{4\to 1} \subset \mathcal{A}^*$, such that*

$$\min_{A\in\mathcal{A}_{1\to 2}} V_2(Az) \le \frac{1}{2} V_1(z),$$
$$\min\left\{\min_{A\in\mathcal{A}_{2\to 3}} V_3(Az),\ \min_{A\in\mathcal{A}_{2\to 3}} V_3(Az)\right\} \le 2V_2(z),$$
$$\min_{A\in\mathcal{A}_{3\to 4}} V_4(Az) \le \frac{1}{2} V_3(z), \quad \min_{A\in\mathcal{A}_{4\to 1}} V_1(Az) \le \frac{1}{3} V_4(z). \tag{6}$$

In the sense of Definition 2, the above inequalities induce the digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ shown in Figure 1 with the node set $\mathcal{V} = \{1, 2, 3, 4\}$ and the edge set $\mathcal{E} = \{(1, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$. The digraph has two simple cycles $\mathcal{C}_1 = (1, 2, 4, 1)$ and $\mathcal{C}_2 = (1, 2, 3, 4, 1)$, and the cycle gains can be calculated as

$$g(\mathcal{C}_1) = \alpha_1\alpha_2\alpha_4 = \frac{1}{2} \times 2 \times \frac{1}{3} = \frac{1}{3} < 1,$$
$$g(\mathcal{C}_2) = \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{1}{2} \times 2 \times 1 \times \frac{1}{3} = \frac{1}{3} < 1.$$

Since the digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ in Figure 1 does not have a sink, and all the simple cycles have gains less than one, by Definition 2, $\{V_1, V_2, V_3, V_4\}$ is a GCLF.

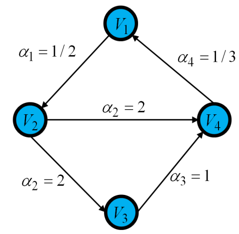

Fig. 1. Example 1. Digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ associated with the inequalities in (6).

Remark 1 *An example of digraphs with no sink is a strongly connected digraph. In general, the associated digraph of a GCLF can be assumed without loss to be strongly connected. To prove this, we introduce some notions in graph theory. A subgraph $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ of $G(\mathcal{V}, \mathcal{E})$ is a strongly connected component (SCC) of $G(\mathcal{V}, \mathcal{E})$ if $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is strongly connected and no other strongly connected subgraph contains $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ as a subgraph. For any digraph $G(\mathcal{V}, \mathcal{E})$, a SCC $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ with no outgoing edges from the nodes of $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is called a terminal SCC. From [43, pp. 17], for any digraph $G(\mathcal{V}, \mathcal{E})$, there exists a terminal SCC. Therefore, if $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with the weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ and $G(\bar{\mathcal{V}}, \bar{\mathcal{E}}, \alpha)$ is a terminal SCC of $G(\mathcal{V}, \mathcal{E}, \alpha)$, then it can be easily proved that $\{V_i\}_{i\in\bar{\mathcal{V}}}$ is also a GCLF associated with $G(\bar{\mathcal{V}}, \bar{\mathcal{E}}, \alpha)$ by the definition of the terminal SCC and Definition 2.*

Given a walk $\mathcal{W} = (v_0, v_1, \ldots)$, define $\mathcal{W}_{[a,b]} := (v_a, \ldots, v_b)$ for $a \le b$, $a, b \in \mathbb{N}$. A decomposition of $\mathcal{W}$ is defined as a sequence of walks $(\mathcal{W}_1, \mathcal{W}_2, \ldots)$ such that $\mathcal{W}_1 = \mathcal{W}_{[i_1, i_2]}$, $\mathcal{W}_2 = \mathcal{W}_{[i_2, i_3]}, \ldots$ and $0 = i_1 < i_2 < \cdots$. The proof of the main result depends on the following lemma, which establishes the fact that the gain of a walk can be expressed as the product of the gains of simple cycles and a path in the given digraph.

Lemma 1 *Suppose that $\mathcal{W} = (v_0, v_1, \ldots, v_{t-1})$ is a walk in $G(\mathcal{V}, \mathcal{E}, \alpha)$. Then, $g(\mathcal{W})$ can be expressed as*

$$g(\mathcal{W}) = g(\mathcal{P}) \prod_{p=1}^{h} g(\mathcal{C}_p),$$

where $h \in \mathbb{N}_+$, $\mathcal{C}_p$, $p \in \{1, 2, \ldots, h\}$, are simple cycles and $\mathcal{P}$ is a path, such that $|\mathcal{W}| = |\mathcal{P}| + \sum_{p=1}^{h} |\mathcal{C}_p|$.

PROOF. If $\mathcal{W} = (v_0, v_1, \ldots, v_{t-1}) =: \mathcal{W}^{[1]}$ is not a path, then there exists a simple cycle $\mathcal{C}_1$ in $\mathcal{W}$. Remove the simple cycle $\mathcal{C}_1$ and get a shorter walk $\mathcal{W}^{[2]}$. For $p \in \mathbb{N}_+$, if $\mathcal{W}^{[p]}$ is not a path, then one can remove a simple cycle $\mathcal{C}_p$ and get a new walk $\mathcal{W}^{[p+1]}$. Noting that the initial walk $\mathcal{W}$ is finite and by the induction argument, we obtain a decomposition of $\mathcal{W}$ which consists of a finite sequence of simple cycles (including loops)

$\mathcal{C}_p$, $p \in \{1, 2, \ldots, h\}$, and a path $\mathcal{P}$. Therefore, the gain of the walk $g(\mathcal{W})$ can be expressed as a product of the gains of the simple cycles and the path. This completes the proof. $\square$

In what follows, it will be proved that the GCLF can be used to certify stabilizability. For easy reference, we formally define the state-feedback switching policy, the corresponding walk on the given digraph, and the switching sequence.

Definition 3 *Let a weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$, be given. Suppose that $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with $G(\mathcal{V}, \mathcal{E}, \alpha)$. For any $x \in \mathbb{R}^n$ and $j \in \mathcal{V}$, define the sets*

$$I(j, x) := \arg\min_{i\in\mathcal{N}_j^+} \min_{A\in\mathcal{A}_{j\to i}} V_i(Ax), \quad \forall j \in \mathcal{V},$$

and

$$\Phi(j, i, x) := \arg\min_{A\in\mathcal{A}_{j\to i}} V_i(Ax), \quad \forall j \in \mathcal{V},\, i \in \mathcal{N}_j^+.$$

Then, the set defined as

$$\begin{aligned}&\sigma(j, i, x) := \{(i_0, \ldots, i_{h-1}) \in \mathcal{M}^h : \\ &A_{i_{h-1}} \cdots A_{i_1} A_{i_0} \in \Phi(j, i, x),\, h \in \mathbb{N}_+\}\end{aligned} \tag{7}$$

is called a state-feedback switching policy associated with the GCLF $\{V_i\}_{i\in\mathcal{V}}$. For any $\xi_0 = z \in \mathbb{R}^n$ and $j_0 \in \mathcal{V}$, if the sequences $\{\xi_t\}_{t=0}^{\infty}$ and $\{j_t\}_{t=0}^{\infty}$ are defined by the inclusions

$$\begin{aligned}&j_{t+1} \in I(j_t, \xi_t),\\ &(i_0, \ldots, i_{h-1}) \in \sigma(j_t, j_{t+1}, \xi_t),\\ &\xi_{t+1} = A_{i_{h-1}} \cdots A_{i_1} A_{i_0} \xi_t, \quad t \in \mathbb{N},\end{aligned} \tag{8}$$

respectively, then ξ_t will be called the state corresponding to the node j_t, and the sequence of nodes $\mathcal{W}_\infty = (j_0, j_1, j_2, \ldots)$ represents a walk in $G(\mathcal{V}, \mathcal{E}, \alpha)$ and will be called a walk associated with the switching policy (7). The corresponding switching sequence is

$$\sigma(j_0, z) := (\sigma(j_0, j_1, \xi_0),\, \sigma(j_1, j_2, \xi_1),\, \sigma(j_2, j_3, \xi_2), \ldots). \tag{9}$$

In the following theorem, we show that if the SLS (1) admits a GCLF, then the switching sequence (9) exponentially stabilizes the SLS (1).

Theorem 1 (Graph control Lyapunov theorem) *If $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with the digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ and the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$, then the SLS (1) under the switching sequence (9) is exponentially stable with the parameters*

$$K = \tau^L \left(\delta \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right)^{\frac{1}{2}} \gamma^{-\frac{\beta+1}{2\eta}}, \quad \phi = \gamma^{\frac{1}{2\eta L}}, \tag{10}$$

where $\tau := \max_{\mu\in\mathcal{M}} \|A_\mu\|$, $L := \max\{|\mathcal{A}_{j\to i}| : (j, i) \in \mathcal{E}\}$, η and γ are the maximum length and gain of simple cycles, β and δ are the maximum length and gain of paths, respectively. In particular, if $\alpha_i \in [0, 1)$, $i \in \mathcal{V}$, then the SLS (1) under (9) is exponentially stable with the parameters

$$K = \tau^L \left(\frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{(\min_{i\in\mathcal{V}} \underline{\kappa}_i)(\max_{i\in\mathcal{V}} \alpha_i)}\right)^{1/2},\ \phi = \left(\max_{i\in\mathcal{V}} \alpha_i\right)^{1/2L}.$$

PROOF. Let $j_0 \in \mathcal{V}$ and $\xi_0 \in \mathbb{R}^n$ be arbitrary. Define the walk $\mathcal{W} = (j_0, j_1, \ldots, j_t)$, and the sequence of times $\{k_t\}_{t=0}^{\infty}$ by $k_0 = 0$, and $k_{t+1} = k_t + |\Phi(j_t, j_{t+1}, \xi_t)|$ for $t \in \mathbb{N}_+$, where $\{\xi_t\}_{t=0}^{\infty}$ is the subsequence of the states defined in (8) so that $\xi_t = x(k_t; z, \sigma)$, $\forall t \in \mathbb{N}$. Then, by the definition of the switching policy in (7), the inequalities in (5) are satisfied for all $z = \xi_t$, $t \in \mathbb{N}_+$, and the Lyapunov function value long the trajectory $\{\xi_t\}_{t=0}^{\infty}$ satisfies $V_{j_t}(\xi_t) \le g(\mathcal{W}) V_{j_0}(\xi_0)$, $\forall t \in \mathbb{N}_+$. By Lemma 1, there exists simple cycles $\mathcal{C}_p$, $p \in \{1, 2, \ldots, h\}$, $h \in \mathbb{N}_+$, and a path $\mathcal{P}$ such that $g(\mathcal{W}) = g(\mathcal{P}) \prod_{p=1}^{h} g(\mathcal{C}_p)$ and $|\mathcal{W}| = |\mathcal{P}| + \sum_{p=1}^{h} |\mathcal{C}_p|$. Thus, we have

$$V_{j_t}(\xi_t) \le g(\mathcal{P}) \prod_{p=1}^{h} g(\mathcal{C}_p) V_{j_0}(\xi_0) \le \delta\gamma^h V_{j_0}(\xi_0),$$

where γ is the maximum gain of simple cycles, and δ is the maximum gain of paths.

