Adaptive Unknown Input and State Observers

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***Abstract*— Adaptive schemes for unknown input and state estimation are proposed for a class of uncertain systems with bounded unknown inputs. First, using a Lyapunov approach, conditions are derived that ensure the state and unknown input estimation errors converge to zero for a constant unknown input. Next, combining a Lyapunov approach and linear matrix inequalities, conditions are given that guarantee a prescribed performance level for state and unknown input estimation for a bounded not necessarily constant unknown input.**

I. INTRODUCTION

A. Motivation

The design of the unknown input observers (UIOs) are categorized into two classes: (i) UIOs for the state estimation in the presence of unknown inputs, and (ii) UIOs for the state and unknown input estimation. In this paper, we propose novel adaptive observer architectures for simultaneous unknown input and state estimation.

B. Literature Overview

The unknown input estimation plays an important role in many applications. In [1], a fault detection scheme for detecting faults in hydraulic valves is proposed using an unknown input observer. Early fault detection in hydraulic systems may prevent damage caused by the faulty valve. Sliding mode and higher-order sliding mode observers for unknown input reconstruction are used for fault detection in [2]–[4] and for stress estimation in humans in [5]. In [6], unknown input observer is used to recover hidden messages in the transmitted signal. In [7], state and unknown input estimation is considered for a class of uncertain systems with time varying unknown input.

A Lyapunov-type conditions were developed for the existence of an estimator that can estimate the state and the unknown input to any degree of accuracy in [8]. These conditions are also sufficient for the existence of a sliding mode unknown input observer that asymptotically estimate the state and the unknown input. In [9], high-gain approximate differentiator based sliding mode observer architecture has been proposed for linear systems with unknown inputs that do not satisfy the so-called observer matching condition. The estimation error is proved to be uniformly ultimate bounded. Different designs of the unknown input observer for linear systems were proposed in [10]–[13]. A reduced-order observer for linear systems with unknown inputs was presented in [10] using coordinate transformation matrix where the states and the unknown inputs were estimated. In [11], the state estimation was achieved using reduced-order observer with pole placement capability. A full-order state observer for linear systems with unknown input was proposed in [12]. In [13], a distributed decoupled observer was presented using an equivalent "free of unknown input" system to simplify the design procedure.

The design of observers for a class of nonlinear systems in the presence of bounded disturbance inputs has been proposed in [14], [15]. In [14], linear matrix inequalities are given for the design of state and unknown input observer that guarantees the state estimation error to satisfy a prescribed degree of accuracy using the $\mathscr{L}_\infty$-stability concept. Finally, in [15] a robust state and unknown input estimation scheme is proposed using a sliding mode observer scheme.

An adaptive unknown input observer has been proposed recently in [16]–[18]. The adaptive unknown input observer proposed in [16] uses multiple model observers. It is a modified form of the standard UIO where a bank of parallel observers are constructed to generate residual signals. The estimation error is used as a residual signal to detect and isolate actuator faults when they occur such as in locked actuators or loss of actuator effectiveness. To apply this scheme, n independent measurements should be available for the n-th order system which limits the applicability of the approach. In [17], unknown input observer is used to estimate the torque in the vehicle engine. An adaptive law is used to allow the unknown input to converge to a compact set. In [18], the state estimation is achieved using a robust adaptive UIO for secure communication.

In our paper, a bounded adaptive unknown input estimators are proposed to estimate the unknown input.

C. Paper Contributions and Organization

The contributions of the paper are:

- An adaptive scheme for unknown input and state estimation for a class of uncertain systems with bounded unknown input is presented. Constant and bounded not necessarily constant unknown inputs are analyzed.
- Lyapunov-based conditions for the adaptive state and unknown input estimation for the constant input case are given in terms of linear matrix inequalities (LMIs) which we solve using the CVX program [19], [20].
- We prove that a bounded adaptive estimator asymptotically estimates the constant unknown input.
- We give LMIs conditions for the state and bounded unknown input estimation with guaranteed performance using $\mathscr{L}_\infty$-stability approach presented in [14], where a linear-in-state-error estimator is used.

