Periodic Stabilization of Discrete-Time Controlled Switched Linear Systems

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***Abstract*— The goal of this paper is to study the state-feedback stabilization of controlled discrete-time switched linear systems (SLSs), where the discrete switching control input and the continuous control input coexist. We propose a periodic hybrid-control policies, periodic control Lyapunov functions (PCLFs), and prove that the existence of a quadratic PCLF is a necessary and sufficient condition for the stabilizability of SLSs. Moreover, computational algorithms are developed to find quadratic PCLFs and test the stabilizability.**

I. INTRODUCTION

The goal of this paper is to study stabilization of discrete-time non-autonomous switched linear systems (SLSs). Stability analysis and stabilization of SLSs have been studied extensively over the past decades [1]–[7] mainly based on Lyapunov theories, for instance, switched Lyapunov functions [8], [9], polynomial Lyapunov functions [**?**], and composite Lyapunov functions [2]–[6].

In this paper, we consider periodic control Lyapunov functions (PCLFs) for stabilization of SLSs. Initially, the concept of PCLFs was explored to study the periodic systems [10]–[12] and the multirate sampled-data control systems [13]. It was not until recently that PCLFs found their applications to various problems for general non-periodic systems, for example, the robust stability and stabilization of uncertain linear [14], [15] and nonlinear systems [16], where it was revealed that, in general, the PCLFs provide less conservative stability analysis and control synthesis conditions, and can improve performances of the control systems, such as the robustness and the H_∞ performance. Continuous counterparts were also investigated in [17], [18]. The PCLF approaches relax the limitation of classical Lyapunov theorems which requires the Lyapunov function to decrease monotonically at every sample, and allows the Lyapunov function to decrease at regular time intervals.

A more general class of Lyapunov functions is the so-called non-monotonic Lyapunov functions. To the authors' knowledge, the non-monotonic Lyapunov functions were pioneered in [19], [20] for nonlinear and switching systems by considering the Lyapunov function which decreases on average over several samples. The notion of the non-monotonic Lyapunov functions was recently generalized in [21] by introducing the graph Lyapunov function, in which a finite set of non-monotonic Lyapunov functions are jointly used to certify the stability.

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Recently, PCLFs were used for stabilization of autonomous SLSs in [22]–[25], where only the discrete switching input exists. On the other hand, this paper attempts to generalize the results for non-autonomous SLSs, where the discrete switching control input and the continuous control input coexist. The main contributions can be summarized as follows:

1) We formally define PCLFs for controlled SLSs and prove an associated PCLF theorem, which states that the existence of a PCLF is a necessary and sufficient condition for stabilizability of SLSs;
2) Computational methods based on convex optimizations and dynamic programming are developed to find PCLFs and check the stabilizability;
3) We study connections among the proposed PCLF theorem and controllability/reachability of SLSs studied in [26].

II. PRELIMINARIES AND PROBLEM FORMULATION

The adopted notation is as follows: $\mathbb{N}$ and $\mathbb{N}_+$: sets of nonnegative and positive integers, respectively; $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$: set of all $n\times m$ real matrices; A^T: transpose of matrix A; $A \succ 0$ ($A \prec 0$, $A \succeq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix A; for any matrix M, $M^\dagger$ denotes its pseudoinverse; I_n: $n\times n$ identity matrix; $||\cdot||$: Euclidean norm of a vector or spectral norm of a matrix; $||\cdot||_P$ is the ellipsoid norm on $x\in\mathbb{R}^n$ defined by $||x||_P := \sqrt{x^TPx}$; for a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of the set; $\mathbb{S}^n$: symmetric $n\times n$ matrices; $\mathbb{S}^n_+$: set of symmetric $n\times n$ positive semi-definite matrices; $\mathbb{S}^n_{++}$: set of symmetric $n\times n$ positive definite matrices; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$: minimum and maximum eigenvalues of symmetric matrix A, respectively; given $P\in\mathbb{S}^n_{++}$.

Consider the discrete-time SLS described by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad x(0) = z \in \mathbb{R}^n, \tag{1}$$

where $k\in\mathbb{N}$, $x:\mathbb{N}\to\mathbb{R}^n$ is the state, $u:\mathbb{N}\to\mathbb{R}^m$ is the continuous control, and $\sigma:\mathbb{N}\to\mathcal{M}$ is the mode. For each $i\in\mathcal{M}$, $A_i\in\mathbb{R}^{n\times n}$ and $B_i\in\mathbb{R}^{n\times m}$ are constant matrices, and the pair (A_i, B_i) is called a subsystem or subsystem matrices. Both u and σ are control signals, and to differentiate (1) from the SLS with $u(k)\equiv 0_n$, i.e., the autonomous SLSs which has only the mode as a control signal, (1) will be called a controlled SLS. Define $\xi := (u,\sigma):\mathbb{N}\to\mathbb{R}^m\times\mathcal{M}$. Then, the sequence $\pi_k := (\xi(i))_{i=0}^{k-1}$

