Exponential Stabilization of Discrete-Time Switched Linear Systems

Wei Zhang [a,*], Alessandro Abate [b], Jianghai Hu [a], Michael P. Vitus [b]

[a] *Department of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47906, USA*

[b] *Department of Aeronautics and Astronautics, Stanford University, CA 94305, USA*

Abstract

This paper studies the exponential stabilization problem for discrete-time switched linear systems based on a control-Lyapunov function approach. It is proved that a switched linear system is exponentially stabilizable if and only if there exists a piecewise quadratic control-Lyapunov function. Such a converse control-Lyapunov function theorem justifies many of the earlier synthesis methods that have adopted piecewise-quadratic Lyapunov functions for convenience or heuristic reasons. In addition, it is also proved that if a switched linear system is exponentially stabilizable, then it must be stabilizable by a stationary suboptimal policy of a related switched LQR problem. Motivated by some recent results of the switched LQR problem, an efficient algorithm is proposed, which is guaranteed to yield a control-Lyapunov function and a stabilizing policy whenever the system is exponentially stabilizable.

Key words: Switched systems, Piecewise quadratic Lyapunov functions, Switching stabilization, Optimal control, Control-Lyapunov functions.

1 Introduction

This paper studies the exponential stabilization problem for discrete-time switched linear systems. Specifically, our goal is to develop an efficient and constructive way to design both a switching strategy and a continuous control strategy to exponentially stabilize the system, when none of the subsystems is stabilizable but the entire switched system is exponentially stabilizable. Such a problem is regarded as one of the fundamental problems for switched systems (Liberzon and Morse [1999]), and has attracted considerable research attention (DeCarlo et al. [2000], Liberzon [2003], Savkin and Evans [2001], Sun and Ge [2005], Branicky [1998]).

Previous research has been mainly focused on the switch-

ing stabilization problem of autonomous switched linear systems, whose subsystems have no continuous-control inputs. Many existing results approach the problem by searching for a switching strategy and a Lyapunov or Lyapunov-like function with decreasing values along the closed-loop system trajectory (Skafidas et al. [1999], Pettersson [2003, 2004], Daafouz et al. [2002]). The main idea is first to parameterize the switching strategy and the Lyapunov-like function in terms of certain matrices and then to translate the Lyapunov or multiple-Lyapunov function theorem into matrix inequalities. The solution of these matrix inequalities, if it exists, characterizes a stabilizing switching strategy. If the solution of the matrix inequalities defines a quadratic common Lyapunov function under the proposed switching strategy, then the system is called quadratic stabilizable. It was proved in Skafidas et al. [1999], Savkin and Evans [2001] that the quadratic stabilizability is equivalent to the strict completeness of a certain set of symmetric matrices. From a different perspective, in Wicks et al. [1998], Pettersson and Lennartson [2001], it was shown that the system is quadratic stabilizable if there exists a stable convex combination of the subsystem matrices. The main limitation of these results is their conservatism. Many switched linear systems are asymptotically or exponentially stabilizable without

having a quadratic common Lyapunov function (Liberzon [2003]). In Pettersson [2003], a piecewise quadratic structure was adopted for the Lyapunov function. By taking a so-called "largest-region-function switching strategy", the stabilization problem was formulated as a bilinear matrix inequality (BMI) problem and some heuristics are proposed to solve the BMI problem numerically.

Recently, stabilization of nonautonomous switched linear systems through both switching control and continuous control has also been studied (Daafouz et al. [2002], Lin and Antsaklis [2008], Kar [2002]). The methods were mostly direct extensions of the switching stabilization results for autonomous systems. By associating to each subsystem a feedback gain and a quadratic Lyapunov function, the stabilization problem was also formulated as a matrix inequality problem, where the feedback-gain matrices were part of the design variables.

The extensive use of various Lyapunov functions has sparked a great interest in the study of the converse Lyapunov function theorems for switched linear systems. In Molchanov and Pyatnitskiy [1989], Dayawansa and Martin [1999], it was proved that the exponential stability of a switched linear system under *arbitrary switching* is equivalent to the existence of a piecewise quadratic, or a piecewise linear, or a smooth homogeneous common Lyapunov function. In Hu and Blanchini [2008], several sufficient and necessary conditions based on the composite Lyapunov functions were derived for stability/stabilizability of switched linear systems under arbitrary switching. However, these converse Lyapunov theorems and the equivalent conditions are only true for the arbitrary-switching case; they are far from necessary for the switching-stabilization problem.

Despite the extensive literature in this field, some fundamental questions regarding the stabilization of a switched linear system remain open. As stated in (Shorten et al. [2007]), "necessary and sufficient conditions for the existence of a general (not necessarily quadratic) stabilizing feedback strategy are not known". In addition, a constructive way of finding a stabilizing strategy when the system is known to be exponentially stabilizable is also lacking. This paper proposes a general control-Lyapunov function framework to tackle these open problems. One of the main contributions of this paper is the proof of the **equivalence** of the following statements for a discrete-time switched linear system:

 (i) *The system is exponentially stabilizable;*
 (ii) *There exists a piecewise quadratic control-Lyapunov function that can be expressed as a pointwise minimum of a finite number of quadratic functions;*
 (iii) *There exists a stationary exponentially-stabilizing hybrid-control policy that consists of a homogeneous switching-control law and a piecewise-linear continuous-control law.*

The particular type of Lyapunov functions as described by item (ii) was used in Hu et al. [2008], Geromel and Colaneri [2006] to study the switching stabilization problem; several sufficient conditions in terms of BMIs were also derived. However, the existence of this type of Lyapunov functions has not been established in the literature. The equivalence of the above three statements constitutes a converse *piecewise-quadratic control-Lyapunov function theorem* (Theorem 7). The theorem guarantees that to study the stabilization problem, it suffices to only consider the control-Lyapunov functions of piecewise-quadratic form and the continuous-control laws of piecewise-linear form. This justifies many of the earlier controller-synthesis methods that have adopted these forms for convenience or heuristic reasons.

The above results are proved by establishing a connection between the exponential stabilization problem and the switched LQR problem (Zhang and Hu [2008a]). It is well-known that a linear time-invariant system is exponentially stabilizable if and only if the infinite-horizon value function of the classical LQR problem is a control-Lyapunov function; furthermore, the value function can be asymptotically approximated through the difference Riccati recursion. In this paper, we prove a nontrivial extension of this classical result to switched linear systems. In particular, we show that a switched linear system is exponentially stabilizable *if and only if* there exists a finite integer N such that the N-horizon value function of the switched LQR problem is a control-Lyapunov function. Furthermore, this control-Lyapunov function can be obtained iteratively through the so-called *switched Riccati mapping*. Such a connection is not only the basis for proving the converse control-Lyapunov function theorem, but is also of its own value in the general study of switched linear systems.

Finally, from a designer's perspective, the theoretical results derived in this paper introduce a constructive and efficient way of computing an exponentially-stabilizing feedback policy. It is shown that if the switched linear system is exponentially stabilizable by an arbitrary feedback policy, then it must also be exponentially stabilizable by a stationary suboptimal policy of a related switched LQR problem (Theorem 8). This property transforms the stabilization problem into a switched LQR problem. Motivated by some recent results on the switched LQR problem (Zhang and Hu [2008b], Zhang et al. [2009a]), an efficient algorithm is proposed which can yield a control-Lyapunov function and a stabilizing policy whenever the system is exponentially stabilizable. Such an algorithm improves upon many existing ones that only provide sufficient conditions for stabilizability and may not yield a stabilizing strategy even when the switched system is exponentially stabilizable. As observed in some simulation examples (Section 7), the stabilizing feedback policy can often be computed efficiently.

This paper is organized is as follows. We start with the problem formulation in Section 2 and then develop a solution framework based on the control-Lyapunov function approach in Section 3. In Sections 4, 5 and 6, we discuss the existence and the computations of the control-Lyapunov function and the corresponding stabilizing policy. The derived results and the proposed algorithm are verified in Section 7 through some numerical examples. Finally, some concluding remarks are given in Section 8.

Notation: In this paper, n, p and M are some arbitrary finite positive integers, $\mathbb{Z}^+$ denotes the set of nonnegative integers, $\mathbb{M} \triangleq \{1, \ldots, M\}$ is the set of subsystem indices, I_n is the $n \times n$ identity matrix, $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$, $|\cdot|$ denotes the cardinality of a given set, $\mathcal{A}$ denotes the set of all the positive semidefinite (p.s.d.) matrices, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalues, respectively, of a given p.s.d. matrix. The variable z denotes a generic initial state of system (1).

2 Problem Formulation

We consider the discrete-time switched linear systems described by:

$$x(t+1) = A_{v(t)}x(t) + B_{v(t)}u(t), \ t \in \mathbb{Z}^+, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the continuous state, $v(t) \in \mathbb{M} \triangleq \{1, \ldots, M\}$ is the switching control that determines the discrete mode, and $u(t) \in \mathbb{R}^p$ is the continuous control. The state x and the control u are unconstrained. The sequence of pairs $\{(u(t), v(t))\}_{t=0}^{\infty}$ is called the *hybrid-control sequence*. For each $i \in \mathbb{M}$, A_i and B_i are constant matrices of appropriate dimensions and the pair (A_i, B_i) is called a subsystem. This switched linear system is time invariant in the sense that the set of available subsystems $\{(A_i, B_i)\}_{i=1}^{M}$ is independent of time t. We assume that there is no internal forced switchings, i.e., that the system can stay at or switch to any mode at any time instant. At each time $t \in \mathbb{Z}^+$, denote by $\xi_t \triangleq (\mu_t, \nu_t) : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{M}$ the *(state-feedback) hybrid-control law* of system (1), where $\mu_t : \mathbb{R}^n \to \mathbb{R}^p$ is called the *(state-feedback) continuous-control law* and $\nu_t : \mathbb{R}^n \to \mathbb{M}$ is called the *(state-feedback) switching-control law*. A sequence of hybrid-control laws constitutes an *infinite-horizon control policy*: $\pi \triangleq \{\xi_0, \xi_1, \ldots, \ldots\}$. Denote by Π the set of all admissible policies. If system (1) is driven by a control policy π, then the closed-loop dynamics is governed by

$$x(t+1) = A_{\nu_t(x(t))}x(t) + B_{\nu_t(x(t))}\mu_t(x(t)), \ t \in \mathbb{Z}^+. \quad (2)$$

In this paper, the policy π is allowed to be time-varying and the control law $\xi_t = (\mu_t, \nu_t)$ at each time step can be an arbitrary function of the state. A policy $\pi = \{\xi, \xi, \ldots\}$

with the same control law $\xi_t = \xi$ at each time t is called a *stationary policy*.

