On the Graph Control Lyapunov Function for Stabilization of Discrete-Time Switched Linear Systems

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***Abstract*— The goal of this paper is to study switching stabilization problems for discrete-time switched linear systems (SLSs). To analyze the switching stabilizability, we introduce the notion of graph control Lyapunov functions (GCLFs). The GCLF is a set of Lyapunov functions which satisfy several Lyapunov inequalities associated with a weighted directed graph (digraph). Each Lyapunov function represents each node in the digraph, and each Lyapunov inequality represents edges connecting a node and its out-neighbors. The weight of each directed edge indicates the decent rate of the Lyapunov functions from the tail to the head of the edge. The GCLF is an extension of the graph Lyapunov function for stability analysis of SLSs to switching stabilization problems, and unifies several control Lyapunov functions in a single framework. In this paper, we proposed a GCLF theorem, which provides a necessary and sufficient condition for switching stabilizability and develop computational algorithms to the stabilizability.**

I. Introduction

The switched linear systems (SLSs) are a class of hybrid systems where the system dynamics matrix is switched within a finite set of subsystem matrices. For SLSs, a fundamental problem is to analyze their stability/stabilizability and design the stabilizing controls [1]. The most popular approach is the use of Lyapunov theorems. The simplest one is the common quadratic Lyapunov function (CQLF), which generally leads to conservative conditions.

To reduce the conservatism, more general Lyapunov functions have been studied so far, for instance, multiple Lyapunov functions [2], piecewise quadratic Lyapunov functions (PWQLF) [3]–[9], polyhedral or polytopic Lyapunov functions [10], sum-of-squares polynomial Lyapunov functions [11], convex hull Lyapunov functions [12], [13], and switched Lyapunov functions [14], [15]. Other approaches include the joint spectral radius (JSR) [16], the generating functions approach [17].

Another progress of classical Lyapunov methods is the so-called non-monotonic Lyapunov functions. The value of such functions may not necessarily decrease at each time step along the state trajectories as in the case of classical Lyapunov functions. To the authors' knowledge, the non-monotonic Lyapunov functions were first proposed in [18], [19] for nonlinear and switching systems, recently generalized in [20] to the graph Lyapunov functions, where a finite set of non-monotonic Lyapunov functions are jointly used to certify the stability. A special class of the non-monotonic Lyapunov functions is the periodic or aperiodic Lyapunov functions (PLF or APLF) [21]–[24] whose value decreases periodically or aperiodically in time.

The goal of this paper is to extend the graph Lyapunov function (GLF) [20] for stability analysis of discrete-time SLSs under arbitrary switchings to stabilization problems. To this end, we introduce the notion of the graph control Lyapunov function (GCLF). The GCLF is a set of Lyapunov functions which satisfy several Lyapunov inequalities associated with a weighted digraph. Each Lyapunov function represents each node in the digraph, and each Lyapunov inequality represents the edges connecting a node and its out-neighbors. The weight of each directed edge indicates the decent rate of the Lyapunov functions from the tail to the head of the edge. In this paper, we derive a GCLF theorem, which provides a necessary and sufficient condition for the switching stabilizability and unifies several control Lyapunov theorems. In addition, by means of the GCLF theorem, an exponential convergence rate can be obtained as well. Computational algorithms are proposed for efficient tests of stabilizability and control designs.