or $\pi_\infty := (\xi(i))_{i=0}^\infty$ is called the hybrid-control sequence [5]. In this paper, the state driven by π_k or π_∞ with initial state $z \in \mathbb{R}^n$ will be denoted by $x(\cdot;\, z,\, \pi_k)$ or $x(\cdot;\, z,\, \pi_\infty)$, respectively. When the hybrid-control sequence π_k (or π_∞) possibly depends on the initial state z, it will be denoted by $\pi_{k,\, z}$ (or $\pi_{\infty,\, z}$). The definition of exponential stabilizability of SLSs is given below.

Definition 1 (Exponential stabilizability): The SLS (1) is called exponentially stabilizable if staring from any initial state $x(0) = z \in \mathbb{R}^n$, there exists a hybrid-switching sequence $\pi_{\infty,\, z}$ such that

$$\|x(k;\, z,\, \pi_{\infty,\, z})\| \leq a c^k \|z\| \tag{2}$$

for all $k \in \mathbb{N}$ and for some constants $a \geq 1$ and $c \in [0, 1)$.

For SLSs, any $c \in \mathbb{R}_+$ satisfying (2) for some $a \geq 1$ will be called an *exponential convergence rate*. The *exponential stabilizing rate*, denoted by $c^* \in \mathbb{R}_+$, is the infimum of all such exponential convergence rates. Note that c^* provides a quantitative metric of the SLS's stabilizability. We refer to the notion of exponential stabilizability simply as stabilizability if there is no confusion. Clearly, the SLS (1) is exponentially stabilizable if one of the subsystems is stabilizable. A nontrivial problem is to stabilize the system when none of the subsystems are stabilizable [5]. In this respect, the following assumption is imposed.

Assumption 1: Each subsystem $(A_i,\, B_i)$ is not stabilizable for all $i \in \mathcal{M}$.

Denote time histories of the state up to k and hybrid-control sequence up to $k-1$ by $\mathcal{I}_k := (x(i),\, u(i-1),\, \sigma(i-1))_{i=0}^k$ with $\mathcal{I}_0 := x(i)$, and define the set of all admissible $\mathcal{I}_k$ by $\mathcal{J}_k$. Then, the mapping $\omega_k : \mathcal{J}_k \to \mathbb{R}^m$, $k \in \mathbb{N}$, is called the continuous-control policy, $\theta_k : \mathcal{J}_k \to \mathcal{M}$, $k \in \mathbb{N}$, is called the switching-control policy, and $\psi_k := (\omega_k,\, \theta_k) : \mathcal{J}_k \to \mathbb{R}^m \times \mathcal{M}$, is called the hybrid-control policy of the SLS (1). The goal of this paper is to determine whether or not a given SLS is stabilizable, and if stabilizable, to find a stabilizing hybrid-control policy.

Problem 1:

1) Determine whether or not a given SLS is stabilizable;
2) If stabilizble, then find a hybrid-control policy $\psi_k : \mathcal{I}_k \to \mathbb{R}^m \times \mathcal{M}$, $k \in \mathbb{N}$, such that the SLS (1) with $\pi_\infty = (\psi_k)_{k=0}^\infty$ is stable.

III. Periodic Stabilization

First, we define the periodic control Lyapunov function (PCLF).

Definition 2 (Periodic control Lyapunov function (PCLF)): If there exist a positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$, positive constants $\kappa,\, \kappa_1,\, \kappa_2,\, \kappa_3 \in \mathbb{R}_{++}$, a positive integer $h \in \mathbb{N}_+$, and a hybrid-control policy $\psi_k : \mathcal{I}_k \to \mathbb{R}^m \times \mathcal{M}$, $k \in \{1, \ldots, h-1\}$, such that for all $z \in \mathbb{R}^n$,

$$\kappa_1 ||z||^2 \leq V(z) \leq \kappa_2 ||z||^2, \tag{3}$$

$$V(x(h;\, z,\, \pi_h)) - V(z) \leq -\kappa_3 ||z||^2, \tag{4}$$

$$\|x(s;\, z,\, \pi_h)\| \leq \kappa \|z\| \quad \forall\, s \in \{0, \ldots, h-1\}, \tag{5}$$

hold with $\pi_h = (\psi_k)_{k=0}^{h-1}$, then V is called the h-periodic control Lyapunov function (h-PCLF).

The hybrid-control policy ψ_k, $k \in \{1, \ldots, h-1\}$, in Definition 2 generates the current and future hybrid-control sequence of length h that satisfies (3)-(5). When it is applied to the system repeatedly every h steps, we obtain an infinite hybrid-control sequence.