Definition 1 *The origin of system (2) is exponentially stable if there exist constants $a \geq 1$ and $0 < c < 1$ such that the system trajectory starting from any initial state $x(0) = z$ satisfies:*

$$\|x(t)\|^2 \leq ac^t\|z\|^2, \quad \forall t \in \mathbb{Z}^+.$$

Definition 2 *The system (1) is called* exponentially stabilizable *if there exists a control policy $\pi = \{(\mu_t, \nu_t)\}_{t \geq 0}$ under which the closed-loop system (2) is exponentially stable.*

Clearly, system (1) is exponentially stabilizable if one of the subsystems is stabilizable. A nontrivial problem is to stabilize the system when none of the subsystems are stabilizable. The main purpose of this paper is to develop an efficient and constructive way to solve the following stabilization problem.

Problem 1 (Stabilization Problem) *Suppose that the pair (A_i, B_i) is not stabilizable for any $i \in \mathbb{M}$. Find, if possible, a control policy π under which the closed-loop system (2) is exponentially stable.*

Most stabilization problems studied in the literature (Skafidas et al. [1999], Lin and Antsaklis [2008], Daafouz et al. [2002]) assume *a priori* that the hybrid-control policy is stationary, i.e., $(\mu_t, \nu_t) = (\mu, \nu)$, for any $t \in \mathbb{Z}^+$, and that each discrete mode is associated with only one feedback gain, i.e., $\mu(x) = F_{\nu(x)}x$, for some $\{F_i\}_{i=1}^{M}$. Problem 1 is more general than these problems as it allows for arbitrary (possibly nonstationary) hybrid-control policies. It will be shown in Section 5 that if the system is exponentially stabilizable, then there must exists a stationary stabilizing policy; however, the number of distinct feedback gains may be larger than the number of subsystems M. See Section 6 for more details.

3 A Control-Lyapunov Function Framework

In this section, we propose a special structure for the control-Lyapunov functions (to be defined below) associated with Problem 1. A distinctive feature of this structure is that if system (1) is exponentially stabilizable, we can always find a control-Lyapunov function with such structure. Additionally, the hybrid-control policy generated by the proposed control-Lyapunov functions can be derived analytically and computed efficiently.

3.1 Background

We first recall a version of the Lyapunov theorem for exponential stability.

Theorem 1 (Khalil [2002]) *Suppose that there exists a policy π and a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}^+$ satisfying:*

(i) $\kappa_1 \|z\|^2 \leq V(z) \leq \kappa_2 \|z\|^2$ for any $z \in \mathbb{R}^n$ and some finite positive constants κ_1 and κ_2;
(ii) $V(x(t)) - V(x(t+1)) \geq \kappa_3 \|x(t)\|^2$ for any $t \in \mathbb{Z}^+$ and some constant $\kappa_3 > 0$, where $x(\cdot)$ is the closed-loop trajectory of system (2) under the policy π.

Then system (2) is exponentially stable under π.

Definition 3 (ECLF) *A nonnegative continuous function $V : \mathbb{R}^n \to \mathbb{R}^+$ is called an* exponentially stabilizing control Lyapunov function *(ECLF) of system (1) if for any $z \in \mathbb{R}^n$, we have*

(i) $\kappa_1 \|z\|^2 \leq V(z) \leq \kappa_2 \|z\|^2$ for some finite positive constants κ_1 and κ_2;
(ii) $V(z) - \inf_{\{v \in \mathbb{M}, u \in \mathbb{R}^p\}} V(A_v z + B_v u) \geq \kappa_3 \|z\|^2$ for some constant $\kappa_3 > 0$.

The existence of an ECLF implies the exponential stabilizability of system (1).

Theorem 2 *If system (1) has an ECLF, then it is exponentially stabilizable by a* stationary *policy.*

PROOF. Let $V(z)$ be an ECLF of system (1). By Definition 3, we know that there must exist a constant $\kappa_3 > 0$ and a hybrid-control law $(\mu(\cdot), \nu(\cdot))$ such that

$$V(z) - V(A_{\nu(z)} z + B_{\nu(z)} \mu(z)) \geq \frac{\kappa_3 \|z\|^2}{2}, \qquad (3)$$

for any $z \in \mathbb{R}^n$, where the factor $1/2$ is introduced in case that the infimum in item (ii) of Definition 3 can not be achieved by any $u \in \mathbb{R}^p$. Let $\pi = \{\xi, \xi, \ldots\}$, where $\xi = (\mu, \nu)$. Since $\kappa_3/2 > 0$, V and π satisfy all the conditions in Theorem 1. Hence, system (1) is exponentially stabilizable by the stationary policy π. $\quad \square$

3.2 An Important Class of ECLFs

If $V(z)$ is an ECLF, then one can always find a stabilizing control policy as in the proof of Theorem 2. Such a policy is exponentially stabilizing, but may result in a large control action. A systematic way to stabilize the system with a reasonable control effort is to choose the hybrid control (u, v) that minimizes the abstract energy at the next step $V(A_v z + B_v u)$ plus a control energy expense. Toward this purpose, we introduce the following hybrid-control law:

$$\begin{aligned} \xi_V(z) &= (\mu_V(z), \nu_V(z)) \\ &= \underset{u \in \mathbb{R}^p, v \in \mathbb{M}}{\arg\inf} \left[V(A_v z + B_v u) + u^T R_v u \right], \quad (4) \end{aligned}$$

where for each $v \in \mathbb{M}$, $R_v = R_v^T \succ 0$ characterizes the penalizing metric for the continuous control u in mode v. Since the quantity inside the bracket is bounded from below, is continuous in u and grows to infinity as $\|u\| \to \infty$, the minimizer of (4) always exists and the following lemma follows immediately.

Lemma 1 *Let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a nonnegative continuous function satisfying the first condition of Definition 3. Let $\xi_V = (\mu_V, \nu_V)$ be defined by V through (4). If for all $z \in \mathbb{R}^n$,*

$$V(z) - V(A_{\nu_V(z)} z + B_{\nu_V(z)} \mu_V(z)) \geq \kappa_3 \|z\|^2, \quad (5)$$

for some constant $\kappa_3 > 0$, then V is an ECLF of system (1) and the system is exponentially stabilizable by the stationary policy $\{\xi_V, \xi_V, \ldots\}$.

If there exists a function satisfying the conditions in Lemma 1, we can use (4) to construct a stabilizing policy with a reasonable control effort. The challenge is how to find such a function. In the rest of this paper, we will focus on a particular class of piecewise quadratic functions as candidates for the ECLFs of system (1). Each of these functions can be written as a pointwise minimum of a finite number of quadratic functions as follows:

$$V_{\mathcal{H}}(z) = \min_{P \in \mathcal{H}} z^T P z, \qquad (6)$$

where $\mathcal{H}$ is a finite set of positive definite matrices, hereby referred to as an *FPD set*. The main reason that we focus on functions of the form (6) is that this form is sufficiently rich in terms of characterizing the ECLFs of system (1). It will be shown in Section 5 that the system is exponentially stabilizable if and only if there exists an ECLF of the form (6) satisfying (5).

3.3 Control Laws Corresponding to $V_{\mathcal{H}}$

With the particular structure of the candidate ECLFs as defined in (6), the control law defined in (4) can be derived in closed form. Its expression is closely related to the Riccati equation and the Kalman gain of the classical LQR problem. For each subsystem $i \in \mathbb{M}$, define a mapping $\rho_i^0 : \mathcal{A} \to \mathcal{A}$ as:

$$\rho_i^0(P) = A_i^T P A_i - A_i^T P B_i (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \quad (7)$$

which is the difference Riccati recursion of subsystem i with a zero state-weighting matrix. For each subsystem $i \in \mathbb{M}$ and each p.s.d. matrix P, the Kalman gain is defined as

$$K_i(P) \triangleq (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \qquad (8)$$

Lemma 2 *Let $\mathcal{H}$ be an arbitrary FPD set. Let $V_{\mathcal{H}} : \mathbb{R}^n \to \mathbb{R}^+$ be defined by $\mathcal{H}$ through (6). Then the control*

law defined in (4) is given by

$$\xi_{V_\mathcal{H}}(z) = \left(-K_{i_\mathcal{H}(z)}\left(P_\mathcal{H}(z)\right)z, i_\mathcal{H}(z)\right), \qquad (9)$$

where $K.(\cdot)$ is the Kalman gain defined in (8) and

$$(P_\mathcal{H}(z), i_\mathcal{H}(z)) = \underset{P \in \mathcal{H}, i \in \mathbb{M}}{\arg\min}\, z^T \rho_i^0(P)z. \qquad (10)$$

PROOF. By (4), to find $\xi_{V_\mathcal{H}}$, we need to solve the following optimization problem:

$$f(z) \triangleq \inf_{u \in \mathbb{R}^p, i \in \mathbb{M}} \left[\min_{P \in \mathcal{H}} (A_i z + B_i u)^T P(A_i z + B_i u) + u^T R_i u\right]$$

$$= \min_{i \in \mathbb{M}, P \in \mathcal{H}} \left\{ \inf_{u \in \mathbb{R}^p} \left[(A_i z + B_i u)^T P(A_i z + B_i u)\right.\right.$$

$$\left.\left. + u^T R_i u\right]\right\}. \quad (11)$$

For each $i \in \mathbb{M}$ and $P \in \mathcal{H}$, the quantity inside the square bracket is quadratic in u. Thus, the optimal value of u can be easily computed as $u^* = -K_i(P)z$, where $K_i(P)$ is the Kalman gain defined in (8). Substituting u^* into (11) and simplifying the resulting expression yield $f(z) = z^T \rho_{i_\mathcal{H}(z)}^0(P_\mathcal{H}(z))z$, where $P_\mathcal{H}(z)$ and $i_\mathcal{H}(z)$ are defined in (10). $\quad\square$