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The paper is organized as follows. The problem statement and the dynamics of the uncertain system are given in Section II. In Section III, our proposed adaptive observer and the conditions for estimating a constant unknown input are derived, and the error dynamics are formulated. In Section IV, linear state and unknown inputs error dynamics are constructed, LMIs conditions for the state and bounded unknown input estimation are obtained, and the $\mathscr{L}_\infty$-stability concept is utilized to give the estimation performance. Finally, a two-loop autopilot example illustrating our results is presented in Section V.

II. PROBLEM STATEMENT

We consider a class of dynamical systems modeled by

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) \tag{1a}$$
$$y(t) = Cx(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u_1(t) \in \mathbb{R}^{m_1}$ is the control input, $u_2(t) \in \mathbb{R}^{m_2}$ is the unknown input, and $y(t) \in \mathbb{R}^p$ is the measured output. The system matrices are $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, and $C \in \mathbb{R}^{p\times n}$.

Our objective is to design adaptive state and unknown input estimation schemes for the dynamical system with constant and bounded not necessarily constant unknown inputs using available input-output information.

III. OBSERVER DESIGN

In this section, we propose a scheme for adaptive state estimation in the presence of constant unknown input. The proposed method also allows for reconstruction of the constant unknown input.

A. Proposed Observer Architecture

The proposed observer for system model (1) is given by

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) + B_1 u_1 + B_2 \hat{u}_2 \tag{2a}$$
$$\hat{y} = C\hat{x}, \tag{2b}$$

where $\hat{x}(t)$ is the state $x(t)$ estimate, and $\hat{u}_2$ is an adaptive estimator of u_2. The observer gain matrices L and F are obtained from the following conditions

$$(A - LC)^\top P + P(A - LC) \prec 0, \tag{3a}$$
$$B_2^\top P = FC, \tag{3b}$$
$$P = P^\top \succ 0, \tag{3c}$$

where the matrix F will be defined later. For system theoretical interpretation of conditions (3a), and (3b), we refer to [21], [22]. The adaptive estimator of the unknown input has the form

$$\dot{\hat{u}}_2 = \Gamma\sigma, \tag{4}$$

where $\Gamma = \operatorname{diag}\{\Gamma_1, \cdots, \Gamma_{m_2}\}, \Gamma_i > 0$, for $i = 1, 2, \cdots, m_2$, and $\sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_{m_2} \end{bmatrix}^\top = F(y - \hat{y})$.

Remark 1: Conditions (3a) and (3b) have been proven by Corless and Tu [7] and Edwards and Spurgeon [8] to be equivalent to the following two conditions:

Condition 1: $\operatorname{rank}(CB_2) = \operatorname{rank}(B_2)$.

Condition 2: For every complex number λ with nonnegative real part,

$$\operatorname{rank}\begin{bmatrix} A - \lambda I & B_2 \\ C & O \end{bmatrix} = n + \operatorname{rank}(B_2).$$

The existence of observers for continuous-time systems where the system has two types of inputs and outputs (measured and unmeasured) has been investigated by Hautus in [23], in which the concepts of strong and strong* detectability have been introduced. Hautus showed that the strong observability implies the strong detectability and that the existence of the state observer is equivalent to the strong* detectability. Hautus gave the conditions for the existence of a strong observer to estimate unknown input using only measured output. The existence conditions for our proposed adaptive unknown input and state observers are the same as Hautus' conditions for strong* detectability. Conditions 1 and 2 are necessary and sufficient for the existence of the strong observer of Hautus [23].

B. Error dynamics

Let the state estimation error be $e = x - \hat{x}$. Then, the observation error dynamics have the form

$$\dot{e} = (A - LC)e + B_2(u_2 - \hat{u}_2). \tag{5}$$

We now give conditions for an adaptive state and unknown input reconstruction in the presence of constant unknown input.

Theorem 1: Suppose u_2 in the plant model given by (1) is constant and B_2 is a full column rank matrix. If there exist a symmetric matrix $P \succ 0$ and matrices L and F such that the conditions in (3) are satisfied, then the state observation error e converges to zero.