Definition 3 (h-PHCS): If a V is a h-PCLF, then the associated h-periodic hybrid-control sequence (h-PHCS) is defined by concatenating the hybrid-control policy ψ_k, $k \in \{1, \ldots, h-1\}$, in Definition 2 at time instants $k = ht$, $t \in \mathbb{N}$, i.e.,

$$\pi_\infty = \{\pi_h,\, \pi_h,\, \pi_h, \ldots\} \tag{6}$$

with $\pi_h = (\psi_k)_{k=0}^{h-1}$.

The following lemma is an extension of the PCLF theorem in [25, Theorem 1] for controlled SLSs (1).

Lemma 1 (PCLF theorem): If there exists a h-PCLF V, then the SLS (1) is exponentially stabilizable with parameters $a = \kappa \left(\frac{\kappa_2}{\kappa_1}\right)^{1/2} \left(\frac{\kappa_2}{\kappa_2 - \kappa_3}\right)$ and $c = \left(1 - \frac{\kappa_3}{\kappa_2}\right)^{\frac{1}{2h}}$ under the h-PHCS (6), where $\kappa \in \mathbb{R}_{++}$ is some constant that satisfies (5).

Proof: The proof is a straightforward extension of [25, Theorem 1], and thus, omitted here. ■

Throughout the paper, we will consider the quadratic function $V(z) = z^T P z = ||z||_P^2$ with a given $P \in \mathbb{S}_{++}^n$. Define the hybrid-control sequence associated with V

$$\tilde{\pi}_h(z) := (\omega_k(z),\, \theta_k(z))_{k=0}^{h-1} = \arg\min_{\pi_h \in \Pi_h} ||x(h;\, z,\, \pi_h)||_P, \tag{7}$$

which generates the current and future hybrid-control sequence of length h to minimize V after h steps, where Π_h is the set of all admissible hybrid-control sequences of the form π_h. For simplicity, we assume that the optimal solution (7) is unique. Concatenating (7) at time instants $k = ht$, $t \in \mathbb{N}$, we also define the infinite hybrid-control sequence

$$\tilde{\pi}_\infty = \{\tilde{\pi}_h,\, \tilde{\pi}_h,\, \tilde{\pi}_h, \ldots\} \tag{8}$$

Alternatively, (8) can be though of as an infinite hybrid-control sequence generated by the state-feedback policy

$$\begin{aligned}&(\omega_k(z),\, \theta_k(z))\\ &= \arg\min_{\pi \in \Pi_{h-(h \bmod k)}} ||x(h - (h \bmod k);\, z,\, \pi)||_P.\end{aligned}$$

Then, the sequence (8) can be equivalently expressed as

$$\tilde{\pi}_\infty = (\omega_k(x(k)),\, \theta_k(x(k)))_{k=0}^\infty.$$

Moreover, define the nonlinear operator $\mathcal{T}_h : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathcal{T}_h(z) := x(h;\, z,\, \tilde{\pi}_h(z)).$$

Motivated by [27], we introduce the quantity

$$\rho(\mathcal{T}_h) := \sup_{\substack{z \in \mathbb{R}^n \\ ||z||_P \neq 0}} \frac{||\mathcal{T}_h(z)||_P}{||z||_P} = \sup_{\substack{z \in \mathbb{R}^n \\ ||z||_P = 1}} ||\mathcal{T}_h(z)||_P,$$

where the second equality follows from the homogeneity of $\mathcal{T}_h$. Since the set $\{z \in \mathbb{R}^n : ||z||_P = 1\}$ is compact and $||\mathcal{T}_h(z)||_P$ is continuous in z, the sup is attained and can be replaced with max. Now, we claim that if $\rho(\mathcal{T}_h) < 1$, then V is an h-PCLF, and $\tilde{\pi}_\infty$ in (8) is an associated h-PHCS.

Theorem 1: If there exists $h \in \mathbb{N}$, $h \geq 1$, such that $\rho(\mathcal{T}_h) < 1$, then V is an h-PCLF, and the SLS (1) is exponentially stabilizable with parameters

$$a = \kappa \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \left(\frac{\lambda_{\max}(P)}{\lambda_{\max}(P) + \rho(\mathcal{T}_h)^2 - 1}\right)$$

and

$$c = \left(\frac{\lambda_{\max}(P) + \rho(\mathcal{T}_h)^2 - 1}{\lambda_{\max}(P)}\right)^{\frac{1}{2h}}$$

under the associated h-PHCS, $\tilde{\pi}_\infty$ defined in (8), where $\kappa \in \mathbb{R}_{++}$ is some constant satisfying (5).

To prove Theorem 1, it is convenient to introduce the intermediate result given below.