3.4   Identifying an ECLF

To check whether a function V is an ECLF, we shall verify condition (5). If V is defined by a FPD set $\mathcal{H}$, then the verification process can be greatly simplified by introducing another FPD set $\mathcal{F}_\mathcal{H}$ defined as:

$$\mathcal{F}_\mathcal{H} = \{\rho_i^0(P) : i \in \mathbb{M} \text{ and } P \in \mathcal{H}\}. \qquad (12)$$

In other words, $\mathcal{F}_\mathcal{H}$ contains all the possible matrices obtained through equation (7) as i ranges over $\mathbb{M}$ and P ranges over $\mathcal{H}$. Similar to (6), we can define

$$V_{\mathcal{F}_\mathcal{H}}(z) = \min_{P \in \mathcal{F}_\mathcal{H}} z^T P z. \qquad (13)$$

Theorem 3 *For an arbitrary FPD set $\mathcal{H}$, the function $V_\mathcal{H}$ is an ECLF of system (1) and the stationary policy $\pi_{V_\mathcal{H}} = \{\xi_{V_\mathcal{H}}, \xi_{V_\mathcal{H}}, \ldots\}$ is exponentially stabilizing if*

$$V_\mathcal{H}(z) - V_{\mathcal{F}_\mathcal{H}}(z) \geq \kappa_3 \|z\|^2, \qquad (14)$$

for all $z \in \mathbb{R}^n$ and some constant $\kappa_3 > 0$.

PROOF. Obviously, $V_\mathcal{H}$ satisfies the first condition of Definition 3. Let $z \in \mathbb{R}^n$ be arbitrary but fixed. Denote by $(\hat{P}, \hat{i})$ the minimizer in (10) for this fixed z. Suppose

that the system starts from z at time 0 and is driven by the control law $\xi_{V_\mathcal{H}}$ as defined in (9). Let $\hat{u} = -K_{\hat{i}}(\hat{P})z$ and $\hat{x}(1) = A_{\hat{i}}z + B_{\hat{i}}\hat{u}$ be the continuous control at time 0 and the state at time 1, respectively. Plugging equations (7) and (8) into $\hat{u}$, we have

$$V_\mathcal{H}(\hat{x}(1)) = \min_{P \in \mathcal{H}} \left[\hat{x}^T(1) \cdot P \cdot \hat{x}(1)\right] \leq \hat{x}^T(1) \cdot \hat{P} \cdot \hat{x}(1)$$

$$\leq \hat{x}^T(1) \cdot \hat{P} \cdot \hat{x}(1) + \hat{u}^T R_{\hat{i}} \hat{u} = z^T \rho_{\hat{i}}^0(\hat{P})z.$$

Considering (10) and (12), we have $V_\mathcal{H}(\hat{x}(1)) \leq z^T \rho_{\hat{i}}^0(\hat{P})z = V_{F_\mathcal{H}}(z)$, which implies $V_\mathcal{H}(z) - V_\mathcal{H}(\hat{x}(1)) \geq V_\mathcal{H}(z) - V_{\mathcal{F}_\mathcal{H}}(z) \geq \kappa_3 \|z\|^2$. Therefore, $V_\mathcal{H}$ also satisfies (5) and the desired result follows from Lemma 1. $\quad\square$

For a given function $V_\mathcal{H}$ of the form (6), to see whether it is an ECLF, it suffices to check condition (14). Since both $V_\mathcal{H}$ and $V_{\mathcal{F}_\mathcal{H}}$ are homogeneous, we only need to consider z on the unit sphere in $\mathbb{R}^n$ to verify (14). In $\mathbb{R}^2$, a practical way of checking (14) is to plot the functions $V_\mathcal{H}(z)$ and $V_{\mathcal{F}_\mathcal{H}}(z)$ along the unit circle to see whether $V_\mathcal{H}(z)$ is uniformly above $V_{\mathcal{F}_\mathcal{H}}(z)$. In higher dimensional state spaces, there is no general way to efficiently verify this condition. Nevertheless, a sufficient convex condition can be obtained.

Corollary 1 (Convex Test) *The function $V_\mathcal{H}$ is an ECLF of system (1) and the stationary policy $\pi_{V_\mathcal{H}}$ is exponentially stabilizing if for each $P \in \mathcal{H}$, there exist nonnegative constants α_j, $j = 1, \ldots, k$, such that*

$$\sum_{j=1}^k \alpha_j = 1, \text{ and } P \succ \sum_{j=1}^k \alpha_j \hat{P}^{(j)}, \qquad (15)$$

where $k = |\mathcal{F}_\mathcal{H}|$ and $\{\hat{P}^{(j)}\}_{j=1}^k$ is an enumeration of $\mathcal{F}_\mathcal{H}$.

PROOF. Let $\{P^{(i)}\}_{i=1}^{|\mathcal{H}|}$ be an enumeration of $\mathcal{H}$. Let $z \in \mathbb{R}^n$ be arbitrary. If $z = 0$, then (14) is trivially satisfied. Suppose that $z \neq 0$. By (15), for each $i = 1, \ldots, |\mathcal{H}|$, we have

$$z^T P^{(i)} z > \sum_{j=1}^k \alpha_j \left(z^T \hat{P}^{(j)} z\right) \geq z^T \hat{P}^{(j_i)} z$$

for some $\hat{P}^{(j_i)} \in \mathcal{F}_\mathcal{H}$. Thus,

$$V_\mathcal{H}(z) = \min_{1 \leq i \leq |\mathcal{H}|} z^T P^{(i)} z > \min_{1 \leq i \leq |\mathcal{H}|} z^T \hat{P}^{(j_i)} z \geq V_{F_\mathcal{H}}(z).$$

Since the above holds for all $z \neq 0$, we have $V_\mathcal{H}(z) - V_{F_\mathcal{H}}(z) \geq \kappa_3 \|z\|^2$, for some constant $\kappa_3 > 0$. Therefore, inequality (14) is always satisfied and the desired result follows from Theorem 3. $\quad\square$

Remark 1 *Notice that the matrices in $\mathcal{H}$ are assumed to be known (see Section 5 for the computation of the set $\mathcal{H}$), and thus the set $\mathcal{F}_{\mathcal{H}}$ can be easily obtained using (12). With known $\mathcal{H}$ and $\mathcal{F}_{\mathcal{H}}$, checking the condition in (15) is an LMI feasibility problem, which can be solved using the various convex optimization algorithms described in (Boyd and Vandenberghe [2004]).*

To summarize, in this section we have proposed a special piecewise-quadratic structure in (6) for the candidate ECLFs associated with Problem 1. The sufficient condition in Corollary 1 provides an easy way to check whether a function satisfying (6) is an ECLF of system (1). If it is, then the hybrid-control law given in (9) must be exponentially stabilizing. In the rest of this paper, we shall focus on how to efficiently find such an ECLF when the system is exponentially stabilizable.

4 A Converse ECLF Theorem Using Dynamic Programming

By focusing on the ECLF of the form (6) and the control law of the form (4), the stabilization problem can be viewed as a switched LQR problem (to be introduced below). The main purpose of this section is to prove the following converse ECLF theorem.

Theorem 4 (Converse ECLF Theorem I) *System (1) is exponentially stabilizable if and only if $V^*(z)$ is an ECLF of system (1) that satisfies condition (5), where V^* is the infinite-horizon value function of the switched LQR problem to be defined in Section 4.1.*

The proof of the above theorem will be given in Section 4.3. Before presenting the proof, we shall briefly recall the formulation of the switched LQR problem in Section 4.1 and some properties of its value functions in Section 4.2. Interested readers are referred to Zhang and Hu [2008b], Zhang et al. [2009a] for an in-depth discussion on the switched LQR problem.

4.1 The Switched LQR Problem

Let $Q_i = Q_i^T \succ 0$ and $R_i = R_i^T \succ 0$ be the weighting matrices for the state and the control, respectively, for subsystem $i \in \mathbb{M}$. Define the running cost as

$$L(x, u, v) = x^T Q_v x + u^T R_v u, \qquad (16)$$

for $x \in \mathbb{R}^n, u \in \mathbb{R}^p, v \in \mathbb{M}$. Denote by $J_\pi(z)$ the total cost, possibly infinite, starting from $x(0) = z$ under policy π, i.e.,

$$J_\pi(z) = \sum_{t=0}^{\infty} L(x(t), \mu_t(x(t)), \nu_t(x(t))). \qquad (17)$$

Define the *infinite-horizon value function* (Bertsekas [2001]) as

$$V^*(z) = \inf_{\pi \in \Pi} J_\pi(z). \qquad (18)$$

Since the running cost is always nonnegative, the infimum always exists. The function $V^*(z)$ will be infinite if $J_\pi(z)$ is infinite for all the policies $\pi \in \Pi$. As a natural extension of the classical LQR problem, the *Discrete-time Switched LQR problem* (DSLQR) is defined as follows.