Noting that $V_{j_0}(\xi_0) \le \bar{\kappa}_{j_0} \|\xi_0\|^2$ and $V_{j_t}(\xi_t) \ge \min_{i\in\mathcal{V}} \underline{\kappa}_i \|\xi_t\|^2$ and combining them, we have

$$\|\xi_t\|^2 \le \delta \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i} \gamma^h \|\xi_0\|^2.$$

Since $h \ge \frac{t-\beta}{\eta}$ and $\gamma < 1$, it follows that

$$\|\xi_t\|^2 \le \delta \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i} \gamma^{-\frac{\beta}{\eta}} \gamma^{\frac{t}{\eta}} \|\xi_0\|^2,$$

which gives $\|\xi_t\| \le rc^t \|z\|$, where

$$r = \left(\delta \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right)^{\frac{1}{2}} \gamma^{-\frac{\beta}{2\eta}}, \quad c = \gamma^{\frac{1}{2\eta}}.$$

To obtain an exponential convergence rate of the SLS, for any $k \in \mathbb{N}$, select $t \in \mathbb{N}$ such that $k \in [k_t, k_{t+1})$. Then, we have

$$\begin{aligned}\|x(k;\, z,\, \sigma)\| &= \|x(k_t + k - k_t;\, z,\, \sigma)\| \\ &\leq \tau^{(k-k_t)} \|x(k_t;\, z,\, \sigma)\| \\ &\leq \tau^L \|\xi_t\| \leq \tau^L r c^t \|z\|,\end{aligned}$$

where we have used $\tau \geq 1$ in (3). Again, as $t \geq (k/L) - 1$ and $c < 1$, we obtain

$$\|x(k;\, z,\, \sigma)\| \leq \tau^L r c^{\left(\frac{k}{L}-1\right)} \|z\| = \frac{\tau^L r}{c} c^{k/L} \|z\|.$$

Therefore, the SLS (1) under the switching sequence (9) is exponentially stable with the parameters in (10). The proof for the case $\alpha_i \in [0,\, 1)$, $i \in \mathcal{V}$, is similar, so omitted for brevity. □

The result proves that the existence of the GCLF is sufficient condition for the stabilizability of the SLS (1). Next, we prove that it is also a necessary condition.

Theorem 2 (Converse GCLF theorem) *Suppose that the SLS (1) is exponentially switching stabilizable. Let a digraph $G(\mathcal{V},\, \mathcal{E})$ with no sink and the positive definite matrices $P_i \succ 0$, $i \in \{1, 2, \ldots, |\mathcal{V}|\}$, be arbitrarily given. Then, the set of quadratic functions $V_i(x) = x^T P_i x$, $i \in \mathcal{V}$, is a QGCLF associated with $G(\mathcal{V},\, \mathcal{E},\, \alpha)$ with some parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{V}$. In other words, there exist $\mathcal{A}_{j\to i} \subset \mathcal{A}^*$, $(j,\, i) \in \mathcal{E}$, such that the inequalities (5) associated with $G(\mathcal{V},\, \mathcal{E},\, \alpha)$ is satisfied, and all the simple cycles of $G(\mathcal{V},\, \mathcal{E},\, \alpha)$ have the cycle gains less than one.*

PROOF. Consider the set of words $\mathcal{A}_{j\to i} = \mathcal{A}^{h_j}$ with $h_j \in \mathbb{N}_+$ for all $(j,\, i) \in \mathcal{E}$. Since the SLS (1) is exponentially stabilizable, there exist $K \in [0,\, \infty)$ and $\phi \in [0,\, 1)$ such that (2) holds. Thus, we have

$$\begin{aligned}&\min_{i\in\mathcal{N}_j^+} \min_{(\mu_0,\cdots,\mu_{h_j-1})\in\mathcal{M}^{h_j}} V_i(A_{\mu_{h_j-1}} \cdots A_{\mu_0} z) \\ &\leq \min_{i\in\mathcal{N}_j^+} \min_{(\mu_0,\cdots,\mu_{h_j-1})\in\mathcal{M}^{h_j}} \|A_{\mu_{h_j-1}} \cdots A_{\mu_0} z\|^2 \lambda_{\max}(P_i) \\ &\leq \min_{i\in\mathcal{N}_j^+} \lambda_{\max}(P_i) K^2 \phi^{2h_j} \|z\|^2 \\ &\leq \alpha_j V_j(z), \quad \forall j \in \mathcal{V},\end{aligned}$$

where

$$\alpha_j = \frac{\min_{i\in\mathcal{N}_j^+} \lambda_{\max}(P_i)}{\lambda_{\min}(P_j)} K^2 \phi^{2h_j}.$$

By increasing h_j, we can make $\alpha_j < 1$ for all $j \in \mathcal{V}$. Therefore, $\{V_i\}_{i\in\mathcal{V}}$ is a QGCLF. □

4 Alternative Sufficient Conditions

Theorem 1 and Definition 3 provide the answers to both Problem 1 and Problem 2. From a computational point of view, to find the gains of all the simple cycles, one needs to enumerate all the simple cycles in $G(\mathcal{V},\, \mathcal{E},\, \alpha)$. For small-scale digraphs, some algorithms are available to enumerate all the simple cycles, for example, those in [44] and the CIRCUIT-FINDING ALGORITHM in [45], whose complexity grows rapidly with the size of the digraphs. For large-scale digraphs, we will develop in the next section a sufficient test based on the JSR theory [24] for postivie SLSs to check the stabilizability without enumerating all the simple cycles of the digraph. The JSR is a natural extension of the spectral radius of the LTI systems, and characterizes the maximum exponential growth rate of the SLSs under arbitrary switching [24]. To this end, we formally define a positive SLS (PSLS) associated with a weighted digraph.

Definition 4 *Consider the weighted digraph $G(\mathcal{V},\, \mathcal{E},\, \alpha)$ with the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$, and let*

$$G(\mathcal{V},\, \mathcal{E}_1,\, \alpha), \ldots,\, G(\mathcal{V},\, \mathcal{E}_Q,\, \alpha)$$

be subgraphs of $G(\mathcal{V},\, \mathcal{E},\, \alpha)$ whose edge sets are disjoint and $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_Q$. Assume that the matrices $E_1, \ldots,\, E_Q \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ are the adjacency matrices of $G(\mathcal{V},\, \mathcal{E}_1,\, \alpha), \ldots,\, G(\mathcal{V},\, \mathcal{E}_Q,\, \alpha)$, respectively. A discrete-time positive switched linear system (PSLS) associated with the adjacency matrices is defined as

$$v(t+1) = E_{\theta(t)} v(t), \quad v(0) = s \in \mathbb{R}_+^{|\mathcal{V}|}, \tag{11}$$

where $t \in \mathbb{N}$, $v(t) \in \mathbb{R}_+^{|\mathcal{V}|}$, is the state and $\theta(t) \in \mathcal{Q} := \{1, 2, \ldots,\, Q\}$ is the mode of the PSLS (11).

Example 2 *Consider the GCLF in Example 1 again with the associated digraph $G(\mathcal{V},\, \mathcal{E},\, \alpha)$ in Figure 1. The corresponding adjacency matrix is*

$$E = \begin{bmatrix} 0 & 0 & 0 & \alpha_4 \\ \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}. \tag{12}$$

There are several ways to construct the subsystem matrices of the PSLS (11). For example, the edge set $\mathcal{E} = \{(1,\, 2),\, (2,\, 3),\, (2,\, 4),\, (3,\, 4),\, (4,\, 1)\}$ can be partitioned into $\mathcal{E}_1 = \{(1,\, 2),\, (2,\, 3),\, (2,\, 4)\}$ and

$\mathcal{E}_2 = \{(3,\,4),\,(4,\,1)\}$, and the adjacency matrices of the digraphs $G(\mathcal{V},\,\mathcal{E}_1,\,\alpha)$, $G(\mathcal{V},\,\mathcal{E}_2,\,\alpha)$ are

$$E_1 = \sum_{(j,\,i)\in\mathcal{E}_1} \alpha_j e_i e_j^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \end{bmatrix}, \tag{13}$$

$$E_2 = \sum_{(j,\,i)\in\mathcal{E}_2} \alpha_j e_i e_j^T = \begin{bmatrix} 0 & 0 & 0 & \alpha_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \end{bmatrix}. \tag{14}$$

The two adjacency matrices form two subsystem matrices of the PSLS (11)*. For different partitions of the edge set $\mathcal{E}$, different PSLSs are obtained.*

Definition 5 (JSR [24]) *The joint spectral radius (JSR) of the set of matrices $\Sigma := \{E_1,\,E_2,\ldots,\,E_Q\}$ is defined by $\rho(\Sigma) := \lim_{k\to\infty} \max_{A\in\Sigma^k} \|A\|^{1/k}$.*

It is known that if the matrix norm $\|\cdot\|$ is submultiplicative, i.e., $\|AB\| \le \|A\|\|B\|$, $\forall A,\,B \in \mathbb{R}^{n\times n}$, then the limit in Definition 5 exists [24, Lemma 1.2] and the limiting value does not depend on the norm used. For further details on the JSR, the reader is referred to [24]. In the following theorem, we provide a sufficient condition based on the JSR of the PSLS (11) for the GCLF.