Proof: Consider the ideal error system dynamics

$$\dot{e} = (A - LC)e. \tag{6}$$

By (3a), $V = e^\top Pe$ is a Lyapunov function of (6). Let $Q = -((A - LC)^\top P + P(A - LC))$. Note that by (3a), $Q = Q^\top \succ 0$. Then, $\dot{V} = -e^\top Qe < 0$. We proceed by evaluating the derivative of V on the trajectories of (5) to obtain

$$\dot{V} = -e^\top Qe + 2e^\top PB_2(u_2 - \hat{u}_2).$$

Let the augmented Lyapunov function candidate be

$$V_a = V + (u_2 - \hat{u}_2)^\top \Gamma^{-1}(u_2 - \hat{u}_2) > 0$$

in the augmented space $(e, u_2 - \hat{u}_2)$. Evaluating the time derivative of V_a on the trajectories of (5) gives

$$\dot{V}_a = \dot{V} + \frac{d}{dt}((u_2 - \hat{u}_2)^\top \Gamma^{-1}(u_2 - \hat{u}_2)). \tag{7}$$

Let $\Delta u_2 = [\Delta u_{2_1} \quad \Delta u_{2_2} \quad \cdots \quad \Delta u_{2_{m_2}}]^\top = u_2 - \hat{u}_2$, and $\sigma = [\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_{m_2}]^\top = B_2^\top Pe$. Then,

$$\dot{V} = -e^\top Qe + 2\sigma^\top \Delta u_2 = -e^\top Qe + 2\sum_{i=1}^{m_2} \sigma_i \Delta u_{2_i}. \tag{8}$$

Taking into account the assumption that u_2 is constant, the second part of (7) becomes

$$\frac{d}{dt}(\Delta u_2^\top \Gamma^{-1} \Delta u_2) = -2\Delta u_2^\top \Gamma^{-1} \dot{\hat{u}}_2 = -2\sum_{i=1}^{m_2} \frac{1}{\Gamma_i} \Delta u_{2_i} \dot{\hat{u}}_{2_i}. \quad (9)$$

Combining (8) and (9) gives

$$\dot{V}_a = -e^\top Q e + 2\sum_{i=1}^{m_2} \Delta u_{2_i} (\sigma_i - \frac{1}{\Gamma_i} \dot{\hat{u}}_{2_i}). \quad (10)$$

Note that if $\dot{\hat{u}}_{2_i} = \Gamma_i \sigma_i$, then

$$\dot{V}_a = -e^\top Q e \le 0 \quad (11)$$

in the $(e, \Delta u_2)^\top$ space, which implies that e and Δu_2 are bounded. We now use the Lyapunov-like lemma, see, for example, [24], [25]. For this, we need to show that $\dot{V}_a(e(t), \Delta u_2(t))$ is uniformly continuous in time. Taking the second time derivative of V_a gives $\ddot{V}_a = -2e^\top Q \dot{e}$, which is bounded, since e and $\dot{e}$ are bounded. Therefore, $\dot{V}_a$ is uniformly continuous, and by the Lyapunov-like lemma,

$$\lim_{t\to\infty} \dot{V}_a \to 0. \quad (12)$$

From (11) and (12), we have to have $\lim_{t\to\infty} e(t) \to 0$, which completes the proof. ■

C. Practical implementation of the adaptation law

To ensure the boundedness of the estimates, we employ the following unknown input estimator

$$\frac{d\hat{u}_{2_i}}{dt} = \begin{cases} 0 & \text{if } \hat{u}_{2_i} \ge \overline{u}_{2_i} \text{ and } \sigma_i > 0 \\ 0 & \text{if } \hat{u}_{2_i} \le \underline{u}_{2_i} \text{ and } \sigma_i < 0 \\ \Gamma_i \sigma_i & \text{otherwise} \end{cases}$$
$$\triangleq \text{Proj}_{\hat{u}_{2_i}}(\Gamma_i \sigma_i), \quad (13)$$

where $\overline{u}_2 = [\overline{u}_{2_1} \ \ \overline{u}_{2_2} \ \ \cdots \ \ \overline{u}_{2_{m_2}}]$ and $\underline{u}_2 = [\underline{u}_{2_1} \ \ \underline{u}_{2_2} \ \ \cdots \ \ \underline{u}_{2_{m_2}}]$ are the upper and lower bounds of the unknown input u_2.