Problem 2 (DSLQR problem) *For a given initial state $z \in \mathbb{R}^n$, find the infinite-horizon policy $\pi \in \Pi$ that minimizes $J_\pi(z)$ subject to equation (2).*

4.2 The Value Functions of the DSLQR Problem

Dynamic programming solves the DSLQR problem by introducing a sequence of value functions. Define the N-horizon value function $V_N : \mathbb{R}^n \to \mathbb{R}$ as:

$$V_N(z) = \inf_{\substack{u(t) \in \mathbb{R}^p, v(t) \in \mathbb{M} \\ 0 \leq t \leq N-1}} \left\{ \sum_{t=0}^{N-1} L(x(t), u(t), v(t)) \right|$$

$$\text{subject to (1) with } x(0) {=} z \right\}. \qquad (19)$$

For any function $V : \mathbb{R}^n \to \mathbb{R}^+$ and any control law $\xi = (\mu, \nu) : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{M}$, denote by $\mathcal{T}_\xi$ the operator that maps V to another function $\mathcal{T}_\xi[V]$ defined as:

$$\mathcal{T}_\xi[V](z) = L(z, \mu(z), \nu(z)) + V(A_{\nu(z)} z + B_{\nu(z)} \mu(z)), \; \forall z \in \mathbb{R}^n.$$

Similarly, define the operator $\mathcal{T}$ by

$$\mathcal{T}[V](z) {=} \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} \{L(z, u, v) {+} V(A_v z {+} B_v u)\}, \; \forall z {\in} \mathbb{R}^n.$$

The equation defined above is called the *one-stage value iteration* of the DSLQR problem. We denote by $\mathcal{T}^k$ the composition of the mapping $\mathcal{T}$ with itself k times, i.e., $\mathcal{T}^{k+1}[V](z) = \mathcal{T}\left[\mathcal{T}^k[V]\right](z)$ for all $k \in \mathbb{Z}^+$ and $z \in \mathbb{R}^n$. Some standard results of Dynamic Programming are summarized in the following lemma.

Lemma 3 (Bertsekas [2001]) *Let $V_0(z) = 0$ for all $z \in \mathbb{R}^n$. Then*

(i) $V_N(z) = \mathcal{T}^N[V_0](z)$ for all $N \in \mathbb{Z}^+$ and $z \in \mathbb{R}^n$;
(ii) $V_N(z) \to V^*(z)$ pointwise in $\mathbb{R}^n$ as $N \to \infty$.
(iii) The infinite-horizon value function satisfies the Bellman equation, i.e., $\mathcal{T}[V^*](z) = V^*(z)$ for all $z \in \mathbb{R}^n$.
(iv) If $R_v \succ 0$ for all $v \in \mathbb{M}$, then there exists a stationary optimal policy, i.e., there exists a hybrid-control law ξ^* such that $\mathcal{T}_{\xi^*}[V^*](z) = V^*(z), \forall z \in \mathbb{R}^n$.

To derive the value function of the DSLQR problem, we introduce a few definitions. Denote by $\rho_i : \mathcal{A} \to \mathcal{A}$ the *Riccati Mapping* of subsystem $i \in \mathbb{M}$, i.e.,

$$\begin{aligned}\rho_i(P) = Q_i &+ A_i^T P A_i \\&- A_i^T P B_i (R_i + B_i^T P B_i)^{-1} B_i^T P A_i.\end{aligned} \quad (20)$$

Definition 4 *Let $2^{\mathcal{A}}$ be the power set of $\mathcal{A}$. The mapping $\rho_{\mathbb{M}} : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ defined by: $\rho_{\mathbb{M}}(\mathcal{H}) = \{\rho_i(P) : i \in \mathbb{M} \text{ and } P \in \mathcal{H}\}$ is called the* Switched Riccati Mapping *associated with Problem 2.*

Definition 5 *The sequence of sets $\{\mathcal{H}_k\}_{k=0}^N$ generated iteratively by $\mathcal{H}_{k+1} = \rho_{\mathbb{M}}(\mathcal{H}_k)$ with initial condition $\mathcal{H}_0 = \{0\}$ is called the* Switched Riccati Sets *associated with Problem 2.*

The sequence of switched Riccati sets always starts from a singleton set $\{0\}$ and evolves according to the switched Riccati mapping. For any finite N, the set $\mathcal{H}_N$ consists of up to M^N p.s.d. matrices. An important fact about the DSLQR problem is that its value functions are completely characterized by the switched Riccati sets.

Theorem 5 (Zhang and Hu [2008b]) *The N-horizon value function for the DSLQR problem is given by*

$$V_N(z) = \min_{P \in \mathcal{H}_N} z^T P z. \quad (21)$$

Remark 2 *Clearly, for any finite N, the value function V_N is a piecewise quadratic function of the form (6).*

4.3 Proof of Theorem 4

We first introduce some notations. Define

$$\begin{aligned}\lambda_Q^- &= \min_{i \in \mathbb{M}}\{\lambda_{\min}(Q_i)\}, \quad \lambda_Q^+ = \max_{i \in \mathbb{M}}\{\lambda_{\max}(Q_i)\}, \\\lambda_R^- &= \min_{i \in \mathbb{M}}\{\lambda_{\min}(R_i)\}, \quad \lambda_R^+ = \max_{i \in \mathbb{M}}\{\lambda_{\max}(R_i)\}, \\\sigma_A^+ &= \max_{i \in \mathbb{M}}\left\{\sqrt{\lambda_{\max}(A_i^T A_i)}\right\}.\end{aligned}$$

Denote by $I_B^+ \subset \mathbb{M}$ the set of indices of nonzero B matrices, i.e., $I_B^+ \triangleq \{i \in \mathbb{M} : \|B_i\| \neq 0\}$. Let $\sigma_{\min}^+(\cdot)$ be the smallest *positive* singular value of a nonzero matrix. If $I_B^+ \neq \emptyset$, define $\hat{\sigma}_B = \min_{i \in I_B^+}\{\sigma_{\min}^+(B_i)\}$.

Our first task is to relate the exponential stabilizability to the boundedness of the value function V^*.

Lemma 4 *Suppose that system (1) is exponentially stabilizable. Then there exists a positive constant $\beta < \infty$ such that $\lambda_Q^- \|z\|^2 \leq V^*(z) \leq \beta\|z\|^2$, for all $z \in \mathbb{R}^n$. Fur-*

thermore, one possible choice of the bound β is given by

$$\beta = \begin{cases} \frac{a\lambda_Q^+}{1-c}, & \text{if } I_B^+ = \emptyset \\ \left(\lambda_Q^+ + \lambda_R^+ \frac{2[c+(\sigma_A^+)^2]}{\hat{\sigma}_B^2}\right) \cdot \frac{a}{1-c}, & \text{otherwise,} \end{cases} \quad (22)$$

where $a \in [1, \infty)$ and $c \in (0, 1)$ are the constants such that the closed-loop trajectory satisfies $\|x(t)\|^2 \leq ac^t\|x(0)\|^2$ for all $t \in \mathbb{Z}^+$.

PROOF. See Appendix A. □

We now prove the main result of this section, namely Theorem 4 (the converse ECLF Theorem I).

PROOF. [Theorem 4] The "if" part follows directly from Theorem 2. To show the "only if" part, suppose that system (1) is exponentially stabilizable. By Lemma 4, $V^*(z)$ satisfies the first condition of Definition 3. Furthermore, by Lemma 3, there exists a control law $\xi^* = (\mu^*, \nu^*)$ such that $V^*(z) = \mathcal{T}_{\xi^*}[V^*](z)$. This implies that

$$\begin{aligned}V^*(z) - V^*(A_{\nu^*(z)}z &+ B_{\nu^*(z)}\mu^*(z)) \\&- [\mu^*(z)]^T R_{\nu^*(z)}[\mu^*(z)] \geq \lambda_Q^- \|z\|^2.\end{aligned}$$

Let $\xi_{V^*} = (\hat{\mu}, \hat{\nu})$ be defined as in (4) with V replaced by V^*. Then we have

$$\begin{aligned}&V^*(z) - V^*\left(A_{\hat{\nu}(z)}z + B_{\hat{\nu}(z)}\hat{\mu}(z)\right) \\&\geq V^*(z) - V^*\left(A_{\hat{\nu}(z)}z + B_{\hat{\nu}(z)}\hat{\mu}(z)\right) - [\hat{\mu}(z)]^T R_{\hat{\nu}(z)}[\hat{\mu}(z)] \\&\geq V^*(z) - V^*\left(A_{\nu^*(z)}z + B_{\nu^*(z)}\mu^*(z)\right) \\&\qquad - [\mu^*(z)]^T R_{\nu^*(z)}[\mu^*(z)] \geq \lambda_Q^- \|z\|^2, \quad (23)\end{aligned}$$

where the second inequality follows from the definition of ξ_{V^*} in (4). Thus, V^* also satisfies condition (5). In summary, V^* is an ECLF satisfying (5). □

By Theorem 4, whenever system (1) is exponentially stabilizable, $V^*(z)$ can be used as an ECLF to construct an exponentially stabilizing control law ξ_{V^*}. However, from a design view point, such an existence result is not very useful as V^* can seldom be obtained exactly. In the next section, we will develop an efficient algorithm to compute an approximation of V^* which is also guaranteed to be an ECLF of system (1).

5 Efficient Computation of ECLFs

5.1 Approximation of V^* as an ECLF

Although the infinite-horizon value function V^* is in general difficult to obtain exactly, it may be approximated by some simple function that can be efficiently computed. If the approximating function is uniformly close to V^*, then it will also serve as an ECLF of system (1). By part (ii) of Lemma 3, the finite-horizon value function V_N converges pointwise to V^* as $N \to \infty$. This motivates us to use V_N to approximate V^* for large N. To guarantee that V_N will eventually become an ECLF, we need to first ensure that the convergence of V_N to V^* is uniform on a compact set, say the unit ball.

Theorem 6 (Zhang and Hu [2008a]) *If* $V^*(z) \leq \beta\|z\|^2$ *for some* $\beta < \infty$, *then*

$$|V_{N_1}(z) - V_N(z)| \leq \alpha_\beta \gamma_\beta^N \|z\|^2,$$

for any $N_1 \geq N \geq 1$, *where*

$$\gamma_\beta = \frac{1}{1 + \lambda_Q^-/\beta} < 1 \quad and \quad \alpha_\beta = \max\{1, \frac{\sigma_A^+}{\gamma_\beta}, \beta\}. \quad (24)$$

By this theorem, the N-horizon value function V_N approaches V^* exponentially fast as $N \to \infty$. Therefore, as we increase N, V_N will eventually become an ECLF of system (1).