Proposition 1: For a given $z \in \mathbb{R}^n$, the hybrid-control policy $\tilde{\pi}_h(z)$ in (7) can be computed by

$$(\theta_{h-1}(z), \ldots, \theta_0(z)) = \arg\min_{\mathbf{i} \in \mathcal{M}^h} z^T (\Phi_{\mathbf{i}}^T P \Phi_{\mathbf{i}}) z, \tag{9}$$

and

$$\begin{bmatrix} \omega_0(z) \\ \vdots \\ \omega_{h-1}(z) \end{bmatrix} = F_{(\theta_{h-1}(z), \ldots, \theta_0(z))} z \in \mathbb{R}^{hm}, \tag{10}$$

where

$$\begin{aligned} &A_{\mathbf{i}} := A_{i_h} \cdots A_{i_2} A_{i_1} \in \mathbb{R}^{n \times n}, \\ &B_{\mathbf{i}} := \begin{bmatrix} A_{i_h} \cdots A_{i_3} A_{i_2} B_{i_1} & \cdots & A_{i_h} B_{i_{h-1}} & B_{i_h} \end{bmatrix} \in \mathbb{R}^{n \times hm}, \\ &F_{\mathbf{i}} := -[P^{1/2} B_{\mathbf{i}}]^\dagger P^{1/2} A_{\mathbf{i}} \in \mathbb{R}^{hm \times n}, \\ &\Phi_{\mathbf{i}} := A_{\mathbf{i}} + B_{\mathbf{i}} F_{\mathbf{i}} \in \mathbb{R}^{n \times n}, \\ &\mathbf{i} := (i_h, \ldots, i_2, i_1). \end{aligned} \tag{11}$$

Proof: By compositions of the state transition (1) from $k = 0$ to $k = h$, we have $x(h) = A_{\mathbf{i}} z + B_{\mathbf{i}} \mathbf{u}$ for any given $\mathbf{u} := \begin{bmatrix} u(0)^T & \cdots & u(h-1)^T \end{bmatrix}^T$ and $\mathbf{i} \in \mathcal{M}^h$. Thus, it is easy to prove that the optimal solution of (7) is obtained by solving

$$\min_{\mathbf{i} \in \mathcal{M}^h} \min_{\mathbf{u} \in \mathbb{R}^{mh}} (A_{\mathbf{i}} z + B_{\mathbf{i}} \mathbf{u})^T P (A_{\mathbf{i}} z + B_{\mathbf{i}} \mathbf{u}). \tag{12}$$

For any fixed $\mathbf{i} \in \mathcal{M}^h$, (12) is a least square problem, and an optimal solution is $\mathbf{u} = -[P^{1/2} B_{\mathbf{i}}]^\dagger P^{1/2} A_{\mathbf{i}} z$. Defining $F_{\mathbf{i}} := -[P^{1/2} B_{\mathbf{i}}]^\dagger P^{1/2} A_{\mathbf{i}}$ and substituting $F_{\mathbf{i}} z$ for $\mathbf{u}$ in (12), we obtain the right-hand side of (9). This completes the proof. ■

Now, the proof of Theorem 1 is given below.

Proof: (Proof of Theorem 1) If $\rho(\mathcal{T}_h) < 1$, then by definition, we have $||\mathcal{T}_h(z)||_P \leq \rho(\mathcal{T}_h) \cdot ||z||_P$, $\forall z \in \mathbb{R}^n$, which implies that (4) of Definition 2 holds with $-\kappa_3 = \rho(\mathcal{T}_h)^2 - 1 < 0$. Since V is quadratic, (3) holds with $\kappa_1 = \lambda_{\min}(P)$ and $\kappa_2 = \lambda_{\max}(P)$. To prove (5), observe that for any $\mathbf{i} \in \mathcal{M}^h$, the corresponding continuous-control sequence in a vector form is $\mathbf{u} = F_{\mathbf{i}} z \in \mathbb{R}^{hm}$ and is bounded, i.e., $||\mathbf{u}|| = (z^T (F_{\mathbf{i}}^T F_{\mathbf{i}}) z)^{1/2} \leq \gamma ||z||$ for some $\gamma \in \mathbb{R}_{++}$. Therefore, the state within a finite time interval is bounded by some constant. This proves (5). Hence, V is an h-PCLF, and by Lemma 1, the SLS (1) is exponentially stabilizable with the parameters given in the statement. ■

Proposition 2: If the SLS (1) is exponentially stabilizable with parameters $a \geq 1$ and $c \in [0, 1)$, then for any $P \in \mathbb{S}^n_{++}$, it holds that

$$\rho(\mathcal{T}_h) \leq (\lambda_{\max}(P)/\lambda_{\min}(P))^{1/2} a c^h, \quad \forall h \in \mathbb{N}. \tag{13}$$