Theorem 3 *For the digraph $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ in Definition 4, if*

(1) $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ has no sink;
(2) there exist $\mathcal{A}_{j\to i} \subset \mathcal{A}^*$, $(j,\,i) \in \mathcal{E}$, and the set of functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, $i \in \mathcal{V}$, such that the inequalities in (5) *associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ is satisfied;*
(3) $\rho(\Sigma) < 1$,

then $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$. Moreover, the SLS (1) *is exponentially stabilizable with the parameters*

$$K = \sqrt{\tau^L \rho(\Sigma)^{-T} \max\left\{\tau^{T-1},\, \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right\}}, \quad \phi = \rho(\Sigma)^{\frac{1}{2L}}.$$

PROOF. See Appendix A. □

Example 3 *Consider the GCLF in Example 1 and Example 2 again, and assume $L = 3$. Different exponential convergence rate ρ can be obtained by using different PSLSs. For instance, if we consider the PSLS* (11) *with the single subsystem matrix* (12)*, then the JSR reduces to the spectral radius, and we have $\rho(\Sigma) = 0.8910$ with $\Sigma = \{E\}$. By Theorem 3, the exponential convergence rate is $\phi = \rho(\Sigma)^{1/L} = 0.9810$. On the other hand, if we consider the PSLS with the subsystem matrices* (14)*, then an upper bound on the JSR can be obtained through the numerical method in [25, Theorem 3] as $\rho(\Sigma) = 0.7114$, and the exponential convergence rate is computed as $\phi = \rho(\Sigma)^{1/L} = 0.9448$. Finally, the exponential convergence rate obtained by using Theorem 1 is $\phi = 0.8327$.*

From Theorem 3, the JSR of the PSLS (11) gives a measure on how fast the exponential convergence of the SLS (1). In addition, Theorem 3 is a sufficient condition for a given functions $\{V_i\}_{i\in\mathcal{V}}$ to be a GCLF, and may not be necessary in general. A question that arises is about how conservative the condition is. In the following result, it is proved that with the rank one selection of the PSLS (11), it is also necessary. For the proof, we follow the result [46, Theorem 2.2].

Proposition 2 *Suppose that $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ and the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$. Then, for the PSLS* (11) *with $\{E_1,\ldots,\,E_Q\} = \{\alpha_j e_i e_j^T\}_{(j,\,i)\in\mathcal{E}}$, $\rho(\Sigma) \le \gamma^{1/\eta} < 1$ holds, where $\gamma \in \mathbb{R}_+$ and $\eta \in \mathbb{N}_+$ are the maximum gain and the maximum length of the simple cycles in $G(\mathcal{V},\,\mathcal{E},\,\alpha)$, respectively.*

PROOF. Definition 5 gives

$$\begin{aligned}\rho(\Sigma) &= \limsup_{k\to\infty} \max_{A\in\Sigma^k} \|A\|_\infty^{1/k} = \limsup_{k\to\infty} \max_{A\in\Sigma^k} \|A\mathbf{1}\|_\infty^{1/k} \\ &= \limsup_{k\to\infty} \max_{\mathcal{W}\in\mathcal{V}^{k+1}} g(\mathcal{W})^{1/k},\end{aligned}$$

where $\mathcal{W}$ a walk of length k in $G(\mathcal{V},\,\mathcal{E},\,\alpha)$, and $\mathbf{1} \in \mathbb{R}^{|\mathcal{V}|}$ is the vector whose entries are ones. By Lemma 1, for each $k \in \mathbb{N}_+$, there exist a path $\mathcal{P}^{[k]}$ and simple cycles $\mathcal{C}_l^{[k]}$, $l \in \{1,\,2,\ldots,\,h^{[k]}\}$, $h^{[k]} \in \mathbb{N}_+$ in $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ such that $g(\mathcal{W}) = g(\mathcal{P}^{[k]}) \prod_{l=1}^{h^{[k]}} g(\mathcal{C}_l^{[k]})$. Therefore, we have

$$\begin{aligned}&\limsup_{k\to\infty} \max_{\mathcal{W}\in\mathcal{V}^{k+1}} g(\mathcal{W})^{1/k} \\ &\le \limsup_{k\to\infty} (\delta\gamma^{h[k]})^{1/k} \\ &\le \limsup_{k\to\infty} \left(\delta\gamma^{(k-\beta)/\eta}\right)^{1/k} \\ &= \lim_{k\to\infty} \left(\delta\gamma^{-\beta/\eta}\right)^{1/k} \gamma^{1/\eta} \\ &= \gamma^{1/\eta} < 1,\end{aligned}$$

where $\beta \in \mathbb{N}_+$ is the maximum length of the paths, $\delta \in \mathbb{R}_+$ is the maximum gain of the paths, and the third line follows from $h[k] \geq \frac{k-\beta}{\eta}$. Thus, the proof is concluded. $\square$

The JSR of the PSLS (11) gives a measure on how fast the exponential convergence of the SLS (1). Unfortunately, the problem of determining if $\rho(\Sigma) < 1$ is NP-hard [24, Section 2.2], [47]. Nevertheless, there exist many over approximation procedures, for instance, the Kronecker lifting [25, Theorem 3] for the positive SLSs and the generating function approach [26] for general SLSs. In the following result, we can compute the exponential convergence rate using the Lyapunov method for the PSLS (11), for instance, [48, Theorem 1], [49].

Proposition 3 *Let a weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$ be given, $G(\mathcal{V}, \mathcal{E}_1, \alpha), \ldots, G(\mathcal{V}, \mathcal{E}_Q, \alpha)$ be any disjoint subgraphs of $G(\mathcal{V}, \mathcal{E}, \alpha)$ such that $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_Q$, and $E_1, E_2, \ldots, E_Q$ be the corresponding adjacency matrices, respectively. If*

(1) $G(\mathcal{V}, \mathcal{E}, \alpha)$ has no sink;
(2) there exist $\mathcal{A}_{j\to i} \subset \mathcal{A}^*$, $(j, i) \in \mathcal{E}$, and the set of functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, $i \in \mathcal{V}$, such that the GCLI (5) associated with $G(\mathcal{V}, \mathcal{E}, \alpha)$ is satisfied;
(3) there exist vectors λ_i, $i \in \mathcal{V}$, scalers $\beta_1 \in \mathbb{R}_{++}$, $\beta_2 \in \mathbb{R}_{++}$ and $\varphi \in [0, 1)$ such that

$$\begin{aligned} &\beta_1 \mathbf{1} \preceq \lambda_p \preceq \beta_2 \mathbf{1}, \quad E_i^T \lambda_j \preceq \varphi \lambda_i, \\ &(j, i) \in \{1, 2, \ldots, Q\}^2, \end{aligned} \tag{15}$$

where $\mathbf{1} \in \mathbb{R}^{|\mathcal{V}|}$ is the vector whose entries are ones,

then $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with $G(\mathcal{V}, \mathcal{E}, \alpha)$. Moreover, the SLS (1) is exponentially stabilizable with the parameters

$$K = \left(\frac{\tau^L}{\varphi}\frac{\beta_2}{\beta_1}\frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}\underline{\kappa}_i}}\right)^{1/2}, \quad \phi = \varphi^{\frac{1}{2L}}.$$

PROOF. Consider the PSLS (11) associated with any set of subgraphs $G(\mathcal{V}, \mathcal{E}_1, \alpha), \ldots, G(\mathcal{V}, \mathcal{E}_Q, \alpha)$ defined in Definition 4, and denote by $v(t; \theta, s)$ the state trajectory of the PSLS (11) under the arbitrarily switching sequence $\theta := (\theta(0), \theta(1), \ldots)$ and the initial state $s \in \mathbb{R}_+^{|\mathcal{V}|}$. By the stability condition of the PSLS in [48, Theorem 1], [49], (15) implies that the Lyapunov function $F_i(v) := v^T \lambda_i$ satisfies the Lyapunov inequality

$$\begin{aligned} &\beta_1 \|v\|_1 \leq F_i(v) \leq \beta_2 \|v\|_1, \\ &F_j(E_i^T v) \leq \varphi F_i(v), \quad \forall (i, j) \in \{1, 2, \ldots, Q\}^2, \end{aligned} \tag{16}$$

for all $v \in \mathbb{R}_+^{|\mathcal{V}|}$, where $\|v\|_1 := \sum_{i=1}^{|\mathcal{V}|} |v_i|$ is the 1-norm. Therefore, $F_{\theta(t)}(v(t; \theta, s)) \leq \varphi^t F_{\theta(0)}(v(0; \theta, s))$ holds. Combining this inequality with (16), one has

$$\begin{aligned} &\|v(t; \theta, s)\|_1 \leq \frac{\beta_2}{\beta_1}\varphi^t \|v(0; \theta, s)\|_1 \\ &\Leftrightarrow \frac{\|E_{\theta(t-1)} \cdots E_{\theta(0)} s\|_1}{\|s\|_1} \leq \frac{\beta_2}{\beta_1}\varphi^t, \ \forall s \in \mathbb{R}_+^{|\mathcal{V}|}, \end{aligned}$$

which implies $\|E_{\theta(t-1)} \cdots E_{\theta(0)}\|_1 \leq \frac{\beta_2}{\beta_1}\varphi^t$, where $\|\cdot\|_1$ is the induced matrix 1-norm. Since $(\theta(0), \theta(1), \ldots, \theta(t-1)) \in \mathcal{Q}^t$ is arbitrary, by the definition of the JSR, we have

$$\rho(\Sigma) = \lim_{t\to\infty} \max_{A\in\Sigma^t} \|A\|_1^{1/t} \leq \lim_{t\to\infty} \left(\frac{\beta_2}{\beta_1}\right)^{1/t} \varphi = \varphi < 1,$$

where $\Sigma = \{E_1, E_2, \ldots, E_Q\}$. By Theorem 3, the SLS is stabilizable, and an exponential convergence rate is given by $\varphi^{\frac{1}{2L}}$. The parameter $K = \left(\frac{\tau^L}{\varphi}\frac{\beta_2}{\beta_1}\frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right)^{1/2}$ can be obtained following similar lines as in the proof of Theorem 3, thus omitted here. $\square$

5 Examples of GCLFs

The GCLF includes several control Lyapunov functions including the quadratic control Lyapunov function and PWQCLFs [11–14, 16]. In this subsection, by studying connections between the GCLF and other Lyapunov functions, we unify the classical Lyapunov theorems.