Theorem 7 (Converse ECLF Theorem II) *If system (1) is exponentially stabilizable, then there exists a constant* $\beta < \infty$ *such that* $V_N(z)$ *is an ECLF satisfying condition (5) for all* $N \geq N_\beta$, *where*

$$N_\beta = \frac{\ln \lambda_Q^-/\alpha_\beta}{\ln \gamma_\beta} \in (0, \infty), \quad (25)$$

with γ_β *and* α_β *defined in (24).*

PROOF. Define

$$\begin{aligned} \xi_N^*(z) &= (\mu_N^*, \nu_N^*) \\ &\triangleq \underset{u \in \mathbb{R}^p, v \in \mathbb{M}}{\arg\inf} \{L(z, u, v) + V_N(A_v z + B_v u)\}. \quad (26) \end{aligned}$$

By Lemma 3 and equation (26), we know that

$$V_{N+1}(z) = \mathcal{T}[V_N](z) = \mathcal{T}_{\xi_N^*}[V_N](z), \forall z \in \mathbb{R}^n.$$

We now fix an arbitrary $z \in \mathbb{R}^n$ and let $u^* = \mu_N^*(z)$, $v^* = \nu_N^*(z)$ and $x^*(1) = A_{v^*} z + B_{v^*} u^*$. Then,

$$V_{N+1}(z) - V_N(x^*(1)) - (u^*)^T R_{v^*}(u^*) \geq \lambda_Q^- \|z\|^2. \quad (27)$$

By Lemma 4, the exponential stabilizability implies the existence of a positive constant $\beta < \infty$ such that $V^*(z) \leq \beta\|z\|^2$, $\forall z \in \mathbb{R}^n$. Let γ_β and α_β be defined in terms of β as in (24). By Theorem 6, $V_{N+1}(z) \leq V_N(z) + \alpha_\beta \gamma_\beta^N \|z\|^2$. Substituting this inequality into (27) yields

$$V_N(z) - V_N(x^*(1)) - (u^*)^T R_{v^*}(u^*) \geq (\lambda_Q^- - \alpha_\beta \gamma_\beta^N)\|z\|^2.$$

Let N_β be defined as in (25). Since $\gamma_\beta < 1$ and $\alpha_\beta \geq \beta > \lambda_Q^-$, we have $N_\beta \in (0, \infty)$. For an arbitrary $N \geq N_\beta$, define $\xi_{V_N} = (\hat\mu, \hat\nu)$ as in (4) with V replaced by V_N. Similar to (23), one can obtain

$$V_N(z) - V_N\big(A_{\hat\nu(z)} z + B_{\hat\nu(z)} \hat\mu(z)\big) \geq \kappa\|z\|^2,$$

where $\kappa \triangleq (\lambda_Q^- - \alpha_\beta \gamma_\beta^N) > 0$, since $N \geq N_\beta$. Hence, by Lemma 1, V_N is an ECLF satisfying (5) for all $N \geq N_\beta$. $\square$

Theorem 7 implies that when system (1) is exponentially stabilizable, the ECLF not only exists but can also be chosen to be a piecewise quadratic function of the form (6). Furthermore, as N increases, the N-horizon value function V_N will eventually become an ECLF.

5.2 Numerical Relaxation

By Theorem 7, to obtain an ECLF, we only need to compute the switched Riccati set $\mathcal{H}_N$ for large N. However, this method may not be computationally feasible as the size of $\mathcal{H}_N$ grows exponentially fast as N increases (See Definition 5). Fortunately, if a small numerical relaxation is allowed, an approximation of V_N can be efficiently computed (Zhang et al. [2009a]).

Definition 6 (Numerical Redundancy) *A matrix* $\hat{P} \in \mathcal{H}_N$ *is called* (numerically) ϵ-redundant *with respect to* $\mathcal{H}_N$ *if*

$$\min_{P \in \mathcal{H}_N \setminus \{\hat{P}\}} z^T P z \leq \min_{P \in \mathcal{H}_N} z^T (P + \epsilon I_n) z, \text{ for any } z \in \mathbb{R}^n.$$

Remark 3 *Numerical redundancy is closely related to the completeness concept for a certain set of matrices (Skafidas et al. [1999], Savkin and Evans [2001]). It can be verified that $\hat{P}$ is ϵ-redundant in $\mathcal{H}_N$ if and only if the set of matrices $\{P^{(j)} + \epsilon I_n - \hat{P}\}_{j=1}^{|\mathcal{H}_N|-1}$ is complete, where $\{P^{(j)}\}_{j=1}^{|\mathcal{H}_N|-1}$ is an enumeration of $\mathcal{H}_N \setminus \{\hat{P}\}$. The direct definition adopted in this paper is to emphasize its role in simplifying the computations of the ECLFs.*

Definition 6 ensures that, when ϵ is small, one can disregard the ϵ-redundant matrices in $\mathcal{H}_N$ without causing much error in defining the value function V_N. This will

Algorithm 1 $[Algo_\epsilon(\cdot)]$

Set $\mathcal{H}_k^\epsilon = \emptyset$.
for each $P \in \mathcal{H}_k$ **do**
  if P does NOT satisfies the condition in *Lemma 5* with respect to $\mathcal{H}_k^\epsilon$ **then**
    $\mathcal{H}_k^\epsilon = \mathcal{H}_k^\epsilon \cup \{P\}$;
  end if
end for
Return $\mathcal{H}_k^\epsilon$.

reduce the set $\mathcal{H}_N$ to a smaller one, which is called the ϵ-*Equivalent Subset*.

Definition 7 (ϵ-ES) *The set $\mathcal{H}_N^\epsilon$ is called an ϵ-Equivalent-Subset (ϵ-ES) of $\mathcal{H}_N$ if $\mathcal{H}_N^\epsilon \subset \mathcal{H}_N$ and for all $z \in \mathbb{R}^n$,*

$$\min_{P \in \mathcal{H}_N} z^T P z \leq \min_{P \in \mathcal{H}_N^\epsilon} z^T P z \leq \min_{P \in \mathcal{H}_N} z^T (P + \epsilon I_n) z.$$

To simplify the computation of the ECLFs, for a given tolerance ϵ, we want to prune out as many ϵ-redundant matrices as possible in $\mathcal{H}_N$. The following lemma provides a sufficient condition for testing the ϵ-redundancy of a given matrix.

Lemma 5 (Redundancy Test) *$\hat{P}$ is ϵ-redundant in $\mathcal{H}_N$ if there exist nonnegative constants $\{\alpha_i\}_{i=1}^{k-1}$ such that $\sum_{i=1}^{k-1} \alpha_i = 1$ and $\hat{P} + \epsilon I_n \succeq \sum_{i=1}^{k-1} \alpha_i P^{(i)}$, where $k = |\mathcal{H}_N|$ and $\{P^{(i)}\}_{i=1}^{k-1}$ is an enumeration of $\mathcal{H}_N \setminus \{\hat{P}\}$.*

The condition in Lemma 5 can be easily verified using various existing convex optimization algorithms (Boyd and Vandenberghe [2004]). To compute an ϵ-ES of $\mathcal{H}_N$, one needs to remove the matrices in $\mathcal{H}_N$ that satisfy the condition in Lemma 5. The detailed procedure is summarized in Algorithm 1. Denote by $Algo_\epsilon(\mathcal{H}_N)$ the ϵ-ES of $\mathcal{H}_N$ returned by the algorithm. To further reduce the complexity, one can apply the algorithm after every switched Riccati mapping, which results in the sequence of the *relaxed switched Riccati sets* $\{\mathcal{H}_k^\epsilon\}_{k=0}^N$:

$$\mathcal{H}_{k+1}^\epsilon = Algo_\epsilon(\rho_\mathbb{M}(\mathcal{H}_k^\epsilon)), \text{ for } k \leq N-1 \tag{28}$$
$$\text{with} \quad \mathcal{H}_0^\epsilon = \mathcal{H}_0.$$

The piecewise quadratic function defined based on $\mathcal{H}_N^\epsilon$ is ϵ-close to V_N but easier to compute as $\mathcal{H}_N^\epsilon$ typically contains much fewer matrices than $\mathcal{H}_N$. This is demonstrated through the following example.
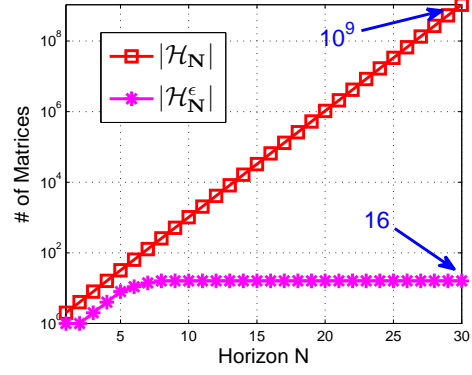

Fig. 1. Evolution of $|\mathcal{H}_N^\epsilon|$ with $\epsilon = 10^{-3}$.

Example 1

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Q_i = I_2, R_i = 1, i = 1, 2. \tag{29}$$

Clearly, neither subsystem is stabilizable. As shown in Fig. 1, a direct computation of $\{\mathcal{H}_k\}_{k=0}^N$ results in a combinatorial complexity on the order of 10^9 for $N = 30$. However, if the relaxed iteration (28) with $\epsilon = 10^{-3}$ is used, eventually $\mathcal{H}_N^\epsilon$ contains only 16 matrices. This example shows that the numerical relaxation may dramatically simplify the computation of $\mathcal{H}_N$. However, before using this relaxation, we shall assure that the relaxation error does not grow unbounded as N increases. Define $V_N^\epsilon(z) = \min_{P \in \mathcal{H}_N^\epsilon} z^T P z$. It is proved in Zhang et al. [2009a] that the total error between $V_N^\epsilon(z)$ and $V_N(z)$ can be bounded uniformly with respect to N.