II. Preliminaries

A. Notation

The adopted notation is as follows: $\mathbb{N}$ and $\mathbb{N}_+$: sets of nonnegative integers and positive integers, respectively; $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$: sets of real numbers, nonnegative real numbers, and positive real numbers, respectively; $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$: set of all $n\times m$ real matrices; A^T: transpose of matrix A; $A\succ 0$ ($A\prec 0$, $A\succeq 0$, and $A\preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix A; I_n: $n\times n$ identity matrix; $||\cdot||$: Euclidean norm of a vector or spectral norm of a matrix; $\mathbb{S}^n$, $\mathbb{S}^n_+$, and $\mathbb{S}^n_{++}$: sets of symmetric $n\times n$ matrices, positive semi-definite matrices, and positive definite matrices, respectively; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$: minimum and maximum eigenvalues of symmetric matrix A, respectively; $e_j\in\mathbb{R}^n$, $j\in\{1, 2, \ldots, n\}$ is the j-th unit vector (all components are 0 except for the j-th component which is 1); for a matrix A, we write $[A]_{ij}$ to denote the matrix entry in the i-th row and j-th column; $\mathrm{diag}(M_1, M_2, \ldots, M_n)$: the matrix with the matrices $M_1, M_2, \ldots, M_n$ on the block-diagonal and zeros elsewhere.

B. Graph theory

A directed graph or digraph $G(\mathcal{V}, \mathcal{E})$ is defined by a set of the nodes $\mathcal{V} := \{1, 2, \ldots, m\}$ and a set of ordered pairs

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of the nodes $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ which represents the set of directed edges, where $(j, i) \in \mathcal{E}$ indicates the edge from the node $j \in \mathcal{V}$ to the node $i \in \mathcal{V}$. For a given node $j \in \mathcal{V}$ of a digraph, $\mathcal{N}_j^- := \{i \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ is called the in-neighbor of the node j, $\mathcal{N}_j^+ := \{i \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ is called the out-neighbor of the node j, $\left|\mathcal{N}_j^-\right|$ is the number of incoming edges adjacent to j and is called the indegree of the node j, and $\left|\mathcal{N}_j^+\right|$ is the number of outgoing edges adjacent to it and is called its outdegree of the node j. A node of a digraph with zero indegree is called a source and a node with zero outdegree is called a sink. The adjacency matrix $E \in \mathbb{R}^{m\times m}$ of $G(\mathcal{V}, \mathcal{E})$ is defined as a matrix with $[E]_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $[E]_{ij} = 0$ otherwise. A walk in a digraph $G(\mathcal{V}, \mathcal{E})$ is a sequence of nodes $\mathcal{W} := (v_1, v_2, \dots, v_k) \in \mathcal{V}^k$ such that $(v_i, v_{i+1}) \in \mathcal{E}$, $i \in \{1, 2, \dots, k-1\}$. The length of the walk $\mathcal{W}$, denoted by $|\mathcal{W}|$, is the number of edges, i.e., $|\mathcal{W}| = k - 1$. A closed walk in $G(\mathcal{V}, \mathcal{E})$ is a walk $\mathcal{W} := (v_1, v_2, \dots, v_k) \in \mathcal{V}^k$ in $G(\mathcal{V}, \mathcal{E})$ such that $v_k = v_1$. A path $\mathcal{P} := (v_1, v_2, \dots, v_k) \in \mathcal{V}^k$ in a digraph $G(\mathcal{V}, \mathcal{E})$ is a walk in $G(\mathcal{V}, \mathcal{E})$ such that $v_i = v_j$ if and only if $i = j$, $(i, j) \in \{1, 2, \dots, k\}^2$. A simple cycle $\mathcal{C} := (v_1, v_2, \dots, v_k) \in \mathcal{V}^k$ in a digraph $G(\mathcal{V}, \mathcal{E})$ is a closed walk in $G(\mathcal{V}, \mathcal{E})$ such that $k \geq 3$, $v_i = v_j$ if and only if $i = j$, $(i, j) \in \{1, 2, \dots, k-1\}^2$, and $v_k = v_1$. A loop in $G(\mathcal{V}, \mathcal{E})$ is an edge $(v_1, v_2) \in \mathcal{E}$ such that $v_1 = v_2$. In this paper, we will regard the loops as simple cycles with two nodes. A digraph $G(\mathcal{V}, \mathcal{E})$ is strongly connected if for every $v_1 \in \mathcal{V}$ and $v_2 \in \mathcal{V}$, there is a path starting at v_1 and ending at v_2. A subgraph $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ of $G(\mathcal{V}, \mathcal{E})$ is a strongly connected component (SCC) of $G(\mathcal{V}, \mathcal{E})$ if $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is strongly connected and no other strongly connected subgraph contains $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$.