Periodically and aperiodically quadratic control Lyapunov function: First of all, the *periodically quadratic control Lyapunov function (PQCLF)* [30–32, 50] is the quadratic function $V_1(x) := x^T P_1 x$, $P_1 \in \mathbb{S}_{++}^n$, such that

$$\min_{A\in\mathcal{A}^h} V_1(Az) \leq \alpha V_1(z), \quad \forall z \in \mathbb{R}^n \backslash \{0_n\} \tag{17}$$

for some $\alpha \in [0, 1)$. The inequality (17) corresponds to the inequalities in (5) associated with $G(\mathcal{V}, \mathcal{E})$, where $G(\mathcal{V}, \mathcal{E})$ is a digraph with one node $\mathcal{V} = \{1\}$ and one edge $\mathcal{E} = \{(1, 1)\}$. Since the edge is a loop, the digraph has no sink. Therefore, $V_1(z)$ is a GCLF of the SLS (1). If $\mathcal{A}_{1\to 1} = \mathcal{A}^h$ is replaced with $\mathcal{A}_{1\to 1} = \mathcal{A}^{[1, h]}$, i.e.,

$$\min_{A\in\mathcal{A}^{[1,\,h]}} V_1(Az) \leq \alpha V_1(z), \quad \forall z \in \mathbb{R}^n \backslash \{0_n\}, \tag{18}$$

then, V_1 called the *aperiodic control Lyapunov function (APCLF)* [30–32, 32, 51]. An example of the APCLF is given below.

Piecewise quadratic control Lyapunov function: For the quadratic functions $V_i(z) := z^T P_i z$, $P_i \in \mathbb{S}_{++}^n$, $i \in \mathcal{V}$, the piecewise quadratic function of the form

$V_{\min}(z) := \min_{i\in\mathcal{V}} V_i(z)$ is called *piecewise quadratic control Lyapunov function (PWQCLF)* [13,16] if

$$\min_{A\in\mathcal{A}^1} V_{\min}(Az) \le \alpha V_{\min}(z), \quad \forall z\in\mathbb{R}^n\backslash\{0_n\} \tag{19}$$

is satisfied for some $\alpha\in[0,\,1)$. It can be easily proved that if $V_{\min}(z)$ is a PWQCLF, then $\{V_i\}_{i\in\mathcal{V}}$ is a QGCLF associated with $G(\mathcal{V},\,\mathcal{E})$, where $\mathcal{E}=\mathcal{V}\times\mathcal{V}$ (complete digraph). There is another class of PWQCLFs in [14], which satisfy

$$V_{\min}(A_j z) \le \alpha V_j(z), \quad \forall z\in\mathbb{R}^n\backslash\{0_n\},\, j\in\mathcal{V}. \tag{20}$$

It can be proved that the PWQCLF satisfying (20) corresponds to a QGCLF associated with $G(\mathcal{V},\,\mathcal{E})$, $\mathcal{E}=\mathcal{V}\times\mathcal{V}$ because the inequality (20) can be viewed as a special case of the inequalties in (5) with $\mathcal{A}_{j\to i}=\{A_j\}$, $(j,\,i)\in\mathcal{E}$. In addition, it can be proved that the PWQCLF $V_{\min}(z)$ satisfying (20) also satisfies (19). If $\mathcal{A}^1$ is replaced by $\mathcal{A}^h$ or $\mathcal{A}^{[1,\,h]}$ in (19), then $V_{\min}(z)$ is called the periodically or aperiodically piecewise quadratic control Lyapunov function (PPWQCLF or APPWQCLF). It can be prove that if $V_{\min}(z)$ is a PPWQCLF, then $\{V_i\}_{i\in\mathcal{V}}$ is a QGCLF associated with $G(\mathcal{V},\,\mathcal{E})$, $\mathcal{E}=\mathcal{V}\times\mathcal{V}$.

Multiple control Lyapunov function: With modifications of the Lyapunov inequalities in (5) and Theorem 1, the multiple Lyapunov function [9,10] for the discrete-time SLSs can be interpreted as a GCLF as well. Roughly speaking, the multiple Lyapunov function defined in [10, Definition 1, Theorem 1] is a GCLF associated with $G(\mathcal{V},\,\mathcal{E})$, which satisfies the Lyapunov inequalities defined in Equation (5) for a partition of the state-space, where $G(\mathcal{V},\,\mathcal{E})$ is complete and all the nodes have loops, $\mathcal{A}_{j\to i}=\mathcal{A}^1$, $\forall(j,\,i)\in\mathcal{E}$, $j=i$, and $\mathcal{A}_{j\to i}=\mathcal{A}^0$, $\forall(j,\,i)\in\mathcal{E}$, $j\neq i$.

6 Numerical computation

This section describes a computational method to find the parameters $\alpha\in\mathbb{R}_+^{|\mathcal{V}|}$ in Theorem 1. Consider the digraph $G(\mathcal{V},\,\mathcal{E})$, the set of matrices $\{P_i\}_{i\in\mathcal{V}}\subset\mathbb{S}^n_{++}$, and the set of quadratic functions $\{V_i\}_{i\in\mathcal{V}}$ with $V_i(z):=z^T P_i z$, $i\in\mathcal{V}$. Define

$$\begin{aligned}&\alpha_{\min,\,j} := \min \alpha \\ &\text{subject to} \\ &\min_{i\in\mathcal{N}_j^+}\min_{A\in\mathcal{A}_{j\to i}} z^T A^T P_i A z \le \alpha z^T P_j z, \quad \forall z\in\mathbb{R}^n.\end{aligned} \tag{21}$$

To compute an over approximation of (21), we consider the following problem.

Problem 3 (SDP approximation) *Let $G(\mathcal{V},\,\mathcal{E})$, $\{P_i\}_{i\in\mathcal{V}}\subset\mathbb{S}^n_{++}$, and $\mathcal{A}_{j\to i}\subset\mathcal{A}^*$, $(j,\,i)\in\mathcal{E}$, be given. For $j\in\mathcal{V}$, solve the semidefinite programming (SDP) problem associated with $G(\mathcal{V},\,\mathcal{E})$*

$$\begin{aligned}&\alpha_{SDP,\,j} := \min_{\lambda_{(A,\,i,\,j)}\in\mathbb{R},\,\alpha_j\in\mathbb{R}} \alpha_j \quad \text{subject to} \\ &\sum_{A\in\mathcal{A}_{j\to i},\,i\in\mathcal{N}_j^+} \lambda_{(A,\,j,\,i)} A^T P_i A \preceq \alpha_j P_j, \\ &\sum_{A\in\mathcal{A}_{j\to i},\,i\in\mathcal{N}_j^+} \lambda_{(A,\,i,\,j)} = 1, \\ &\lambda_{(A,\,i,\,j)} \ge 0, \quad \forall A\in\mathcal{A}_{j\to i},\, i\in\mathcal{N}_j^+,\end{aligned} \tag{22}$$

where $\lambda_{(A,\,i,\,j)}$ is a scalar indexed by $(A,\,i,\,j)\in\mathcal{A}_{j\to i}\times\mathcal{N}_j^+\times\mathcal{V}$.

Proposition 4 *$\alpha_{\min,\,j}\le\alpha_{SDP,\,j}$ for all $j\in\mathcal{V}$.*

PROOF. The proof is easily completed by noting that the minimum of quadratic functions is always over estimated by any average value of the quadratic functions. □

Remark 2 *The sufficient SDP test is an extension of the existing SDP tests for different control Lyapunov functions, for example, those in [14, Theorem 3], [16, Corollary 1], [12,13].*

The following two results prove some properties of the SDP test in Problem 3. The first result is regarding the conservatism entailed in Problem 3.