Proof: Suppose that the SLS (1) is exponentially stabilizable with parameters $a \geq 1$ and $c \in [0, 1)$. Then, for any initial state $x(0) = z \in \mathbb{R}^n$, there exists a hybrid-control sequence $\pi_{\infty, z}$ such that $||x(k; z, \pi_{\infty, z})|| \leq a c^k ||z||$ holds for all $k \in \mathbb{N}$. Combining the inequality with $\sqrt{\lambda_{\min}(P)} ||x|| \leq ||x||_P \leq \sqrt{\lambda_{\max}(P)} ||x||$, one gets

$$||x(k; z, \pi_\infty)||_P^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} a^2 c^{2k} ||z||_P^2, \quad \forall z \in \mathbb{R}^n,$$

which implies $\rho(\mathcal{T}_h) \leq (\lambda_{\max}(P)/\lambda_{\min}(P))^{1/2} a c^k$, and $\rho(\mathcal{T}_h) \to 0$ as $h \to \infty$. ■

As a corollary, a converse PCLF theorem is given below.

Corollary 1 (Converse PCLF theorem): If the SLS (1) is exponentially stabilizable with parameters $a \geq 1$ and $c \in [0, 1)$, then for any $P \in \mathbb{S}^n_{++}$, $V(z) = z^T P z$ is an h-PCLF of the SLS satisfying $\rho(\mathcal{T}_h) < 1$ for all $h > \bar{h}$, where

$$\bar{h} := \left\lceil \frac{\ln \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \ln a^2}{\ln(1/c^2)} \right\rceil$$

and $\lceil x \rceil$ stands for the minimum integer greater than $x \in \mathbb{R}$.

Proof: If the SLS (1) is exponentially stabilizable with parameters $a \geq 1$ and $c \in [0, 1)$, then by Proposition 2, (13) holds, and by direct calculation, we can prove that the right-hand side of the last inequality is negative if $h > \bar{h}$. This completes the proof. ■

For a stabilizable SLS under h-PHCS (8) and given $h \in \mathbb{N}_+$ such that $\rho(\mathcal{T}_h) < 1$, its exponential stabilizing rate is defined as

$$\begin{aligned} c_h^* := \inf_{\tilde{\pi}_\infty(\cdot)} \{ c \geq 0 \ : \ &\text{there exists } a < \infty \text{ such that} \\ &\|x(k; z, \tilde{\pi}_\infty(z))\| \leq a c^k \|z\|, \ \forall z \in \mathbb{R}^n, \forall k \in \mathbb{N} \}. \end{aligned}$$

Obviously, $c^* \leq c_h^*$ holds. In addition, by Theorem 1, an overestimation of c_h^* can be expressed in terms of $\rho(\mathcal{T}_h)$.

Proposition 3: If $\rho(\mathcal{T}_h) < 1$, then

$$c_h^* \leq \left(\frac{\lambda_{\max}(P) + \rho(\mathcal{T}_h)^2 - 1}{\lambda_{\max}(P)}\right)^{\frac{1}{2h}}. \tag{14}$$

Proof: Straightforward from the exponential convergence rate c in the statement of Theorem 1. ■

A natural question is whether the exponential stabilizing rate c^* defined in Section II can be achieved by using an h-PHCS (8), i.e., whether $c^* = c_h^*$ for some $h \geq 1$. The next result shows that asymptotically this is indeed the case.

Proposition 4: If $\rho(\mathcal{T}_h) < 1$, then $\lim_{h \to \infty} c_h^* = c^*$.

Proof: Without loss of generality, let $P = I_n$. By Proposition 2 and Proposition 3, $c_h^* \leq \rho(\mathcal{T}_h)^{\frac{1}{h}}$ and $\rho(\mathcal{T}_h) \leq$

ac^h for any parameters $a \geq 1$ and $c \in [0, 1)$ for exponential stabilizability. Combining the two inequalities, we have $c_h^* \leq a^{1/h}c$. From the definition of c^*, any $\varepsilon > 0$ such that $c^* + \varepsilon < 1$ is a exponential convergence rate. Thus, we have $c_h^* \leq a^{1/h}(c^* + \varepsilon)$. Taking the limit $h \to \infty$ and noting that $c_h^* \geq c^*$ and that $\varepsilon > 0$ is arbitrary, we obtain the desired conclusion. ■

IV. Numerical Computation

Proposition 1 suggests computational strategies to check the stabilizability of SLSs. Define the set

$$\mathcal{P}_h := \{\Phi_{\mathbf{i}}^T P \Phi_{\mathbf{i}} \in \mathbb{S}_+^n : \mathbf{i} \in \mathcal{M}^h\}. \tag{15}$$

Motivated by [5]–[7] , we develop a convex optimization-based algorithm to compute an over estimation of c_h^*.

Problem 2: Let $P \in \mathbb{S}_{++}^n$ be given. Solve

$$\gamma_h^* := \underset{\gamma,\, \alpha_1, \ldots, \alpha_k \in \mathbb{R}}{\text{minimize}} \quad \gamma \tag{16}$$

subject to

$$\sum_{i=1}^{k} \alpha_i P^{(i)} \preceq \gamma P, \tag{17}$$

$$\sum_{i=1}^{k} \alpha_i = 1, \quad \alpha_i \geq 0,$$

where $k = |\mathcal{P}_h|$, $\{P^{(i)}\}_{i=1}^k$ is an enumeration of $\mathcal{P}_h$.