Lemma 1 ([25, page 17]): Every digraph $G(\mathcal{V}, \mathcal{E})$ can be partitioned into SCCs $G(\mathcal{V}_1, \mathcal{E}_1), \dots, G(\mathcal{V}_k, \mathcal{E}_k)$ with disjoint sets of vertices $\mathcal{V}_1, \dots, \mathcal{V}_k$.

Definition 1: For any digraph $G(\mathcal{V}, \mathcal{E})$, a SCC $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ with no outgoing edges from the nodes of $G(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is called a terminal SCC.

Lemma 2 ([25, page 17]): For any digraph $G(\mathcal{V}, \mathcal{E})$, there exists a terminal SCC.

Given a digraph $G(\mathcal{V}, \mathcal{E})$, define a mapping $w : \mathcal{E} \to \mathbb{R}$, where $w(j, i)$, $(j, i) \in \mathcal{E}$, represents the weight of the edge $(j, i) \in \mathcal{E}$. The weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha_j \in \mathbb{R}_+$, $j \in \mathcal{V}$, is defined so that $w(j, i) = \alpha_j$ if $(j, i) \in \mathcal{E}$, and $w(j, i) = 0$ otherwise. Every notion for the digraph can be similarly applied to the weighted digraph. The adjacency matrix $E \in \mathbb{R}^{m\times m}$ of the weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ is defined as the matrix with $[E]_{ij} = \alpha_j$ if $(j, i) \in \mathcal{E}$ and $[E]_{ij} = 0$ otherwise. The gain $g(\mathcal{W})$ of the walk $\mathcal{W} = (v_0, v_1, \dots, v_{k-1}) \in \mathcal{V}^k$ in the weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ is defined by the product of the weights of the edges along the walk, i.e., $g(\mathcal{W}) := \prod_{t=0}^{k-2} w(v_t, v_{t+1})$. The cycle gain $g(\mathcal{C})$ of the simple cycle $\mathcal{C}$ is defined in a similar way.

C. Problem formulation

Consider the discrete-time (autonomous) SLS

$$x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = z \in \mathbb{R}^n, \tag{1}$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state, $\sigma(k) \in \mathcal{M} := \{1, 2, \dots, M\}$ is called the mode, and A_i, $i \in \mathcal{M}$, are the subsystem matrices. An infinite switching sequence will be denoted by $\sigma := (\sigma(0), \sigma(1), \dots)$. Starting from $x(0) = z \in \mathbb{R}^n$ and under the switching sequence σ, the trajectory of the SLS (1) is denoted by $x(k; z, \sigma)$, $k \in \mathbb{N}$. In this paper, we assume that the switching sequence σ can be determined by the designer, i.e., σ is the control input. The notion of exponential switching stabilizability is presented below.

Definition 2 (Exponential switching stabilizability): The SLS (1) is called exponentially switching stabilizable with the parameters K and ρ if there exist $K \in [0, \infty)$ and $\rho \in [0, 1)$ such that starting from any initial state $x(0) = z \in \mathbb{R}^n$, there exists a switching sequence σ for which the trajectory $x(k; z, \sigma)$ satisfies

$$||x(k; z, \sigma)|| \leq K\rho^k ||z||, \quad \forall k \in \mathbb{N}, \tag{2}$$

The exponential stabilizing rate ρ^* is the infimum of all ρ for which (2) holds. Throughout the paper, any $\rho \in [0, 1)$ satisfying (2) will be called *the exponential convergence rate*. In addition, we often call the exponential switching stabilizability simply by exponential stabilizability or stabilizability if there is no confusion. The problem addressed in this paper is stated as follows.

Problem 1 (Stabilizability problem): Determine the stabilizability of the SLS (1).

As a byproduct of our development, we also solve the following design problem.