Lemma 6 (Zhang et al. [2009a]) *If $V^*(z) \leq \beta \|z\|^2$ for some $\beta < \infty$, then*

$$V_N(z) \leq V_N^\epsilon(z) \leq V_N(z) + \epsilon \eta_\beta \|z\|^2, \tag{30}$$

where

$$\eta_\beta = \frac{1 + (\beta/\lambda_Q^- - 1)\gamma_\beta}{1 - \gamma_\beta}, \tag{31}$$

with γ_β defined in (24).

The above lemma indicates that by choosing ϵ small enough, V_N^ϵ can approximate V_N uniformly on the unit ball with arbitrary accuracy. This warrants V_N^ϵ as an ECLF for large N and small ϵ.

Theorem 8 (Converse ECLF Theorem III) *If system (1) is exponentially stabilizable, then there exists a*

positive constant $\beta < \infty$ such that $V_N^\epsilon(z)$ is an ECLF satisfying condition (5) for all $N \geq \tilde{N}_\beta$ and all $\epsilon \leq \epsilon_\beta$, where

$$\tilde{N}_\beta = \frac{\ln\left(\lambda_Q^-/(2\alpha_\beta)\right)}{\ln \gamma_\beta} > 0, \quad and \quad \epsilon_\beta = \frac{\lambda_Q^-}{2\eta_\beta} > 0, \quad (32)$$

with γ_β and α_β defined in (24) and η_β defined in (31).

PROOF. See Appendix B. $\square$

Algorithm 2 [Computation of ECLF]

Specify proper values for ϵ, $\epsilon_{\min}$ and $N_{\max}$ and let $\mathcal{H}_0 = \{0\}$.
while $\epsilon > \epsilon_{\min}$ **do**
   for $N = 1$ to $N_{\max}$ **do**
     $\mathcal{H}_N = Algo_\epsilon(\rho_\mathbb{M}(\mathcal{H}_{N-1}))$
     if $\mathcal{H}_N^\epsilon$ satisfies the condition of Corollary 1
     then
       stop and return the set $\mathcal{H}_N^\epsilon$ which characterizes the ECLF V_N^ϵ.
     end if
   end for
   $\epsilon = \epsilon/2$
end while

5.3   Overall Algorithm

In summary, if system (1) is exponentially stabilizable, an ECLF of the form (6) defined by $\mathcal{H}_N^\epsilon$ can always be found for sufficiently large N and sufficiently small ϵ. To compute such an ECLF, we start from a reasonable guess of ϵ and perform the relaxed switched Riccati iteration (28). After each iteration, we check whether the condition of Corollary 1 is met. If so, an ECLF is found; otherwise we should continue with iteration (28). Since the system may not be exponentially stabilizable, an upper limit for the iteration number N_{max} should be imposed. When the maximum iteration number $N_{\max}$ is reached, we should reduce ϵ and restart iteration (28) from $N = 0$. This process is repeated until an ECLF is found or ϵ reaches a predefined lower limit $\epsilon_{\min}$. The above procedure of constructing an ECLF is summarized in Algorithm 2. This algorithm guarantees to yield an ECLF and hence a stabilizing policy, provided that the system (1) is exponentially stabilizable and that $\epsilon_{\min}$ is sufficiently small and $N_{\max}$ is sufficiently large.

6   The Stationary Stabilizing control Policy

In this section, we study some properties of the stabilizing policy associated with the ECLF V_N^ϵ. Define ξ_N^ϵ according to (4) with V replaced by V_N^ϵ. By Lemma 2,

ξ_N^ϵ is uniquely characterized by the set $\mathcal{H}_N^\epsilon$ through the following equation:

$$\xi_N^\epsilon(z) \triangleq (\mu_N^\epsilon(z), \nu_N^\epsilon) = \left(-K_{i_N^\epsilon(z)}\left(P_N^\epsilon(z)\right)z, i_N^\epsilon(z)\right),$$
$$\text{with} \quad (P_N^\epsilon(z), i_N^\epsilon(z)) = \underset{P \in \mathcal{H}_N^\epsilon, i \in \mathbb{M}}{\arg\min} z^T \rho_i^0(P)z, \quad (33)$$

where $K_\cdot(\cdot)$ is the Kalman gain defined in (8).

By Lemma 1 and Theorem 8, if system (1) is exponentially stabilizable, then it must be stabilizable by the stationary policy $\pi_N^\epsilon \triangleq \{\xi_N^\epsilon, \xi_N^\epsilon, \ldots\}$ for sufficiently large N and sufficiently small ϵ. In particular, if $\mathcal{H}_N^\epsilon$ and V_N^ϵ are the FPD set and the corresponding ECLF returned by Algorithm 2, then π_N^ϵ must be exponentially stabilizing. Therefore, Algorithm 2 can be used to compute both the ECLF and the corresponding stabilizing policy.

6.1   Properties of ξ_N^ϵ

Since both $\mathcal{H}_N^\epsilon$ and $\mathbb{M}$ contain finitely many elements, the minimizer $(P_N^\epsilon(z), i_N^\epsilon(z))$ in (33) must be piecewise constant. For each pair $(P, i) \in \mathcal{H}_N^\epsilon \times \mathbb{M}$, define a subset of $\mathbb{R}^n$ as:

$$\Omega_N^\epsilon(P, i) = \left\{z \in \mathbb{R}^n : (P, i) = \underset{\hat{P} \in \mathcal{H}_N^\epsilon, \hat{i} \in \mathbb{M}}{\arg\min} z^T \rho_{\hat{i}}^0(\hat{P})z\right\}. \quad (34)$$

The set $\Omega_N^\epsilon(P, i)$ such defined is called a *decision region* associated with ξ_N^ϵ in the sense that the points within the same decision region correspond to the same pair of feedback gain $K_i(P)$ and switching control i under the control law ξ_N^ϵ. According to (34), a decision region must be homogeneous. This implies that the control law ξ_N^ϵ is also homogeneous. Furthermore, it follows immediately
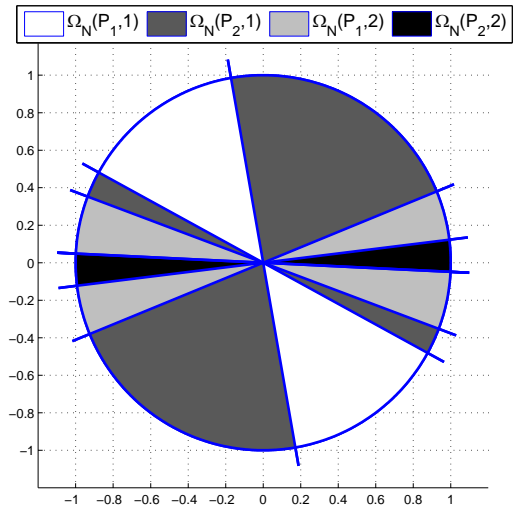

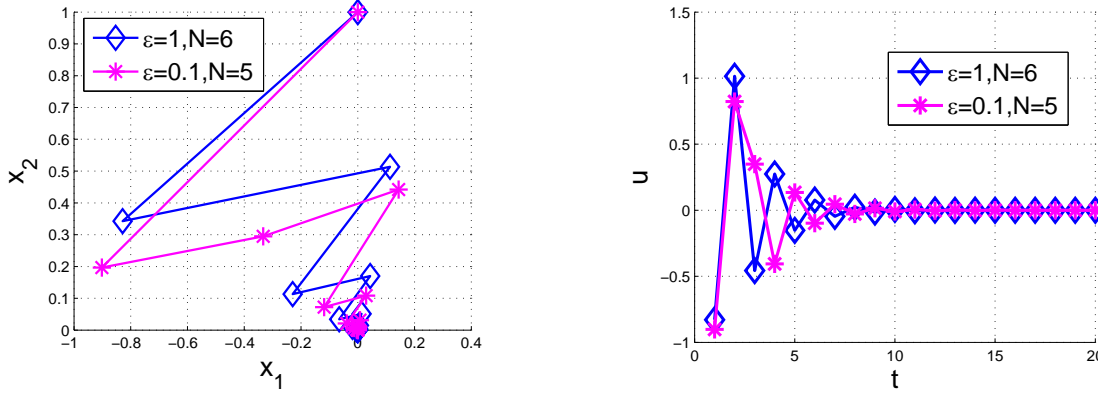Fig. 2. Typical Decision regions

10

Fig. 3. Simulation results for Example 1. Left figure: phase-plane trajectories generated by the ECLFs V_6^1 and $V_5^{0.1}$ starting from the same initial condition $x_0 = [0, 1]^T$. Right figure: the corresponding continuous controls.

from (33) that the continuous-control law μ_N^ϵ is piecewise linear with a constant feedback gain within each decision region. Note that a decision region $\Omega_N^\epsilon(P, i)$ may be disconnected except at the origin and the union of all the decision regions covers the entire space $\mathbb{R}^n$. For example, if $\mathbb{M} = \{1, 2\}$ and $\mathcal{H}_N$ contains two matrices P_1 and P_2, then there will be four homogeneous decision regions as shown in Fig. 2.

The decision regions that have the same switching control constitute a *switching region*. For each $i \in \mathbb{M}$, the switching region $S_N^\epsilon(i)$ is defined as:

$$S_N^\epsilon(i) = \cup_{P \in \mathcal{H}_N^\epsilon} \Omega_N^\epsilon(P, i). \tag{35}$$

The states that reside in the same switching region evolve through the same subsystem; however, they may be controlled by different feedback gains.

In summary, the control law ξ_N^ϵ divides the state space into at most $M \cdot |\mathcal{H}_N^\epsilon|$ homogeneous decision regions, each of which corresponds to a pair of feedback gain and switching control. These decision regions are exactly characterized by the matrices in the relaxed switched Riccati set $\mathcal{H}_N^\epsilon$. For a given state value z, by comparing the values of $z^T \rho_i^0(P) z$ for each pair of $(P, i) \in \mathcal{H}_N^\epsilon \times \mathbb{M}$, one can easily determine which decision region the state z belongs to. At time t, if $x(t) \in \Omega_N^\epsilon(P, i)$, then the hybrid control action at this time step is: $u(t) = -K_i(P)x(t)$ and $v(t) = i$. Therefore, after obtaining the set $\mathcal{H}_N^\epsilon$ from Algorithm 2, the hybrid control sequence and the closed-loop trajectory starting from any initial state can be easily computed.