We now show that we also have $\lim_{t\to\infty} e(t) \to 0$ for the above u_2 estimator. Substituting (13) into (10) yields

$$\dot{V}_a = -e^\top Q e + 2\sum_{i=1}^{m_2} \Delta u_{2_i} (\sigma_i - \frac{1}{\Gamma_i} \text{Proj}_{\hat{u}_{2_i}}(\Gamma_i \sigma_i)).$$

It is easy to verify that $\Delta u_{2_i}(\sigma_i - \frac{1}{\Gamma_i}\text{Proj}_{\hat{u}_{2_i}}(\Gamma_i\sigma_i)) \le 0$. Therefore, $\dot{V}_a \le -e^\top Q e$. By the Lyapunov-like lemma, $\lim_{t\to\infty} e(t) \to 0$.

D. Estimating the unknown input

Applying Theorem 1 with the adaptation law (13), we now show that $\hat{u}_2 \to u_2$ as $t \to \infty$. To proceed, we need to show that $\dot{e}$ is uniformly continuous. Note that $\dot{e}$ is uniformly continuous if $\ddot{e}$ is bounded. Taking the second derivative of e, we obtain $\ddot{e} = (A - LC)\dot{e} - B_2 \dot{\hat{u}}_2$. Since $\dot{\hat{u}}_{2_i} = \text{Proj}_{\hat{u}_{2_i}}(\Gamma_i \sigma_i)$ and $\dot{e}$ are bounded, $\ddot{e}$ is bounded and hence, $\dot{e}$ is uniformly continuous. By the Lyapunov-like lemma, $\lim_{t\to\infty} \dot{e}(t) \to 0$. In the steady state, $e = 0$ and $\dot{e} = 0$. But $\dot{e} = (A - LC)e + B_2(u_2 - \hat{u}_2)$, so $B_2(u_2 - \hat{u}_2) = 0$. For B_2 of full column rank, $\hat{u}_2 = u_2$ in the steady state.

IV. State and unknown input estimation with guaranteed performance

In this section, we extend our adaptive state and unknown input estimation to the case when u_2 is a bounded unknown input not necessarily constant.

A. Guaranteed performance

Assumption 1: The unknown input u_2 is bounded with bounded derivative.

Letting $\zeta = [e, \ \ \Delta u_2]^\top$ and then combining (4) and (5), we obtain

$$\dot{\zeta} = \tilde{A} \ \zeta + \tilde{B} \ \dot{u}_2, \quad (14)$$

where

$$\tilde{A} = \begin{bmatrix} A - LC & B_2 \\ -\Gamma B_2^\top P & O \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} O \\ I_{m_2} \end{bmatrix}, \quad (15)$$

and where O denotes a zero matrix. To proceed, we define $\mathscr{L}_\infty$-stability with performance level (p.l.) γ for the system (14).

Definition 1: The system

$$\dot{\zeta} = \tilde{A} \ \zeta + \tilde{B} \ \dot{u}_2 \quad (16a)$$
$$z = H\zeta = [O \ \ I_{m2}] \ \zeta, \quad (16b)$$

where z is the output and $H \in \mathbb{R}^{m_2 \times (n+m_2)}$, is globally uniformly $\mathscr{L}_\infty$-stable with performance level γ if the following conditions are satisfied:

1) $\tilde{A}$ has eigenvalues in the open left half plane.
2) For every initial condition $\zeta(t_0) = \zeta_0$, where $t_0 \ge 0$, and every bounded unknown input derivative $\dot{u}_2(\cdot)$, there exists a bound $\beta(\zeta_0, ||\dot{u}_2(\cdot)||_\infty)$ such that
$$||\zeta(t)|| \le \beta(\zeta_0, ||\dot{u}_2(\cdot)||_\infty), \quad \forall t \ge t_0. \quad (17)$$
3) For zero initial condition, $\zeta(t_0) = 0$, and every bounded unknown input derivative $\dot{u}_2(\cdot)$, we have
$$||z(t)|| \le \gamma ||\dot{u}_2(\cdot)||_\infty, \quad \forall t \ge t_0. \quad (18)$$
4) For every initial condition, $\zeta(t_0) = \zeta_0$, and every bounded unknown input derivative $\dot{u}_2(\cdot)$, we have
$$\limsup_{t\to\infty} ||z(t)|| \le \gamma ||\dot{u}_2(\cdot)||_\infty. \quad (19)$$

For more details on the $\mathscr{L}_\infty$-stability with level of performance, we refer to [26]. For zero initial error, γ is defined as the upper bound on the $\mathscr{L}_\infty$ gain.