Problem 2 (Control design problem): If the SLS (1) is stabilizable, then find a state-feedback switching policy under which the SLS (1) is stable.

It is trivial that if one of the subsystem matrices is Schur stable, then the SLS (1) is stabilizable. To avoid the triviality, the following assumption is made in this paper.

Assumption 1: Each of the subsystem matrix A_i, $i \in \mathcal{M}$, is not Schur stable.

As a result, we have

$$\tau := \max_{i\in\mathcal{M}} \|A_i\| \geq 1. \tag{3}$$

Lastly, some notions in [20] will be briefly reviewed. Hereafter, we will think of the set of system submatrices $\mathcal{A} := \{A_1, \dots, A_M\}$ as a finite *alphabet* and we will refer to a finite product of matrices from this set as a *word*. The set of all words $A_{i_{k-1}} \cdots A_{i_1} A_{i_0}$ of length $k \in \mathbb{N}$ is denoted by $\mathcal{A}^k := \{A_{i_{k-1}} \cdots A_{i_0}\}_{(i_0, \cdots, i_{k-1}) \in \mathcal{M}^k}$ with $\mathcal{A}^0 := \{I_n\}$, the set of all finite words is denoted by $\mathcal{A}^* := \bigcup_{h\in\mathbb{Z}_+} \mathcal{A}^h$, and the set of all words with length from $k_1 \in \mathbb{N}$ to $k_2 \in \mathbb{N}$, $k_2 \geq k_1$, is denoted by $\mathcal{A}^{[k_1, k_2]} := \bigcup_{i\in\{k_1, k_1+1, \dots, k_2\}} \mathcal{A}^i$.

III. Graph control Lyapunov function

In this section, a formal definition of the graph control Lyapunov function and the corresponding stabilization theorems are presented. The *graph control Lyapunov function (GCLF)* is defined as follows.

Definition 3 (Graph control Lyapunov function (GCLF)): Let a weighted digraph $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ with the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$, be given. A set of nonnegative continuous functions $V_i : \mathbb{R}^n \to \mathbb{R}_+,\, i \in \mathcal{V}$, satisfying

$$\underline{\kappa}_i \|z\|^2 \le V_i(z) \le \overline{\kappa}_i \|z\|^2, \quad \forall z \in \mathbb{R}^n, \tag{4}$$

for some positive constants $\underline{\kappa}_i,\, \overline{\kappa}_i \in \mathbb{R}_{++},\, i \in \mathcal{V}$, will be called a graph control Lyapunov function (GCLF) associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ if

1) there exist $\mathcal{A}_{j\to i} \subset \mathcal{A}^*,\, (j,\, i) \in \mathcal{E}$, such that the inequalities

$$\min_{i\in\mathcal{N}_j^+} \min_{A\in\mathcal{A}_{j\to i}} V_i(Az) \le \alpha_j V_j(z),$$
$$\forall z \in \mathbb{R}^n \backslash \{0_n\},\, j \in \mathcal{V} \tag{5}$$

associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ are satisfied;
2) all the simple cycles in $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ (including the loops) have the cycle gains strictly less than 1;
3) $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ has no sink.

When the GCLF $\{V_i\}_{i\in\mathcal{V}}$ consists of quadratic functions, then it will be called a quadratic GCLF (QGCLF).

Remark 1: The role of the weighted digraph $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ in Definition 3 is to provide information for constructing the inequalities in (5). In other words, $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ represents a network that defines how the Lyapunov function values in different nodes should evolve along walks when certain switching sequences associated with the walks are applied to the SLS.

Remark 2: For small scale weighted digraphs, the second condition of Definition 3 can be easily checked by hand. For larger digraphs, it is almost impossible to check the second condition manually. However, there are graph theoretic algorithms for enumerations of simple cycles.