The problem of simultaneously determining $P \in \mathbb{S}_{++}^n$ and the solution to Problem 2 is a nonconvex problem, while for a fixed P, i.e., $P = I_n$, the problem is a convex linear matrix inequality problem. Once a solution to (16) is obtained, it can be used to estimate c_h^*.

Proposition 5: If (16) is feasible, then $\rho(\mathcal{T}_h) \leq (\gamma_h^*)^{1/2}$, holds. In particular, $c_h^* \leq \left(\frac{\lambda_{\max}(P)+\gamma_h^*-1}{\lambda_{\max}(P)}\right)^{\frac{1}{2h}}$.

Proof: The proof follows the same lines of approaches as [5, Corollary 1] or [3, Theorem 3]; hence it is omitted here. ■

Example 1: Consider the SLS (1) with

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

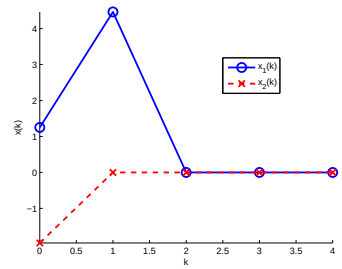
$$A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

taken from [5, Example 1]. Solving Problem 2 yields $\gamma_1^* = 2.0187$ and $\gamma_2^* = 0$. For $h = 2$, the closed-loop subsystem matrices in (11) are

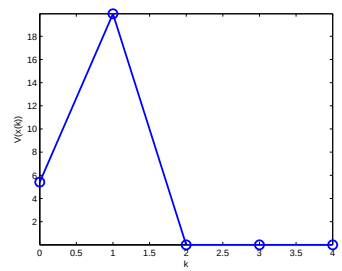
$$\Phi_{(1,1)} = \begin{bmatrix} 3.2 & -1.6 \\ -1.6 & 0.8 \end{bmatrix}, \quad \Phi_{(1,2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Phi_{(2,1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Phi_{(2,2)} = \begin{bmatrix} 0 & 0 \\ 0 & 2.25 \end{bmatrix}.$$

One can observe that $\Phi_{(1,2)}$ and $\Phi_{(2,1)}$ are zero matrices. This implies that the system can be stabilized in two steps. With $x(0) = \begin{bmatrix} 1.2527 & -1.9606 \end{bmatrix}^T$, simulation results including the state trajectories and the evolution of the PCLF value are shown in Figure 1.



(a)



(b)

Fig. 1. Example 1. (a) Time histories of the state variables $x_1(k)$ and $x_2(k)$; (b) Time histories of the Lyapunov function.

The approach in Problem 2 requires the pseudoinverse of matrices, which can be computationally demanding for large-scale problems. For efficient computations, the dynamic programming approach [6], [28] can be used. To this end, we will introduce some useful notions from [5], [6]. Define the value function $J_k : \mathbb{R}^n \to \mathbb{R}$, $k \in \{0, 1, \ldots, h\}$, as

$$J_k(z) := \inf_{\pi_k \in \Pi_k} ||x(k;\, z,\, \pi_k)||_P^2, \quad k \in \{0, 1, \ldots, h\},$$

with $J_0(z) := z^T P z$. By standard results of dynamic programming [28], for any finite integer $h \in \mathbb{N}_+$, the value function can be computed recursively using the one-stage value iteration

$$J_{k+1}(z) = \inf_{u \in \mathbb{R}^m,\, \sigma \in \mathcal{M}} J_k(A_\sigma z + B_\sigma u),$$
$$\forall z \in \mathbb{R}^n,\, k \in \{0, 1, \ldots, h-1\}.$$

The following result states that the value function can be characterized by the switched Riccati sets (SRSs) proposed in [5], [6].

Definition 4 (Switched Riccati set): Define $\rho_i(P) := (A_i - B_i(P^{1/2}B_i)^\dagger P^{1/2}A_i)^T P (A_i - B_i(P^{1/2}B_i)^\dagger P^{1/2}A_i)$. The sequence of sets $\{\mathcal{H}_k\}_{k=0}^h$ generated iteratively by $\mathcal{H}_{k+1} = \{\rho_i(X) \in \mathbb{S}_+^n : i \in \mathcal{M},\, X \in \mathcal{H}_k\}$ with $\mathcal{H}_0 = \{P\}$ is called the switched Riccati sets (SRSs).