Proposition 5 *Given a SLS for which $\det(A_i)\ge 1$ for all $i\in\mathcal{M}$. Then, for any $G(\mathcal{V},\,\mathcal{E})$, $P_i\in\mathbb{S}^n_{++}$, $i\in\mathcal{V}$, and $\mathcal{A}_{j\to i}\subset\mathcal{A}^*$, $(j,\,i)\in\mathcal{E}$, there exists $j\in\mathcal{V}$ such that $\alpha_{SDP,\,j}\ge 1$.*

PROOF. See Appendix B. □

See [26, section IV] for an example of switching stabilizable SLSs for which each subsystem matrix has determinant no less than one. For these SLSs, there always exists $j\in\mathcal{V}$ such that $\alpha_{\min,\,j}\ge 1$. Therefore, Proposition 1 cannot be used to identify the stabilizability. Moreover, if the SLS (1) is stabilizable, $\det(A_i)\ge 1$ for all $i\in\mathcal{M}$, and if one tries to find a PQCLF using the SDP (22) ($\mathcal{E}=\{(1,\,1)\}$ and $\mathcal{V}=\{1\}$), the gain of the simple cycle in $G(\mathcal{V},\,\mathcal{E},\,w)$ with the parameters $\alpha_{SDP,\,j}\in\mathbb{R}_+$, $\forall j\in\mathcal{V}$, is always larger than or equal to one for any $P_1\in\mathbb{S}^n_{++}$. This means that the conservatism of the SDP (22) will not be entirely vanished when the PQCLF or APQCLF is considered. For the

PQCLF/APQCLF case, can we give an explicit condition on the SLS such that for some $P_1 \in \mathbb{S}^n_{++}$, the gain of the simple cycle is less than one? A clear answer was given in [32, Theorem 22]. Before presenting the result, we introduce the notion of the periodic open-loop stabilizability of the SLS (1).

Definition 6 (Periodic open-loop stabilizability) *The SLS* (1) *is called periodic open-loop stabilizable if there exists $A \in \mathcal{A}^*$ such that $\rho(A) < 1$.*

Lemma 2 ([32, Theorem 22]) *For the SDP* (22) *associated with $G(\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{1\}$, $\mathcal{E} = \{(1, 1)\}$, we have $\alpha_{SDP,1} < 1$ for some $h \in \mathbb{N}_+$, $P_1 \in \mathbb{S}^n_{++}$, and $\mathcal{A}_{1\to 1} \subset \mathcal{A}^*$ if and only if the SLS* (1) *is periodic open-loop stabilizable.*

Lemma 2 proves an inherent restriction of the SDP test (22) using the PQCLF or the APQCLF. A question is whether or not the same conclusion can be drawn for the fully generalized digraphs. It is natural to expect that since only one of $\alpha_{SDP,j}$, $j \in \mathcal{V}$, is enforced to be larger than or equal to one by Proposition 5 when $\det(A_i) \geq 1$ for all $i \in \mathcal{M}$, there is still a chance that the gains of all the simple cycles of $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha_{SDP,j}$, $j \in \mathcal{V}$, are strictly less than one so that we can identify the stabilizability. It will be proved later that the answer is negative: the same conclusion as in Lemma 2 holds for the arbitrarily general digraphs. To prove this, we first establish the following result, which proves a convergence property of Problem 3 as the size of the sets of the words increases.

Proposition 6 *Suppose that*

(1) a given digraph $G(\mathcal{V}, \mathcal{E})$ has no sink;
(2) there exist the matrices $P_i = P_i^* \in \mathbb{S}^n_{++}$, $i \in \mathcal{V}$, the words $\mathcal{A}_{j\to i} = \mathcal{A}^*_{j\to i} \subset \mathcal{A}^{[1,h]}$, $\forall (j, i) \in \mathcal{E}$, and the scalars $\lambda_{(A,i,j)} = \lambda^*_{(A,i,j)} \forall A \in \mathcal{A}^*_{j\to i}$, $(j, i) \in \mathcal{E}$, $\alpha_j = \alpha_j^* \in \mathbb{R}_+$, $\forall j \in \mathcal{V}$, such that the constraints of the SDP (22) *associated with $G(\mathcal{V}, \mathcal{E})$ are satisfied;*
(3) the weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha_j = \alpha_j^* \in \mathbb{R}_+$, $\forall j \in \mathcal{V}$, has simple cycles whose gains are all less than one.

Then, for arbitrary $P_i = \tilde{P}_i \in \mathbb{S}^n_{++}$, $i \in \mathcal{V}$,

(1) there exists a digraph $G(\mathcal{V}, \tilde{\mathcal{E}})$ with no sink;
(2) there exist the words $\mathcal{A}_{j\to i}$, $\forall (j, i) \in \tilde{\mathcal{E}}$, and $\alpha_j = \tilde{\alpha}_j \in \mathbb{R}_+$, $\forall j \in \mathcal{V}$, satisfying the constraints of the SDP (22) *associated with $G(\mathcal{V}, \tilde{\mathcal{E}})$;*
(3) the weighted digraph $G(\mathcal{V}, \tilde{\mathcal{E}}, \alpha)$ with the parameters $\tilde{\alpha}_j \in \mathbb{R}_+$, $\forall j \in \mathcal{V}$, has simple cycles whose gains are all less than one.

PROOF. See Appendix C. □

In the following result, we prove that the limitation of the SDP test (22) when the PQCLF or the APQCLF is considered cannot be overcome by the use of the general QGCLF.

Proposition 7 *There exist a digraph $G(\mathcal{V}, \mathcal{E})$, $P_i \in \mathbb{S}^n_{++}$, $i \in \mathcal{V}$, and $\mathcal{A}_{j\to i} \subset \mathcal{A}^*$, $(j, i) \in \mathcal{E}$, such that*

(1) $G(\mathcal{V}, \mathcal{E})$ has no sink;
(2) the gains of all the simple cycles in $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha_{SDP,j}$, $\forall j \in \mathcal{V}$ is less than one,

if and only if the SLS (1) *is periodic open-loop stabilizable.*

PROOF. See Appendix D. □

Example 4 *Consider the SLS* (1) *with*

$$A_1 = \begin{bmatrix} 0.5923 & 0.5283 & 0.7565 \\ 1.5375 & 0.7170 & 1.0567 \\ 0.9333 & 0.2953 & -0.0096 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.1003 & 0.0578 & 0.2078 \\ 1.1190 & -0.6934 & 0.6320 \\ 1.3056 & 0.0255 & -3.2854 \end{bmatrix}.$$

The goal is to determine the stabilizability of the SLS. We generate the quadratic functions $V_i(z) = z^T P_i z$, $i \in \mathcal{V}$, where the matrices P_i $i \in \mathcal{V}$, are randomly chosen from the set of matrices $\mathcal{H}_4$ obtained by the iteration $\mathcal{H}_0 := \{I_n\}$ and $\mathcal{H}_k := \{A_p^T H A_p + I_n : H \in \mathcal{H}_{k-1}, p \in \mathcal{M}\}$ for $k \in \{1, 2, \ldots, 4\}$. Note that the construction of the matrix set $\mathcal{H}_4$ is motivated from the generating function method in [26]. We also randomly generate a digraph $G(\mathcal{V}, \mathcal{E})$ with the adjacency matrix

$$E = \begin{bmatrix} 0&1&1&1&0&1&1 \\ 0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0 \\ 1&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0 \end{bmatrix}.$$

With $\mathcal{A}_{j\to i} = \mathcal{A}^{[1,6]}$, $\forall (q, p) \in \mathcal{E}$, we obtain $\alpha_{SDP,1} = 1.9758$, $\alpha_{SDP,2} = 0.7905$, $\alpha_{SDP,3} = 0.7904$, $\alpha_{SDP,4} = 0.4389$, $\alpha_{SDP,5} = 1.7832$, $\alpha_{SDP,6} = 0.4415$, and

$\alpha_{SDP,7} = 0.3820$. *The adjacency matrix of the weighted digraph* $G(\mathcal{V}, \mathcal{E}, \alpha)$ *is obtained as*

$$E = \begin{bmatrix} 0 & 0.7905 & 0.7904 & 0.4389 & 0 & 0.4415 & 0.3820 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3820 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.9758 & 0 & 0 & 0 & 1.7832 & 0 & 0.3820 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.7832 & 0 & 0 \\ 1.9758 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The digraph has the simple cycles $\mathcal{C}_1 = (1, 7, 2, 1)$, $\mathcal{C}_2 = (1, 4, 1)$, $\mathcal{C}_3 = (1, 7, 4, 1)$, $\mathcal{C}_4 = (1, 7, 1)$, *and the corresponding cycle gains are* $g(\mathcal{C}_1) = 0.5967$, $g(\mathcal{C}_2) = 0.8671$, $g(\mathcal{C}_3) = 0.3313$, $g(\mathcal{C}_4) = 0.7548$, *respectively. Since all the cycle gains are less than one, by Theorem 1, the SLS is stabilizable, and the exponential convergence rate is* $\phi = 0.9960$. *Proposition 3 gives an alternative way to estimate the exponential convergence rate without considering the cycle gains. To investigate the possibility that Proposition 3 can also give a valid result, consider the subgraphs* $G(\mathcal{V}, \mathcal{E}_1, \alpha), \ldots, G(\mathcal{V}, \mathcal{E}_5, \alpha)$ *of* $G(\mathcal{V}, \mathcal{E}, \alpha)$ *with the adjacency matrices* $E_1 = e_1^T e_1 E$, $E_2 = e_2^T e_2 E$, $E_3 = (e_3^T e_3 + e_4^T e_4)E$, $E_4 = (e_5^T e_5 + e_6^T e_6)E$, $E_5 = e_7^T e_7 E$, *respectively. Applying Proposition 3 yields the exponential convergence rate* $\phi = 0.9845$. *On the other hand, the APQCLF with* $V_1(z) = x^T x$ *and* $\mathcal{A}_{1\to 1} = \mathcal{A}^{[1,\,6]}$ *cannot certify the stabilizability, while the APQCLF with* $V_1(z) = x^T x$ *and* $\mathcal{A}_{1\to 1} = \mathcal{A}^{[1,\,7]}$ *gives* $\phi = 0.9714$.