6.2 *Relationships with Other Controllers*

Many hybrid-control laws proposed in the literature (Skafidas et al. [1999], Pettersson [2003], Lin and

Antsaklis [2008]) can be written in the following form:

$$\tilde{\xi}(z) = (\tilde{\mu}(z), \tilde{\nu}(z)) = (F_{\tilde{i}(z)}z, \tilde{i}(z))$$
$$\text{with} \quad \tilde{i}(z) = \arg\min_{i \in \mathbb{M}} z^T Q_i z, \tag{36}$$

where $\{F_i\}_{i \in \mathbb{M}}$ are the feedback gains and $\{Q_i\}_{i \in \mathbb{M}}$ are some symmetric matrices characterizing the decision regions. The control law $\tilde{\xi}(z)$ is exponentially stabilizing if $\{F_i\}_{i \in \mathbb{M}}$ and $\{Q_i\}_{i \in \mathbb{M}}$ satisfy certain matrix inequalities. However, these matrix inequalities are only sufficient conditions for the exponential stabilizability. There may not be a stabilizing control law necessarily of the form (36) even when the switched linear system is exponentially stabilizable.

By a similar argument as in the last subsection, it can be easily verified that (i) $\tilde{\xi}$ divides the state space into at most M homogeneous decision regions; (ii) each switching control is associated with only one feedback gain. Compared with $\tilde{\xi}$, the proposed control law ξ_N^ϵ is more general. The number of decision regions of ξ_N^ϵ may be more than M and the same switching control may be paired with more than one feedback gains.

7 Numerical Examples

7.1 *Example 1 Revisited*

Consider the same two-mode switched system as defined in (29). Neither of the subsystems is stabilizable by itself. However, the switched system is stabilizable through a proper hybrid control. The stabilization problem can be easily solved using Algorithm 2. Starting from $\epsilon = 1$, the algorithm terminates after 5 steps which results in an ECLF V_6^1 defined by the relaxed switched Riccati set $\mathcal{H}_6^1$. Using a smaller relaxation $\epsilon = 0.1$, the algorithm stops after 4 steps resulting in an ECLF $V_5^{0.1}$ defined by the relaxed switched Riccati set $\mathcal{H}_5^{0.1}$. It is worth mentioning
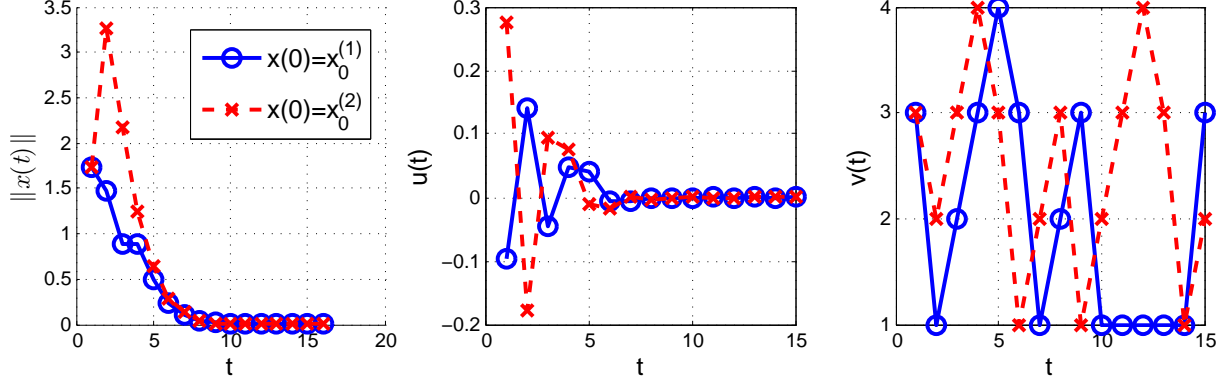
Fig. 4. Simulation results for Example 2 with two different initial conditions: $x_0^{(1)} = [1, 1, 0, -1]^T$ and $x_0^{(2)} = [1, 0, -1, 1]^T$. Left figure: norms of the closed-loop trajectories associated with the two initial conditions; middle figure: the corresponding continuous control sequences; right figure: the corresponding mode sequences.

that $\mathcal{H}_6^1$ contains only two matrices and $\mathcal{H}_5^{0.1}$ contains 3 matrices. With these matrices, starting from any initial position x_0, the control laws corresponding to $\mathcal{H}_6^1$ and $\mathcal{H}_5^{0.1}$ can be computed using equation (33). The closed-loop trajectories generated by these two control laws starting from the same initial position $x_0 = [0, 1]^T$ are plotted on the left of Fig. 3. On the right of the same figure, the continuous control signals corresponding to the two trajectories are plotted. In both cases, the switching signals jump to the other mode at every time step and are not shown in the figure. It can be seen that the ECLF $V_5^{0.1}$ stabilizes the system with a faster convergence speed and a smaller control energy than V_6^1. This is because a smaller relaxation ϵ makes the resulting trajectory closer to the optimal trajectory of the DSLQR problem.

7.2 Example 2

We now consider a multi-dimensional example with four subsystems:

$$A_1 = \begin{bmatrix} \frac{1}{2} & -1 & 2 & 3 \\ 0 & \frac{-1}{2} & 2 & 4 \\ 0 & -1 & \frac{5}{2} & 2 \\ 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{-1}{2} & -1 & 2 & 1 \\ 0 & \frac{3}{2} & -2 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -2 & -1 & 2 & \frac{5}{2} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{-1}{2} \\ 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad A_4 = \begin{bmatrix} \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -2 & \frac{1}{2} \end{bmatrix},$$

$B_1 = B_3 = [1, 2, 3, 4]^T$, $B_2 = B_4 = [4, 3, 2, 1]^T$, $Q_i = I_4$, and $R_i = 1$, for $i = 1, \ldots, 4$. It can be verified that none of the subsystems is stabilizable. Algorithm 2 is used

to solve the stabilization problem with $\epsilon = 1$. The algorithm terminates after 6 steps, resulting in an ECLF V_7^1 defined by the relaxed switched Riccati set $\mathcal{H}_7^1$ which consists of 13 matrices. Compared with the previous two examples, this example requires more matrices to characterize the stabilizing policy due to the increase in the state dimension and the number of subsystems. To test the controller performance, the hybrid-control sequences are computed using $\mathcal{H}_7^1$ based on (33) for two different initial conditions: $x(0) = x_0^{(1)} = [1, 1, 0, -1]^T$ and $x(0) = x_0^{(2)} = [1, 0, -1, 1]^T$. The hybrid-control sequences and the norms of the closed-loop trajectories are plotted in Fig. 4. It can be seen that, for both initial conditions, the system utilizes multiple modes to maintain the stability of the switched system.

8 Conclusion

This paper studies the exponential stabilization problem for discrete-time switched linear systems. It has been proved that if the system is exponentially stabilizable, then there must exist a piecewise quadratic ECLF. More importantly, this ECLF can be chosen to be a finite-horizon value function of a related switched LQR problem. An efficient algorithm has been developed to compute such an ECLF and the corresponding stabilizing policy whenever the system is exponentially stabilizable. As observed in some numerical examples, the ECLF and the stabilizing policy can usually be characterized by only a few p.s.d. matrices, which can be easily computed using the relaxed switched Riccati mapping.

It is worth pointing out that the proposed algorithm can be used for any state dimension; however, it may have difficulty in obtaining the solution within a reasonable amount of time when n is extremely large. Future research will focus on improving the performance of the algorithm in high-dimensional state spaces and on extending the algorithm to solve the robust stabilization problem for uncertain switched linear systems.

A Proof of Lemma 4

The main challenge to prove this lemma is that the stabilizing policy may employ a continuous control sequence $u(t)$ whose norm does not converge to zero exponentially fast. Our strategy is to project out the component of each $u(t)$ that lies in the null space of $B_{v(t)}$ and show that the norm of its orthogonal part converges to zero exponentially fast. To this end, the following lemma is needed.

Lemma 7 *Let $B \in \mathbb{R}^{n \times p}$ be arbitrary but $B \neq 0$. Then for any $u \in \mathbb{R}^p$ in the column space of B^T, i.e., $u \in col(B^T)$, we must have $\|u\| \leq \|Bu\|/\sigma_{\min}^+(B)$.*

Lemma 7 can be easily proved using the singular value decomposition. See Zhang et al. [2009b] for more details. With this result, we are ready to prove Lemma 4.