We now present a lemma from [14] that we use in our proof of the main result of this paper.

Lemma 1: Consider a system with bounded input w and performance output z described by

$$\dot{e} = F(t, e, w) \quad (20a)$$
$$z = G(t, e), \quad (20b)$$

where $e(t) \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_w}$, and $z(t) \in \mathbb{R}^{n_z}$. Suppose there exists a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and scalars $\alpha, \beta_1, \beta_2 > 0$ and $\mu_1, \mu_2 \ge 0$ such that

$$\beta_1 \|e\|^2 \le V(e) \le \beta_2 \|e\|^2, \quad (21)$$

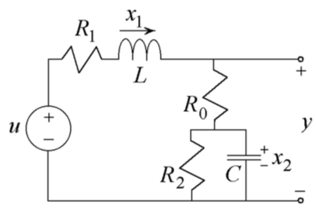

Fig. 1: *RLC* circuit of Example 1.

and

$$DV(e)F(t,e,w) \leq -2\alpha(V(e) - \mu_1||w||^2), \quad (22a)$$
$$||G(t,e)||^2 \leq \mu_2 V(e), \quad (22b)$$

for all $t \geq 0$, where DV denotes the derivative of V. Then system (20) is globally uniformly $\mathscr{L}_\infty$-stable with performance level $\gamma = \sqrt{\mu_1\mu_2}$.
The proof of Lemma 1 is given in [14].

B. Stability of $\tilde{A}$

The stability of $\tilde{A}$ in (15) is critical in the state and unknown input estimation. We investigated if the stability of $\tilde{A}$ is implied by the stability of $(A-LC)$. In the following, we provide a couple of examples to illustrate our discussion.

Example 1: Consider the *RLC* circuit shown in Figure 1. Let x_1 be the current through the inductor and x_2 be the capacitor voltage. The *RLC* circuit is modeled by the following equations,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{R_0+R_1}{L} & -\frac{1}{L}\\ \frac{1}{C} & -\frac{1}{CR_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}u_2,\ y = \begin{bmatrix}R_0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

where $L = 1$ H, $R_1 = R_2 = 1\Omega$, $C = 1$ F, and $R_0 = 0.1\Omega$. Solving (3), we obtain

$$P = \begin{bmatrix}1.815 & 18.15\\ 18.15 & 405.2\end{bmatrix},\quad L = \begin{bmatrix}239.52\\ -10.65\end{bmatrix}.$$

For $\Gamma = 20$, the matrix $\tilde{A}$ has the form

$$\tilde{A} = \begin{bmatrix}-25.05 & -240.52 & 1\\ 2.06 & 9.65 & 0\\ -36.3 & -362.98 & 0\end{bmatrix}.$$

The eigenvalues of $\tilde{A}$ are located at $-6.96 \pm -j14.91$, and -1.48. This $\tilde{A}$ is Hurwitz.

Next, we give an example where the matrix $(A-LC)$ is Hurwitz while $\tilde{A}$ is not Hurwitz.

Example 2: Consider the induction motor model in [15], where

$$A = \begin{bmatrix}-2379.2 & 0 & 0 & 0 & 0\\ 0 & -2.3 & 0 & 0.21 & 0\\ 0 & 0 & -2.3 & 0 & 0.21\\ 0 & 267.5 & 0 & -43.83 & 0\\ 0 & 0 & 267.54 & 0 & -43.83\end{bmatrix},$$

$$B_2 = \begin{bmatrix}68245 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 2\\ 0 & 232.75 & 0\\ 0 & 0 & -232.75\end{bmatrix},\ C = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}.$$