Remark 3 *To find matrices* P_i, $i \in \mathcal{V}$, *simultaneously in Problem 3, it can be extended to a bilinear (or biaffine) matrix inequality (BMI) optimization problem, which has BMI constraints. There are several algorithms to find its local minimums or stationary points, for example, the path-following method [52], the subgradient method [53], the interior point method [54], and the DC (difference of two convex functions) programming [55]. Even though there is no general guideline for deciding digraphs, one can use a simple digraph, for instance, a digraph with two nodes* $\mathcal{V} = \{1, 2\}$ *and two edges* $\mathcal{E} = \{((1, 2), (2, 1)\}$, *and combined with one of the aforementioned methods e.g., the path-following method [52], effective algorithms can be developed.*

7 Conclusion

In this paper, we have extended the work in [29] to deal with the stabilization of the SLSs. The GCLF has been proposed, and it has been proved that the existence of the GCLF is a necessary and sufficient condition for the stabilizability of the SLSs. Moreover it has been proved that the GCLF unifies various Lyapunov functions, for instance, the periodic, aperiodic, and piecewise quadratic control Lyapunov functions. Computational methods to search for the GCLF have been developed based on the SDP and the BMIs. A numerical example has been given to illustrate the proposed algorithm and demonstrate the potential advantage of the GCLF approach. The problem of searching the GCLF has a structure that can be computationally parallelized by the existing multi-agent optimization techniques, for example, the distributed optimization in [56], and this can be a possible subject of the future research.

A Proof of Theorem 3

As the JSR does not depend on the matrix norm used [24], in this proof, it is convenient to use the induced matrix ∞-norm defined by $\|A\|_\infty := \max_{1\le i\le |\mathcal{V}|}(\sum_{j=1}^{|\mathcal{V}|} |A_{ij}|)$. First, we will prove that if the JSR is less than one, then so are the gains of all the simple cycles. Consider an arbitrary walk of length t, $\mathcal{W}_{t+1} = (v_0, v_1, \ldots, v_t)$, in $G(\mathcal{V}, \mathcal{E}, \alpha)$. If $E \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ is the adjacency matrix of $G(\mathcal{V}, \mathcal{E}, \alpha)$, then the gain of $\mathcal{W}_{t+1}$ can be written as

$$\begin{aligned} g(\mathcal{W}_{t+1}) &= \prod_{k=0}^{t-1} g((v_k, v_{k+1})) \\ &= \prod_{k=0}^{t-1} e_{v_{k+1}}^T E e_{v_k} = \left\| e_{v_t} \prod_{k=0}^{t-1} e_{v_{k+1}}^T E e_{v_k} \right\|_\infty \\ &= \left\| \left(\prod_{k=0}^{t-1} e_{v_{k+1}} e_{v_{k+1}}^T E \right) e_{v_0} \right\|_\infty \\ &= \left\| \left(\prod_{k=0}^{t-1} e_{v_{k+1}} e_{v_{k+1}}^T E e_{v_k} e_{v_k}^T \right) e_{v_0} \right\|_\infty \\ &= \left\| \left(\prod_{k=0}^{t-1} \alpha_{v_k} e_{v_{k+1}} e_{v_k}^T \right) e_{v_0} \right\|_\infty . \end{aligned}$$

Note that each adjacency matrix E_p of the subgraph $G(\mathcal{V}, \mathcal{E}_p, \alpha)$ can be expressed as $E_p = \sum_{(j,\,i)\in\mathcal{E}_p} \alpha_j e_i e_j^T$. Therefore, if $\theta(k) \in \mathcal{M}$ is chosen so that $(v_k, v_{k+1}) \in \mathcal{E}_{\theta(k)}$ for all $k \in \{0, 1, \ldots, t-1\}$, then since all the elements of E_p are positive, we have

$$\left\| \left(\prod_{k=0}^{t-1} \alpha_{v_k} e_{v_{k+1}} e_{v_k}^T \right) e_{v_0} \right\|_\infty \le \left\| \prod_{k=0}^{t-1} E_{\theta(k)} e_{v_0} \right\|_\infty .$$

Combining the last two results, we have

$$g(\mathcal{W}_{t+1}) \le \left\| \prod_{k=0}^{t-1} E_{\theta(k)} e_{v_0} \right\|_\infty \le \max_{A\in\Sigma^t} \|A e_{v_0}\|_\infty, \quad \text{(A.1)}$$

where $\Sigma := \{E_1, E_2, \dots, E_Q\}$.

On the other hand, for any $j \in \mathcal{V}$, the definition of the JSR gives

$$\rho(\Sigma) = \lim_{k\to\infty} \max_{A\in\Sigma^k} \|A\|_\infty^{1/k} \geq \lim_{k\to\infty} \max_{A\in\Sigma^k} \|Ae_j\|_\infty^{1/k},$$

where $e_j \in \mathbb{R}^{|\mathcal{V}|}$, $j \in \mathcal{V}$, is the j-th unit vector. This implies that there exists $T \in \mathbb{N}_+$ such that

$$\rho(\Sigma) \geq \max_{A\in\Sigma^t} \|Ae_j\|_\infty^{1/t}, \quad \forall t \geq T. \tag{A.2}$$

Combining (A.1) with (A.2), we have

$$g(\mathcal{W}_{t+1}) \leq \rho(\Sigma)^t, \quad \forall t \geq T. \tag{A.3}$$

Using the last inequality, we will prove that the gains of all simple cycles are less than one. Assume by contradiction that there exists a simple cycle $\mathcal{C}$ in $G(\mathcal{V}, \mathcal{E}, \alpha)$ with $g(\mathcal{C}) \geq 1$, define a walk $\mathcal{W}_\infty = (v_0, v_1, \dots)$ which circles around $\mathcal{C}$ infinitely many times, and $\mathcal{W}_{t+1} = (v_0, v_1, \dots, v_t)$ is a closed walk which is a truncation of $\mathcal{W}_\infty$. Then, the left-hand side of (A.3) should be always larger than or equal to one. Since $\rho(\Sigma)^t \to 0$ as $t \to \infty$, there exists a sufficiently large $t \in \mathbb{N}_+$ such that the right hand side of the last inequality is strictly less than one. This gives us a contradiction. Thus, $G(\mathcal{V}, \mathcal{E}, \alpha)$ does not have a simple cycle with its cycle gain larger than or equal to one. By Definition 2, $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with $G(\mathcal{V}, \mathcal{E}, \alpha)$.

To estimate the exponential convergent rate of the SLS (1), define the sequence generated by $k_0 = 0$, and $k_{t+1} = k_t + |\Phi(j_t, j_{t+1}, \xi_t)|$ for $t \in \mathbb{N}_+$, where $\{\xi_t\}_{t=0}^\infty$ is the subsequence of the states defined in (8) so that $\xi_t = x(k_t; z, \sigma)$, $\forall t \in \mathbb{N}$. If $\mathcal{W}_{t+1} = (v_0, v_1, \dots, v_t)$ is a walk associated with the policy (7), then $V_{v_t}(\xi_t) \leq g(\mathcal{W}_{t+1})V_{v_0}(\xi_0)$, and using (A.3), we have

$$V_{v_t}(\xi_t) \leq \rho(\Sigma)^t V_{v_0}(\xi_0), \quad \forall t \geq T.$$

Combining the last inequality with (4) leads to

$$\|\xi_t\|^2 \leq \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i} \rho(\Sigma)^t \|\xi_0\|^2, \quad \forall t \geq T. \tag{A.4}$$

Since $\|\xi_t\|^2 \leq \tau^{T-1}\|\xi_0\|^2$, $\forall t \in [0, T-1]$, where $\tau := \max_{\mu\in\mathcal{M}} \|A_\mu\|$, it is easy to prove that

$$\rho(\Sigma)^{-T+1} \max\left\{\tau^{T-1}, \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right\} \rho(\Sigma)^t \|\xi_0\|^2.$$

is an upper bound on $\tau^{T-1}\|\xi_0\|^2$ for all $t \in [0, T-1]$ and an upper bound on the right-hand side of (A.4) for all $t \geq T$. Therefore, the following holds for all $t \in \mathbb{N}$:

$$\|\xi_t\|^2 \leq \rho(\Sigma)^{-T+1} \max\left\{\tau^{T-1}, \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right\} \rho(\Sigma)^t \|\xi_0\|^2.$$

For any $k \in \mathbb{N}$, choose $t \in \mathbb{N}$ such that $k_{t+1} \geq k \geq k_t$. Noting $k = k_t + (k - k_t) \leq k_t + L$ and $t \geq k/L - 1$, we have

$$\begin{aligned}
&\|x(k; z, \sigma)\|^2 = \|x(k_t + k - k_t; z, \sigma)\|^2 \\
&\leq \tau^{k-k_t} \|x(k_t; z, \sigma)\|^2 \leq \tau^L \|\xi_t\|^2 \\
&\leq \tau^L \rho(\Sigma)^{-T+1} \max\left\{\tau^{T-1}, \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right\} \rho(\Sigma)^{k/L-1} \|z\|^2,
\end{aligned}$$

and the desired result follows.