PROOF. [Lemma 4] Let $z \in \mathbb{R}^n$ be arbitrary and fixed. Obviously, $V^*(z)$ can be no smaller than the one-step state cost, which implies $V^*(z) \geq \lambda_Q^- \|z\|^2$. To prove that $V^*(z) \leq \beta\|z\|^2$, let $\pi = \{(\mu_t, \nu_t)\}_{t=0}^{\infty}$ be an exponentially stabilizing policy. By Definition 2, the closed-loop trajectory $x(t)$ with initial condition $x(0) = z$ satisfies $\|x(t)\|^2 \leq ac^t\|z\|^2$, for some $a \in [1, \infty)$ and $c \in (0,1)$. Thus, $\sum_{t=0}^{\infty} \|x(t)\|^2 \leq \frac{a}{1-c}\|z\|^2$. Denote by $(u(t), v(t))$ the hybrid-control sequence generated by π, i.e., $u(t) = \mu_t(x(t))$ and $v(t) = \nu_t(x(t))$. If $I_B^+ = \emptyset$, then $u(t)$ can be chosen to be zero for each $t \geq 0$. Thus,

$$V^*(z) = \sum_{t=0}^{\infty} x^T(t) Q_{v(t)} x(t) \leq \frac{a\lambda_Q^+}{1-c}\|z\|^2,$$

which is the desired result with $\beta = \frac{a\lambda_Q^+}{1-c}$. We now suppose that $I_B^+ \neq \emptyset$, which implies that $\hat{\sigma}_B > 0$. Define a new control sequence

$$\tilde{u}(t) = \begin{cases} 0, & \text{if } B_{v(t)} = 0, \\ [u(t)]_{B_{v(t)}^T}, & \text{otherwise,} \end{cases}$$

where $[\cdot]_{B_{v(t)}^T}$ denotes the projection of a given vector onto the column space of $B_{v(t)}^T$. Then $u(t) - \tilde{u}(t)$ is in the null space of $B_{v(t)}$, implying that $B_{v(t)}\tilde{u}(t) = B_{v(t)}u(t)$. As a result, under the new hybrid control sequence $(\tilde{u}(t), v(t))$, the closed-loop trajectory is still $x(t)$. Since $(\tilde{u}(t), v(t))$ is just one choice of the hybrid control sequence, we have

$$V^*(z) \leq \sum_{t=0}^{\infty} L(x(t), \tilde{u}(t), v(t))$$
$$\leq \lambda_Q^+ \frac{a}{1-c}\|z\|^2 + \lambda_R^+ \sum_{t=0}^{\infty} \|\tilde{u}(t)\|^2. \quad \text{(A.1)}$$

Furthermore, by Lemma 7, we have

$$\sum_{t=0}^{\infty} \|\tilde{u}(t)\|^2 \leq \frac{1}{\hat{\sigma}_B^2} \sum_{t=0}^{\infty} \|B_{v(t)}\tilde{u}(t)\|^2$$
$$= \frac{1}{\hat{\sigma}_B^2} \sum_{t=0}^{\infty} \|B_{v(t)}u(t)\|^2$$
$$\leq \frac{1}{\hat{\sigma}_B^2} \sum_{t=0}^{\infty} \|x(t+1) - A_{v(t)}x(t)\|^2$$
$$\leq \frac{2}{\hat{\sigma}_B^2} \left[\frac{ac}{1-c} + (\sigma_A^+)^2 \frac{a}{1-c} \right] \|z\|^2$$
$$\leq \frac{2a[c + (\sigma_A^+)^2]}{\hat{\sigma}_B^2(1-c)} \|z\|^2.$$

This inequality together with (A.1) yields the desired result. $\square$

B Proof of Theorem 8

PROOF. Fix an arbitrary $z \in \mathbb{R}^n$. Define

$$\xi_N^\epsilon(z) = (\mu_N^\epsilon, \nu_N^\epsilon)$$
$$\triangleq \underset{u \in \mathbb{R}^p, v \in \mathbb{M}}{\arg\inf} \{L(z, u, v) + V_N^\epsilon(A_v z + B_v u)\}. \quad \text{(B.1)}$$

Let $\tilde{V}_{N+1}^\epsilon(z) = \mathcal{T}_{\xi_N^\epsilon}[V_N^\epsilon](z)$, i.e.,

$$\tilde{V}_{N+1}^\epsilon(z) = \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} \{L(z, u, v) + V_N^\epsilon(A_v z + B_v u)\}$$
$$= \min_{v \in \mathbb{M}, P \in \mathcal{H}_N^\epsilon} \{z^T \rho_v(P) z\} = \min_{P \in \rho_{\mathbb{M}}(\mathcal{H}_N^\epsilon)} z^T P z. \quad \text{(B.2)}$$

By (28), we know that $\mathcal{H}_{N+1}^\epsilon = Algo_\epsilon(\rho_{\mathbb{M}}(\mathcal{H}_N^\epsilon))$. Then it follows directly from Definition 7 that

$$\tilde{V}_{N+1}^\epsilon(z) = \min_{P \in \rho_{\mathbb{M}}(\mathcal{H}_N^\epsilon)} z^T P z$$
$$\leq \min_{P \in \mathcal{H}_{N+1}^\epsilon} z^T P z = V_{N+1}^\epsilon(z). \quad \text{(B.3)}$$

Let $u^\epsilon = \mu_N^\epsilon(z)$, $v^\epsilon = \nu_N^\epsilon(z)$ and $x^\epsilon(1) = A_{v^\epsilon} z + B_{v^\epsilon} u^\epsilon$. According to (B.2), we have

$$\tilde{V}_{N+1}^\epsilon(z) - V_N^\epsilon(x^\epsilon(1)) - (u^\epsilon)^T R_{v^\epsilon}(u^\epsilon) \geq \lambda_Q^- \|z\|^2. \quad \text{(B.4)}$$

By the exponential stabilizability, there exists a constant $\beta < \infty$ such that $V^*(z) \leq \beta\|z\|^2$, $\forall z \in \mathbb{R}^n$. Then by equation (B.3), Lemma 6 and Theorem 6, we have

$$\tilde{V}_{N+1}^\epsilon(z) \leq V_{N+1}^\epsilon(z) \leq V_{N+1}(z) + \epsilon\eta_\beta\|z\|^2$$
$$\leq V_N(z) + (\alpha_\beta \gamma_\beta^N + \epsilon\eta_\beta)\|z\|^2$$
$$\leq V_N^\epsilon(z) + (\alpha_\beta \gamma_\beta^N + \epsilon\eta_\beta)\|z\|^2.$$

Combining this with inequality (B.4) yields

$$\begin{aligned} &V_N^\epsilon(z) - V_N^\epsilon(x^\epsilon(1)) - (u^\epsilon)^T R_{v^\epsilon}(u^\epsilon) \\ &\geq \tilde{V}_{N+1}^\epsilon(z) - V_N^\epsilon(x^\epsilon(1)) \\ &\qquad - (u^\epsilon)^T R_{v^\epsilon}(u^\epsilon) - (\alpha_\beta \gamma_\beta^N + \epsilon \eta_\beta)\|z\|^2 \\ &\geq (\lambda_Q^- - \alpha_\beta \gamma_\beta^N - \epsilon \eta_\beta)\|z\|^2. \end{aligned}$$

Let $\tilde{N}_\beta$ and ϵ_β be defined as in (32). It can be easily seen that $\lambda_Q^- - \alpha_\beta \gamma_\beta^N - \epsilon \eta_\beta > 0$ for all $N \geq \tilde{N}_\beta$ and $\epsilon \leq \epsilon_\beta$. Then, by a similar argument as in the proof of Theorem 7, we can conclude that V_N^ϵ is an ECLF satisfying (5) for all $N \geq \tilde{N}_\beta$ and $\epsilon \leq \epsilon_\beta$. $\quad \square$

References

D. P. Bertsekas. *Dynamic Programming and Optimal Control*, volume 2. Athena Scientific, 2nd edition, 2001.

S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, USA, 2004. ISBN 0521833787.

M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, 1998.

J. Daafouz, P. Riedinger, and C. Iung. Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. *IEEE Transactions on Automatic Control*, 47(11):1883–1887, Nov. 2002.

W. P. Dayawansa and C. F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Transactions on Automatic Control*, 44(4):751–760, Apr. 1999.

R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of IEEE, Special Issue on Hybrid Systems*, 88(7):1069–1082, 2000.

J. C. Geromel and P. Colaneri. Stability and stabilization of discrete time switched systems. *International Journal of Control*, 79(7):719–728, 2006.

T. Hu and F. Blanchini. Polyhedral functions, composite quadratic functions, and equivalent conditions for stability/stabilization. In *IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 2008.

T. Hu, L. Ma, and Z. Lin. Stabilization of switched systems via composite quadratic functions. *IEEE Transactions on Automatic Control*, 53(11):2571–2585, 2008.

I. N. Kar. Quadratic stabilization of a collection of linear systems. *International Journal of Control*, 33(2):153–160, Feb. 2002.

H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.

D. Liberzon. *Switching in Systems and Control*. Systems & Control: Foundations & Applications. Birkhauser, Boston, 2003.

D. Liberzon and A. S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, Oct. 1999.

H. Lin and P. J. Antsaklis. Hybrid state feedback stabilization with L_2 performance for discrete-time switched linear systems. *International Journal of Control*, 81(7):1114–1124, Jul. 2008.

A. P. Molchanov and Y. S. Pyatnitskiy. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *System and Control Letter*, 13(1):59–64, Jul. 1989.

S. Pettersson. Controller design of switched linear systems. In *Proceedings of the American Control Conference*, pages 3869–3874, Boston, MA, Jun. 2004.

S. Pettersson. Synthesis of switched linear systems. In *IEEE Conference on Decision and Control*, pages 5283–5288, Maui, HI, Dec. 2003.

S. Pettersson and B. Lennartson. Stabilization of hybrid systems using a min-projection strategy. In *Proceedings of the American Control Conference*, pages 223–228, Arlington, VA, Jun. 2001.

A. V. Savkin and R. J. Evans. *Hybrid Dynamical Systems: Controller and Sensor Switching Problems*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2001.

R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49(4):545–592, Nov. 2007.

E. Skafidas, R.J. Evans, A.V. Savkin, and I.R. Petersen. Stability results for switched controller systems. *Automatica*, 35(12):553–564, Apr. 1999.

Z. Sun and S. S. Ge. *Switched Linear Systems: Control and Design*. Springer, London, 2005.

M. A. Wicks, P. Peleties, and R. A. DeCarlo. Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems. *European Journal Control*, 4(2):140–147, Apr. 1998.

W. Zhang and J. Hu. On the value functions of the optimal quadratic regulation problem for discrete-time switched linear systems. In *IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 2008a.

W. Zhang and J. Hu. On Optimal Quadratic Regulation for Discrete-Time Switched Linear Systems. In *International Workshop on Hybrid Systems: Computation and Control*, pages 584–597, St Louis,MO,USA, Apr. 2008b.

W. Zhang, A. Abate, and J. Hu. Efficient suboptimal solutions of switched LQR problems. In *Proceedings of the American Control Conference*, St. Louis, MO, Jun. 2009a.

W. Zhang, J. Hu, and A. Abate. Stabilization of switched linear systems with unstabilizable subsystems. Technical report, Purdue University, 2009b. TR ECE 09-01.