Solving (3), we obtain

$$P = \begin{bmatrix}13.13 & 0 & 0 & 0 & 0\\ 0 & 6597 & -137.57 & 56.69 & -1.18\\ 0 & -137.58 & 6597 & -1.18 & 56.69\\ 0 & 56.69 & -1.18 & 59.25 & -0.059\\ 0 & -1.18 & 56.69 & -0.059 & 59.25\end{bmatrix},$$

$$L = \begin{bmatrix}-1489.5 & 0 & 0\\ 0 & 0.912 & 0.0467\\ 0 & 0.0467 & 0.9119\\ 0 & 151.97 & 0.125\\ 0 & 0.125 & 151.97\end{bmatrix}.$$

Therefore, the matrix $(A-LC)$ is Hurwitz. For $\Gamma = 20I_3$, the matrix $\tilde{A}$ has its eigenvalues located at $-444.876 \pm j1.1059 \times 10^6$, $-98.988 \pm j7981.95$, $-99.113 \pm j7975.34$, 0, and 0. Thus, in this example the matrix $\tilde{A}$ is not Hurwitz.
In conclusion, the stability of $(A-LC)$ does not imply the stability of $\tilde{A}$. The stability of $\tilde{A}$ as a function of its parameters requires further investigation.

C. Sufficient conditions for the state and unknown input estimation

We present now sufficient conditions for the design of the state and unknown input observer when u_2 is a bounded unknown input. We also provide the performance level of the proposed observer.

Theorem 2: Suppose $\dot{u}_2$ is bounded, $\tilde{A}$ is asymptotically stable in the plant model given by (14), and there exist a symmetric matrix $P \succ 0$, matrices L and F such that conditions (3) are satisfied. If there exist $\alpha > 0$, $\mu \geq 0$, a symmetric matrix $\tilde{P} \succ 0$ such that the matrix inequalities

$$\phi \preceq 0 \quad (23a)$$
$$\begin{bmatrix}\tilde{P} & *\\ H & \mu I\end{bmatrix} \succeq 0 \quad (23b)$$

are satisfied where

$$\phi = \begin{bmatrix}\phi_{11} & \tilde{P}\tilde{B}\\ * & -2\alpha I\end{bmatrix} \quad (24)$$

and

$$\phi_{11} = \tilde{P}\tilde{A} + \tilde{A}^\top\tilde{P} + 2\alpha\tilde{P}, \quad (25)$$

then observer (2) yields $\mathscr{L}_\infty$-stable state and unknown input error dynamics with performance level $\gamma = \sqrt{\mu}$ for the performance output $z = H\zeta$.

Proof: We evaluate the Lyapunov derivative of $\tilde{V}(\zeta) = \zeta^\top\tilde{P}\zeta$ on the trajectories of (14) to obtain

$$\dot{\tilde{V}}(\zeta) = D\tilde{V}(\zeta)\dot{\zeta} = 2\zeta^\top\tilde{P}(\tilde{A}\zeta + \tilde{B}\dot{u}_2).$$

Let $q = [\zeta^\top \quad \dot{u}_2^\top]^\top$. Performing manipulations gives

$$\begin{aligned} q^\top\phi q &= \begin{bmatrix}\zeta^\top & \dot{u}_2^\top\end{bmatrix}\begin{bmatrix}\phi_{11} & \tilde{P}\tilde{B}\\ * & -2\alpha I\end{bmatrix}\begin{bmatrix}\zeta\\ \dot{u}_2\end{bmatrix}\\ &= 2\zeta^\top\tilde{P}\tilde{A}\zeta + 2\zeta^\top\tilde{P}\tilde{B}\dot{u}_2 + 2\alpha\zeta^\top\tilde{P}\zeta - 2\alpha\dot{u}_2^\top\dot{u}_2\\ &= D\tilde{V}(\zeta)\dot{\zeta} - 2\alpha||\dot{u}_2||^2 + 2\alpha\tilde{V}(\zeta). \end{aligned}$$

Since $\phi \preceq 0$, then

$$D\tilde{V}(\zeta)\dot{\zeta} - 2\alpha||\dot{u}_2||^2 + 2\alpha\tilde{V}(\zeta) = q^\top\phi q \leq 0. \quad (26)$$

Rearranging (26) gives

$$D\tilde{V}(\zeta)\dot{\zeta} \leq -2\alpha(\tilde{V}(\zeta) - ||\dot{u}_2||^2). \tag{27}$$

Therefore, condition (22a) in Lemma 1 holds with $\mu_1 = 1$.