B Proof of Proposition 5

Using the inequality of arithmetic and geometric means, for any $H \in \mathbb{S}_+^n$, we have $(1/n)\mathrm{trace}(H) \geq \sqrt[n]{\det(H)}$. Therefore, for any $P_j \in \mathbb{S}_{++}^n$, $P_i \in \mathbb{S}_{++}^n$, $i \in \mathcal{N}_j^+$, and $A \in \mathcal{A}_{j\to i}$, $(j, i) \in \mathcal{E}$,

$$\begin{aligned}
&\mathrm{trace}(I_n - P_j^{-1/2} A^T P_i A P_j^{-1/2}) \\
&\leq n - n[\det(P_j^{-1} A^T P_i A P_j^{-1})]^{1/n} \\
&= n - n\left[\frac{\det(P_i)}{\det(P_j)} \det(A)^2\right]^{1/n} \\
&\leq n - n\left[\frac{\min_{i\in\mathcal{N}_j^+} \det(P_i)}{\det(P_j)} \det(A)^2\right]^{1/n} \\
&\leq n - n\left[\frac{\min_{i\in\mathcal{N}_j^+} \det(P_i)}{\det(P_j)}\right]^{1/n}
\end{aligned}$$

where the last inequality follows from the assumption that $\det(A_\mu) \geq 1$ for all $\mu \in \mathcal{M}$. Therefore, if $j^* := \arg\min_{j\in\mathcal{V}} \det(P_j)$, then

$$\begin{aligned}
&\mathrm{trace}(I_n - P_{j^*}^{-1/2} A^T P_i A P_{j^*}^{-1/2}) \\
&\leq n - n\left[\frac{\min_{i\in\mathcal{N}_{j^*}^+} \det(P_i)}{\det(P_{j^*})}\right]^{1/n} \\
&\leq 0.
\end{aligned}$$

This implies that the convex hull of the set $I_n - P_{j^*}^{-1/2} A^T P_i A P_{j^*}^{-1/2}$ for $A \in \mathcal{A}_{j^*\to i}$, $i \in \mathcal{N}_{j^*}^+$ does not intersect $\mathbb{S}_{++}^n$ as matrices in the latter set have positive trace. This in turn implies that the convex hull of the set $P_{j^*} - A^T P_i A$ for $A \in \mathcal{A}_{j^* \to i}$, $i \in \mathcal{N}_{j^*}^+$ does not intersect $\mathbb{S}_{++}^n$, either. By the definition of $\alpha_{SDP,\,j}$, we have $\alpha_{SDP,\,j} \geq 1$.

C Proof of Proposition 6

For any $j \in \mathcal{V}$, define by $\mathcal{T}(k,\,j) \subset \mathcal{V}^{k+1}$ the set of all walks of length k, $\mathcal{W}_{k+1} = (v_0,\,v_1,\ldots,\,v_k)$, in $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ with the initial node $v_0 = j$. For any $j \in \mathcal{V}$, the LMI constraint of the SDP (22) corresponds to a subgraph of which consists of the node j, its out-neighbors, and the directed edges from the node j to its out-neighbors. Now, let $j = j_0 \in \mathcal{V}$. By plugging the right-hand side of the LMIs in (22) for all $j \in \mathcal{V}$ into the left-hand side of the LMI in (22) for $j = j_0 \in \mathcal{V}$, we can obtain

$$\sum_{\substack{A_0 \in \mathcal{A}^*_{j_0 \to j_1},\, j_1 \in \mathcal{N}^+_{j_0} \\ A_1 \in \mathcal{A}^*_{j_1 \to j_2},\, j_2 \in \mathcal{N}^+_{j_1}}} \frac{1}{\alpha^*_{j_1}} \frac{1}{\alpha^*_{j_0}} \lambda^*_{(A_1,\,j_2)} \lambda^*_{(A_0,\,j_1)} A_0^T A_1^T P^*_{j_2} A_1 A_0 \\ \preceq P^*_{j_0},$$

which corresponds to a digraph which consists of the node j_0, its second order out-neighbors, and the directed edges from the node j_0 to its second order out-neighbors, where k-th order out-neighbors of a node $j \in \mathcal{V}$ are defined as all nodes which can be reached from the node j in exactly k hops. Repeating this procedure $k-2$ times more, it can be proved that $\Phi \preceq P_j^*$ holds for some $\Phi \in \mathrm{conv}(\mathcal{C}^*(1,\,j))$, where

$$\mathcal{C}^*(k,\,j) \\ := \left\{ \begin{array}{c} \frac{1}{g(\mathcal{W}_{k+1})} (X_{k-1}\cdots X_0)^T P^*_{v_k} (X_{k-1}\cdots X_0) : \\ X_t \in \mathcal{A}^*_{v_t \to v_{t+1}},\, t \in \{0,\,1,\ldots,\,k-1\}, \\ \mathcal{W}_{k+1} = (v_0,\,v_1,\ldots,\,v_k) \in \mathcal{T}(k,\,j) \end{array} \right\}.$$

By direct manipulations, it can be proved that the last inequality can be rewritten by $\Phi \preceq \tilde{\alpha}_j \tilde{P}_j$ for some $\Phi \in \mathrm{conv}(\tilde{\mathcal{C}}(1,\,j))$, where

$$\tilde{\alpha}_j = \min_{\mathcal{W}_{k+1} \in \mathcal{T}(k,\,q)} g(\mathcal{W}_{k+1}) \frac{\lambda_{\max}(P_j^*)}{\lambda_{\min}(\tilde{P}_j)}, \\ \tilde{\mathcal{C}}(k,\,j) \\ := \left\{ \begin{array}{c} (X_{k-1}\cdots X_0)^T \tilde{P}_{v_k} (X_{k-1}\cdots X_0) : \\ X_t \in \mathcal{A}^*_{v_t \to v_{t+1}},\, t \in \{0,\,1,\ldots,\,k-1\}, \\ (v_0,\,v_1,\ldots,\,v_k) \in \mathcal{T}(k,\,j) \end{array} \right\}.$$

Following similar lines as in the proof of Theorem 1, it can be proved that $\lim_{\mathcal{W}_{k+1} \in \mathcal{T}(k,\,j),\, k \to \infty} g(\mathcal{W}_{k+1}) = 0$, and hence, for a sufficiently large $k = k_j \in \mathbb{N}_+$, we can make $\tilde{\alpha}_j$ arbitrarily small. On the other hand, consider another weighted digraph $G(\mathcal{V},\,\tilde{\mathcal{E}},\,\alpha)$ with the parameters $\tilde{\alpha}_j$, $\forall j \in \mathcal{V}$, where the edge set $\tilde{\mathcal{E}} \subseteq \mathcal{V} \times \mathcal{V}$ is constructed in such a way that each node $j \in \mathcal{V}$ in $G(\mathcal{V},\,\tilde{\mathcal{E}},\,\alpha)$ has out-neighbors which are the k_j-th order out-neighbors of the node $j \in \mathcal{V}$ in $G(\mathcal{V},\,\mathcal{E})$. It is clear that the digraph $G(\mathcal{V},\,\tilde{\mathcal{E}})$ has no sink if $G(\mathcal{V},\,\mathcal{E})$ so does. Suppose that exists a simple cycle $\mathcal{C}$ in $G(\mathcal{V},\,\tilde{\mathcal{E}},\,\alpha)$ such that $g(\mathcal{C}) \geq 1$. If the edge $(j,\,i) \in \tilde{\mathcal{E}}$ is included in $\mathcal{C}$, then by choosing $k_j \in \mathbb{N}_+$ large enough, we can make $g(\mathcal{C}) < 1$. This completes the proof.

D Proof of Proposition 7

To prove the sufficiency, suppose that the SLS (1) is periodic open-loop stabilizable. Then, by Lemma 2, there exist $P_1 \in \mathbb{S}_{++}^n$ and $h \in \mathbb{N}_+$ such that $\alpha_{SDP,\,1} < 1$ for $G(\mathcal{V},\,\mathcal{E})$, $\mathcal{V} = \{1\}$, $\mathcal{E} = \{(1,\,1)\}$ and $\mathcal{A}_{1\to 1} \subset \mathcal{A}^h$. Thus, the statements 1) and 2) are satisfied. This proves the sufficiency.

For the necessity part, suppose that there exist a digraph $G(\mathcal{V},\,\mathcal{E})$, $P_i \in \mathbb{S}_{++}^n$, $i \in \mathcal{V}$, and $\mathcal{A}_{j\to i} \subset \mathcal{A}^*$, $(j,\,i) \in \mathcal{E}$ such that the statements 1) and 2) hold. By Proposition 6, we have $\Phi \preceq \beta_{j,\,k} I_n$ for some $\Phi \in \mathrm{conv}(\mathcal{C}(1,\,j))$, where

$$\beta_{j,\,k} = \min_{\mathcal{W}_{k+1} \in \mathcal{T}(k,\,j)} g(\mathcal{W}_{k+1}) \lambda_{\max}(P_j) \\ \mathcal{C}(k,\,j) \\ := \left\{ \begin{array}{c} (X_{k-1}\cdots X_0)^T (X_{k-1}\cdots X_0) : \\ X_t \in \mathcal{A}_{v_t \to v_{t+1}},\, t \in \{0,\,1,\ldots,\,k-1\}, \\ (v_0,\,v_1,\ldots,\,v_k) \in \mathcal{T}(k,\,j) \end{array} \right\}.$$

Since $\lim_{\mathcal{W}_{k+1} \in \mathcal{T}(k,\,j),\, k\to\infty} g(\mathcal{W}_{k+1}) = 0$, for a sufficiently large $k = \bar{k} \in \mathbb{N}_+$, we get $\beta_{j,\,\bar{k}} < 1$, implying that for the SDP (22) associated with $G(\mathcal{V},\,\mathcal{E})$, $\mathcal{V} = \{1\}$, $\mathcal{E} = \{(1,\,1)\}$, $\alpha_{SDP,\,1} < 1$ holds for $P_1 = I_n$ and some $\mathcal{A}_{1\to 1} \subset \mathcal{A}^*$. By Lemma 2, this ensures that the SLS (1) is periodic open-loop stabilizable. This completes the proof.

References

[1] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Syst. Mag.*, vol. 19, no. 5, pp. 59–70, 1999.

[2] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM rev.*, vol. 49, no. 4, pp. 545–592, 2007.

[3] E. Skafidas, R. J. Evans, A. V. Savkin, and I. R. Petersen, "Stability results for switched controller systems," *Automatica*, vol. 35, no. 4, pp. 553–564, 1999.

[4] S. Pettersson and B. Lennartson, "Stabilization of hybrid systems using a min-projection strategy," in *Proceedings of the American Control Conference*, vol. 1, 2001, pp. 223–228.

[5] M. Wicks, P. Peleties, and R. DeCarlo, "Switched controller synthesis for the quadratic stabilisation of a pair of unstable linear systems," *Eur. J. Control*, vol. 4, no. 2, pp. 140–147, 1998.