Lemma 2 ([6, Theorem 1]): For $t \in \{0, 1, \ldots, h\}$, the value function $J_k(z)$ is represented by

$$J_k(z) = \min_{X \in \mathcal{H}_k} z^T X z.$$

Furthermore, for $z \in \mathbb{R}^n$ and $t \in \{1, 2, \ldots, h\}$, if we define

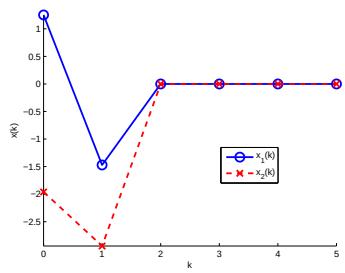
$$(P_t^*(z),\, i_t^*(z)) = \underset{X \in \mathcal{H}_{t-1},\, i \in \mathcal{M}}{\arg\min} z^T \rho_i(X) z,$$

then the h-PHCS (7) at z is $\tilde{\pi}_h(z) = ((K_{i^*_{h-k}(z)}(P^*_{h-k}(z))z,\, i^*_{h-k}(z)))_{k=0}^{h-1}$, where $K_i(P) := -(P_i^{1/2}B_i)^\dagger P_i^{1/2}A_i$.

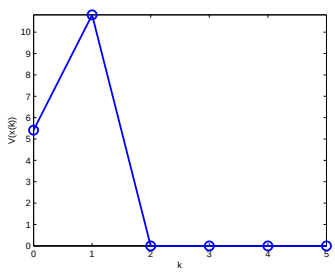
It is easy to prove that Problem 2 with $\mathcal{P}_h$ replaced by the SRS $\mathcal{H}_h$ gives the same γ_h^*. Therefore, the following optimization problem is an equivalent but more efficient alternative to Problem 2.

Problem 3: Let $P \in \mathbb{S}^n_{++}$ be given. Solve (16), where $k = |\mathcal{H}_h|$ and $\{P^{(i)}\}_{i=1}^k$ is an enumeration of $\mathcal{H}_h$.

Example 2: Consider the SLS (1) in Example 1 again. Solving Problem 3 yields $\gamma_1^* = 2.0187$ and $\gamma_2^* = 0$, which are identical to the results in Example 1. The simulation results including time histories of the state and the PCLF value under the h-PHCS (7) are depicted in Figure 2, where the h-PHCS is generated by using Lemma 2.



(a)



(b)

Fig. 2. Example 3. (a) Time histories of the state variables $x_1(k)$ and $x_2(k)$; (b) Time histories of the PCLF value.

V. Controllability and Reachability

In this section, we study connections between the PCLF theorem and the controllability/reachability studied in [26].

Definition 5 ([26, page 1438]):

1) (Controllable state): The state $y \in \mathbb{R}^n$ is controllable (to the origin) if there exist a time instant $k \in \mathbb{N}_+$ and a finite hybrid-control sequence $\pi_{h,\,y}$ depending on y such that $x(h;\, y,\, \pi_{k,\,y}) = 0_n$;
2) (Controllable set): The controllable set of the SLS (1), denoted by $\mathcal{C}$, is the set of states which are controllable;
3) (Controllability): The SLS (1) is said to be controllable (to the origin) if $\mathcal{C} = \mathbb{R}^n$;
4) (Reachable state): The state $y \in \mathbb{R}^n$ is reachable if there exist a finite time instant $k \in \mathbb{N}_+$ and a finite hybrid-control sequence $\pi_{k,\,y}$ depending on y such that $x(h;\, 0_n,\, \pi_{k,\,y}) = y$;
5) (Reachable set): The reachable set of the SLS (1), denoted by $\mathcal{R}$, is the set of states which are reachable;
6) (Reachability): The SLS (1) is said to be reachable if $\mathcal{R} = \mathbb{R}^n$.

For all $\mathbf{i} := (\mathbf{i_h}, \ldots, \mathbf{i_2}, \mathbf{i_1}) \in \mathcal{M}^\mathbf{h}$, define

$$\mathcal{C}(\mathbf{i}) := \{x \in \mathbb{R}^n :\, A_\mathbf{i} x \in \mathrm{Im} B_\mathbf{i}\}, \quad \mathcal{C}_h := \bigcup\nolimits_{\mathbf{i}\in\mathcal{M}^h} \mathcal{C}(\mathbf{i}).$$

Then, the controllability set of the SLS (1) can be represented by

$$\mathcal{C} := \bigcup\nolimits_{h=1}^{\infty} \mathcal{C}_h.$$

In addition, for all $\mathbf{i} := (i_h, \ldots, i_2, i_1) \in \mathcal{M}^h$, if we define

$$\mathcal{R}(\mathbf{i}) := \mathrm{Im} B_\mathbf{i}, \quad \mathcal{R}_h := \bigcup\nolimits_{\mathbf{i}\in\mathcal{M}^h} \mathcal{R}(\mathbf{i}),$$

then, the reachable set can be represented by

$$\mathcal{R} := \bigcup\nolimits_{h=1}^{\infty} \mathcal{R}_h.$$

For completeness, we present [26, Theorem 1, Theorem 2].