Next, taking the Schur complement of (23b), we obtain

$$\tilde{P} - H^\top \mu^{-1} H = \tilde{P} - \mu^{-1} H^\top H \succeq 0. \tag{28}$$

Pre-multiplying (28) by $\zeta^\top$ and post-multiplying it by ζ gives

$$\zeta^\top \tilde{P}\zeta - \mu^{-1}\zeta^\top H^\top H\zeta \geq 0. \tag{29}$$

Rearranging the above gives

$$||H\zeta||^2 \leq \mu\tilde{V}(\zeta). \tag{30}$$

So condition (22b) in Lemma 1 holds for $\mu_2 = \mu$. From (27) and (30), we conclude that the assumptions of Lemma 1 are satisfied. Therefore the state and unknown input error dynamics are $\mathscr{L}_\infty$-stable with performance level $\gamma = \sqrt{\mu}$. ■

We summarize our discussion with an algorithm for the design of the adaptive observer.

Algorithm 1:

1) For the dynamical system (1), solve conditions (3) for (P, L, F) by letting $Y = PL$ and solving the following LMIs for (P, Y, F) using CVX,

$$\begin{aligned} A^\top P + PA - C^\top Y^\top - YC &\prec 0, \\ B_2^\top P = FC, \quad P = P^\top &\succ 0. \end{aligned}$$

2) Choose the estimator gain Γ and set $\dot{\hat{u}}_2 = \Gamma F(y - \hat{y})$.
3) Construct state and unknown input error dynamics system (14) and check that $\tilde{A}$ is Hurwitz.
4) Let $H = [O \quad I_{m_2}]$, choose the design parameter α and solve LMIs (23) for $\tilde{P}$ and μ.

V. Example

In this section, we present an example to illustrate the effectiveness of the proposed observer to estimate the state and unknown input for a nonlinear system with bounded unknown input and bounded unknown input derivative. LMIs of Theorem 2 have been solved using CVX [19], [20].

Example 3: We consider the flight path rate demand missile (two-loop) autopilot system from [27]. The state variables of the system are:
x_1: flight path rate demand,
x_2: pitch rate,
x_3: elevator deflection,
x_4: rate of change of elevator deflection.
The output of the system are state variables x_1 and x_2. The state space model of the two-loop autopilot system has the form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} -\frac{1}{T_a} & \frac{a+\sigma^2 w_b^2}{T_a} & \frac{-k_b\sigma^2 w_b^2}{T_a} & -k_b\sigma^2 {w_b}^2 \\ -\frac{1+w_b^2 T_a^2}{T_a(1+\sigma^2 w_b^2)} & \frac{1}{T_a} & \frac{(T_a^2-\sigma^2)k_b w_b^2}{T_a(1+\sigma^2 w_b^2)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -w_a^2 & -2\zeta_a w_a \end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ k_q w_a^2\end{bmatrix} u_1 + \begin{bmatrix}1\\ 0\\ 1\\ 0\end{bmatrix} u_2, \; y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}.$$

TABLE I: Parameter values for the two-loop autopilot example.

Parameter	Value
T_a	0.36 s
σ^2	0.00029 s^2
w_b	11.77 rad/s
ζ_a	0.6
k_b	-9.91 s^{-1}
w_a	180 rad/s
k_q	-0.07

The numerical values of the model parameters are shown in Table I. The unknown input u_2 is taken to be $u_2 = \sin(10t)$.