[6] D. Liberzon, "Switching in systems and control," 2003.

[7] F. Blanchini and C. Savorgnan, "Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions," *Automatica*, vol. 44, no. 4, pp. 1166–1170, 2008.

[8] A. P. Molchanov and Y. S. Pyatnitskiy, "Criteria of asymptotic stability of differential and difference inclusions encountered in control theory," *Systems Control Lett.*, vol. 13, no. 1, pp. 59–64, 1989.

[9] M. S. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Control*, vol. 43, no. 4, pp. 475–482, 1998.

[10] H. Lin and P. J. Antsaklis, "Hybrid state feedback stabilization with l_2 performance for discrete-time switched linear systems," *Internat. J. Control*, vol. 81, no. 7, pp. 1114–1124, 2008.

[11] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Automat. Control*, vol. 43, no. 4, pp. 555–559, 1998.

[12] L. Xie, S. Shishkin, and M. Fu, "Piecewise Lyapunov functions for robust stability of linear time-varying systems," *Systems Control Lett.*, vol. 31, no. 3, pp. 165–171, 1997.

[13] S. Pettersson, "Synthesis of switched linear systems," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, vol. 5, 2003, pp. 5283–5288.

[14] J. C. Geromel and P. Colaneri, "Stability and stabilization of discrete time switched systems," *Internat. J. Control*, vol. 79, no. 7, pp. 719–728, 2006.

[15] ——, "Stability and stabilization of continuous-time switched linear systems," *SIAM J. Control Optim.*, vol. 45, no. 5, pp. 1915–1930, 2006.

[16] W. Zhang, A. Abate, J. Hu, and M. P. Vitus, "Exponential stabilization of discrete-time switched linear systems," *Automatica*, vol. 45, no. 11, pp. 2526–2536, 2009.

[17] A. Polański, "On absolute stability analysis by polyhedral Lyapunov functions," *Automatica*, vol. 36, no. 4, pp. 573–578, 2000.

[18] A. Papchristodoulou and S. Prajna, "Robust stability analysis of nonlinear hybrid systems," *IEEE Trans. Automat. Control*, vol. 54, no. 5, pp. 1034–1041, 2009.

[19] P. A. Parrilo and A. Jadbabaie, "Approximation of the joint spectral radius using sum of squares," *Linear Algebra Appl.*, vol. 428, no. 10, pp. 2385–2402, 2008.

[20] T. Hu and Z. Lin, "Composite quadratic Lyapunov functions for constrained control systems," *IEEE Trans. Automat. Control*, vol. 48, no. 3, pp. 440–450, 2003.

[21] T. Hu, L. Ma, and Z. Lin, "Stabilization of switched systems via composite quadratic functions," *IEEE Trans. Automat. Control*, vol. 53, no. 11, pp. 2571–2585, 2008.

[22] J. Daafouz, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach," *IEEE Trans. Automat. Control*, vol. 47, no. 11, pp. 1883–1887, 2002.

[23] J.-W. Lee and G. E. Dullerud, "Optimal disturbance attenuation for discrete-time switched and markovian jump linear systems," *SIAM J. Control Optim.*, vol. 45, no. 4, pp. 1329–1358, 2006.

[24] R. Jungers, *The Joint Spectral Radius, Theory and Applications*. Berlin, Germany: Springer-Verlag, 2009.

[25] V. D. Blondel and Y. Nesterov, "Computationally efficient approximations of the joint spectral radius," *SIAM J. Matrix Anal. Appl.*, vol. 27, no. 1, pp. 256–272, 2005.

[26] J. Hu, J. Shen, and W. Zhang, "Generating functions of discrete-time switched linear systems: Analysis, computation, and stability applications," *IEEE Trans. Automat. Control*, vol. 56, no. 5, pp. 1059–1074, 2011.

[27] A. A. Ahmadi, "Non-monotonic lyapunov functions for stability of nonlinear and switched systems: Theory and computation," Ph.D. dissertation, Massachusetts Institute of Technology, 2008.

[28] A. A. Ahmadi and P. Parrilo, "Non-monotonic Lyapunov functions for stability of discrete-time nonlinear and switched systems," in *Proceedings of the 47th IEEE Conference on Decision and Control*, 2008, pp. 614–621.

[29] A. A. Ahmadi, R. M. Jungers, P. A. Parrilo, and M. Roozbehani, "Joint spectral radius and path-complete graph Lyapunov functions," *SIAM J. Control Optim.*, vol. 52, no. 1, pp. 687–717, 2014.

[30] M. Fiacchini and M. Jungers, "Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: A set-theory approach," *Automatica*, vol. 50, no. 1, pp. 75–83, 2014.

[31] M. Fiacchini, A. Girard, and M. Jungers, "On stabilizability conditions for discrete-time switched linear systems," in *Proceedings of the 53rd IEEE Conference on Decision and Control*, 2014, pp. 5469–5474.

[32] ——, "On the stabilizability of discrete-time switched linear systems: novel conditions and comparisons," *IEEE Trans. Automat. Control (in press)*, 2015.

[33] Y. Ebihara, D. Peaucelle, and D. Arzelier, "Periodically time-varying memory state-feedback controller synthesis for discrete-time linear systems," *Automatica*, vol. 47, no. 1, pp. 14–25, 2011.

[34] J. F. Trégouët, D. Peaucelle, D. Arzelier, and Y. Ebihara, "Periodic memory state-feedback controller: New formulation, analysis, and design results," *IEEE Trans. Automat. Control*, vol. 58, no. 8, pp. 1986–2000, 2013.

[35] A. Kruszewski, R. Wang, and T.-M. Guerra, "Nonquadratic stabilization conditions for a class of uncertain nonlinear discrete time TS fuzzy models: A new approach," *IEEE Trans. Automat. Control*, vol. 53, no. 2, pp. 606–611, 2008.

[36] S. Bittanti, P. Bolzern, and P. Colaneri, "The extended periodic Lyapunov lemma," *Automatica*, vol. 21, no. 5, pp. 603–605, 1985.

[37] A. Butz, "Higher order derivatives of Liapunov functions," *IEEE Trans. Automat. Control*, vol. 14, no. 1, pp. 111–112, 1969.

[38] V. Meigoli and S. K. Y. Nikravesh, "Stability analysis of nonlinear systems using higher order derivatives of Lyapunov function candidates," *Systems Control Lett.*, vol. 61, no. 10, pp. 973–979, 2012.

[39] A. A. Ahmadi and P. A. Parrilo, "On higher order derivatives of Lyapunov functions," in *Proceedings of the American Control Conference*, 2011, pp. 1313–1314.

[40] Y. Ebihara, T. Hagiwara, D. Peaucelle, and D. Arzelier, "Robust performance analysis of linear time-invariant uncertain systems by taking higher-order time-derivatives of

the state," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005, pp. 5030–5035.

[41] R. C. L. F. Oliveira, M. C. de Oliveira, and P. L. D. Peres, "Convergent lmi relaxations for robust analysis of uncertain linear systems using lifted polynomial parameter-dependent lyapunov functions," *Systems Control Lett.*, vol. 57, no. 8, pp. 680–689, 2008.

[42] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory.* Philadelphia, PA: SIAM, 1994.

[43] J. Bang-Jensen and G. Z. Gutin, *Digraphs: theory, algorithms and applications.* Springer Science and Business Media, 2008.

[44] P. Mateti and N. Deo, "On algorithms for enumerating all circuits of a graph," *SIAM J. Comput.*, vol. 5, no. 1, pp. 90–99, 1976.

[45] D. B. Johnson, "Finding all the elementary circuits of a directed graph," *SIAM J. Comput.*, vol. 4, no. 1, pp. 77–84, 1975.

[46] A. A. Ahmadi and P. A. Parrilo, "Joint spectral radius of rank one matrices and the maximum cycle mean problem," in *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp. 731–733.

[47] V. D. Blondel and J. N. Tsitsiklis, "The boundedness of all products of a pair of matrices is undecidable," *Systems Control Lett.*, vol. 41, no. 2, pp. 135–140, 2000.

[48] X. Liu, "Stability analysis of switched positive systems: A switched linear copositive Lyapunov function method," *IEEE Trans. Circuits Syst. II*, vol. 56, no. 5, pp. 414–418, 2009.

[49] E. Fornasini and M. E. Valcher, "Stability and stabilizability criteria for discrete-time positive switched systems," *IEEE Trans. Automat. Control*.

[50] D. Lee and J. Hu, "Periodic stabilization of discrete-time switched linear systems," in *Proceedings of the 54th IEEE Conference on Decision and Control*, 2015.

[51] ——, "Graph control Lyapunov function for stabilization of discrete-time switched linear systems," in *Proceedings of the American Control Conference*.

[52] A. Hassibi, J. How, and S. Boyd, "A path-following method for solving BMI problems in control," in *Proceedings of the American Control Conference*, vol. 2, 1999, pp. 1385–1389.

[53] E. Polak and Y. Wardi, "Nondifferentiable optimization algorithm for designing control systems having singular value inequalities," *Automatica*, vol. 18, no. 3, pp. 267–283, 1982.

[54] K. Goh, L. Turan, M. Safonov, G. Papavassilopoulos, and J. Ly, "Biaffine matrix inequality properties and computational methods," in *Proceedings of the American Control Conference*, vol. 1, 1994, pp. 850–855.

[55] Q. T. Dinh, S. Gumussoy, W. Michiels, and M. Diehl, "Combining convex–concave decompositions and linearization approaches for solving BMIs, with application to static output feedback," *IEEE Trans. Automat. Control*, vol. 57, no. 6, pp. 1377–1390, 2012.

[56] A. Nedić, A. Ozdaglar, and P. A. Parrilo, "Constrained consensus and optimization in multi-agent networks," *IEEE Trans. Automat. Control*, vol. 55, no. 4, pp. 922–938, 2010.