Lemma 3:

1) [26, Theorem 1]: The SLS (1) is controllable if and only if there exist $h < \infty$ and $\mathbf{i} \in \mathcal{M}^h$ such that $\mathrm{Im} A_\mathbf{i} \subseteq \mathcal{R}(\mathbf{i})$;
2) [26, Theorem 2]:The SLS (1) is reachable if and only if there exist $h < \infty$ and $\mathbf{i} \in \mathcal{M}^h$ such that $\mathcal{R}(\mathbf{i}) = \mathbb{R}^n$.

The following proposition presents a connectoin between the controllability/reachability and the value of $\rho(\mathcal{T}_h)$.

Proposition 6: Suppose that the SLS (1) is controllable. Then, the following statements are true:

1) For any $P \in \mathbb{S}^n_{++}$, there exists $h < \infty$ such that $\rho(\mathcal{T}^h) = 0$. In other words, for such a h, the state of the closed-loop SLS (1) under the h-PHCS (8) reaches the origin within the finite time h;
2) For any $P \in \mathbb{S}^n_{++}$, there exists $k < \infty$ such that $\gamma_h^* = 0$.

Proof: The statement 1) is obvious from the definitions of the controllability and $\rho(\mathcal{T}_h)$. For the statement 2), observe that $\rho(\mathcal{T}_h) = 0$ from some $h \in \mathbb{N}_+$ implies

$$\min_{\mathbf{i}\in\mathcal{M}^h} \inf_{\mathbf{u}\in\mathbb{R}^{mh}} (A_\mathbf{i} z + B_\mathbf{i}\mathbf{u})^T P (A_\mathbf{i} z + B_\mathbf{i}\mathbf{u}) = 0, \quad \forall z \in \mathbb{R}^n. \tag{18}$$

Let $\mathbf{i}^* \in \mathcal{M}^h$ and $\mathbf{u}^* = F_{\mathbf{i}^*} z$ be an optimal solution that satisfies (18), where $F_\mathbf{i}$, $\mathbf{i} \in \mathcal{M}^h$, is defined in Proposition 1. Then, (18) can be written as

$$z^T (A_{\mathbf{i}^*} z + B_{\mathbf{i}^*} F_{\mathbf{i}^*})^T P (A_{\mathbf{i}^*} + B_{\mathbf{i}^*} F_{\mathbf{i}^*}) z = 0, \quad \forall z \in \mathbb{R}^n,$$

which ensures $A_{\mathbf{i}^*} + B_{\mathbf{i}^*} F_{\mathbf{i}^*} = 0$. This implies that there exists a zero matrix in $\mathcal{P}_k$ defined in (15). Hence, in (16), $\alpha_1, \ldots, \alpha_h \in \mathbb{R}$ can be chosen so that the left-hand side of (17) becomes a zero matrix. Thus, $\gamma_h^* = 0$, and the proof is completed. ■

Remark 1: Since the reachability is a special case of the controllability, the same result of Proposition 6 holds when the SLS (1) is reachable.

Example 3: Consider the SLS in Example 1 again. For $h = 2$, $A_{\mathbf{i}}$ and $B_{\mathbf{i}}$ are

$$A_{(1,1)} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \; A_{(1,2)} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix},$$
$$A_{(2,1)} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}, \; A_{(2,2)} = \begin{bmatrix} \frac{9}{4} & 3 \\ 0 & \frac{9}{4} \end{bmatrix},$$
$$B_{(1,1)} = \begin{bmatrix} A_1 B_1 & B_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix},$$
$$B_{(1,2)} = \begin{bmatrix} A_1 B_2 & B_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$
$$B_{(2,1)} = \begin{bmatrix} A_2 B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 3.5 & 1 \\ 3 & 0 \end{bmatrix},$$
$$B_{(2,2)} = \begin{bmatrix} A_2 B_2 & B_2 \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since $B_{(1,2)}$ and $B_{(2,1)}$ have full column rank, the SLS is reachable, and by Proposition 6, $\rho(\mathcal{T}^h) = 0$, $\gamma_h^* = 0$, and the state trajectory of the SLS under the 2-PHCS (7) reaches the origin within two steps as shown in the simulation of Example 1.

Conclusion

In this paper, we have developed a PCLF theorem and computational algorithms to check the stabilizability of SLSs. The PCLF theorem claims that the existence of a PCLF is a necessary and sufficient condition for stabilizability of SLSs. Agendas for future researches are 1) the defectiveness of controlled SLSs [27], that is, determining whether or not there exists a hybrid-control policy such that the exponential stabilizing rate $c^* \in \mathbb{R}_+$ is achieved, and 2) connections between the stabilizability of autonomous SLSs and controlled SLSs. We remain these problems for subjects of future researches.

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