We apply Theorem 2 for the autopilot model. Note that the conditions of the existence of UIO are satisfied, where $\text{rank}(B2) = \text{rank}(CB_2) = 1$. We use the CVX software to compute (L, F, P) that satisfy the conditions in (3). We obtain

$$L = \begin{bmatrix} -16.088 & -66.420 \\ 69.632 & 222.372 \\ -28.867 & -98.993 \\ 3692.687 & 9301.315 \end{bmatrix}, \quad F = \begin{bmatrix} 8899.1 & 2651.55 \end{bmatrix}.$$

We set $\Gamma = 20$, $\bar{\dot{u}}_2 = 10$, and $\underline{\dot{u}}_2 = -10$. The error estimation dynamics (14) take the form

$$\dot{\zeta} = \tilde{A}\ \zeta + \tilde{B}\ \dot{u}_2 = \begin{bmatrix} 13.3 & 69.3 & 1.1 & 0.4 & 1 \\ -120.24 & -219.6 & -474.1 & 0 & 0 \\ 28.9 & 99 & 0 & 1 & 1 \\ -3692.7 & -9301.3 & -32400 & -216 & 0 \\ -177981.5 & -53031.1 & 0 & 0 & 0 \end{bmatrix} \zeta + \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \dot{u}_2,$$

where $\tilde{A}$ is asymptotically stable with the eigenvalues located at $-101.535 \pm j474.156, -32.823 \pm j133.833$, and -153.569.

We solve the LMIs given by (23), and obtain the observation error performance level $\gamma = 0.124$. In our simulation, we use the initial condition of the system to be $x(0) = [0.5, -10, 5, -3]^\top$ and the initial conditions on the adaptive observer dynamics are zero. The design parameter $\alpha = 1$. We can see from Figures 2 and 3 that the adaptive observer estimates the system states well. The unknown input is reconstructed accurately as can be seen in Figure 4 and 5.

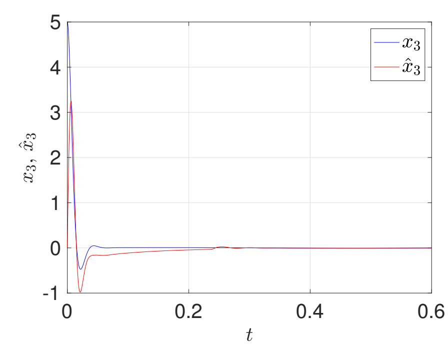

Fig. 2: Plot of the state x_3 and its estimate in Example 3.

VI. Conclusions

An open problem is to investigate the conditions under which the matrix $\tilde{A}$ in (15) is Hurwitz. At present, the design

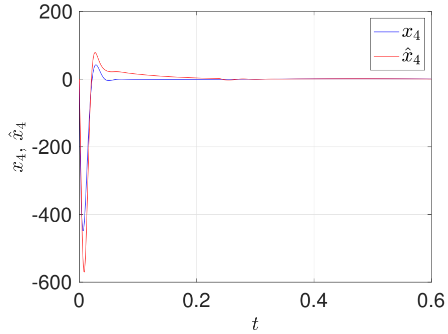

Fig. 3: Plot of the state x_4 and its estimate in Example 3.

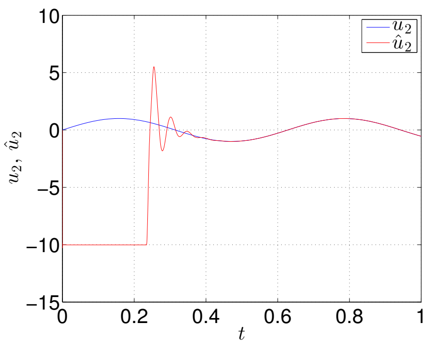

Fig. 4: Plot of unknown input estimation in Example 3.

parameters α, and μ in Theorem 2 are selected by trial and error. More systematic procedure to select the parameters is desired.

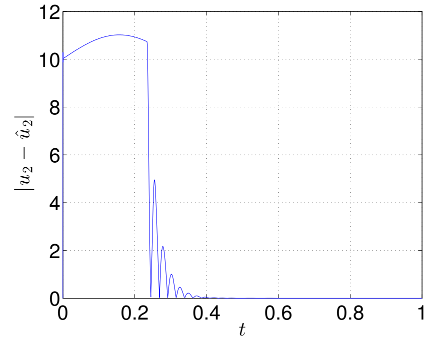

Fig. 5: Plot of the absolute value of the unknown input estimation error in Example 3.

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