Example 1: Consider the SLS (1), and suppose that there exist nonnegative continuous functions $V_1,\, V_2,\, V_3,\, V_4$, satisfying (4) in Definition 3, and the words $\mathcal{A}_{1\to 2},\, \mathcal{A}_{2\to 3},\, \mathcal{A}_{2\to 4},\, \mathcal{A}_{3\to 4},\, \mathcal{A}_{4\to 1} \subset \mathcal{A}^*$, such that

$$\min_{A\in\mathcal{A}_{1\to 2}} V_2(Az) \le \frac{1}{2} V_1(z),$$
$$\min\left\{\min_{A\in\mathcal{A}_{2\to 3}} V_3(Az),\ \min_{A\in\mathcal{A}_{2\to 3}} V_3(Az)\right\} \le 2V_2(z),$$
$$\min_{A\in\mathcal{A}_{3\to 4}} V_4(Az) \le \frac{1}{2} V_3(z), \qquad \min_{A\in\mathcal{A}_{4\to 1}} V_1(Az) \le \frac{1}{3} V_4(z). \tag{6}$$

In the sense of Definition 3, the above inequalities induce the digraph $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ shown in Figure 1 with the node set $\mathcal{V} = \{1,\, 2,\, 3,\, 4\}$ and the edge set $\mathcal{E} = \{(1,\, 2),\, (2,\, 3),\, (2,\, 4),\, (3,\, 4),\, (4,\, 1)\}$. The digraph has two simple cycles $\mathcal{C}_1 = (1,\, 2,\, 4,\, 1)$ and $\mathcal{C}_2 = (1,\, 2,\, 3,\, 4,\, 1)$, and the cycle gains can be calculated as

$$g(\mathcal{C}_1) = \alpha_1\alpha_2\alpha_4 = \frac{1}{2} \times 2 \times \frac{1}{3} = \frac{1}{3} < 1,$$
$$g(\mathcal{C}_2) = \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{1}{2} \times 2 \times 1 \times \frac{1}{3} = \frac{1}{3} < 1.$$

Since the digraph $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ in Figure 1 does not have a sink, and all the simple cycles have gains less than one, by Definition 3, $\{V_1,\, V_2,\, V_3,\, V_4\}$ is a GCLF.

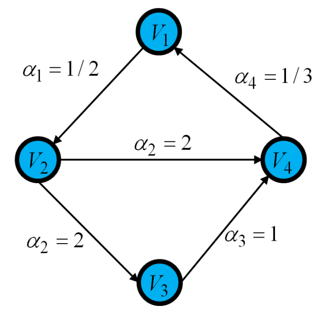

Fig. 1. Example 1. Digraph $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ associated with the inequalities in (6).

An example of the digraphs with no sink is a strongly connected digraph. The following result shows that the digraph of a GCLF can be assumed without loss to be strongly connected.

Proposition 1: Assume that $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with the weighted digraph $G(\mathcal{V},\,\mathcal{E},\,\alpha)$, and let $G(\bar{\mathcal{V}},\,\bar{\mathcal{E}},\,\alpha)$ be a terminal SCC of $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ (see Definition 1). Then, $\{V_i\}_{i\in\bar{\mathcal{V}}}$ is also a GCLF associated with $G(\bar{\mathcal{V}},\,\bar{\mathcal{E}},\,\alpha)$.

Proof: Since the terminal SCC $G(\bar{\mathcal{V}},\,\bar{\mathcal{E}},\,\alpha)$ does not have a sink and $\mathcal{N}_i^+ \subseteq \bar{\mathcal{V}}$ for $i \in \bar{\mathcal{V}}$, all the conditions in Definition 3 are satisfied. This completes the proof. ■

For small scale digraphs, some algorithms are available to enumerate all the simple cycles, for example, those in [26] and the CIRCUIT-FINDING ALGORITHM in [27]. For large scale digraphs, we can use the simplified GCLF theorem whichs lead to numerical algorithms easier to solve.

Proposition 2: Let a weighted digraph $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ with the parameters $\alpha_i \in [0,\, 1),\, i \in \mathcal{V}$, be given. The set of functions $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ if all the conditions of Definition 3 except for the part 3) hold.

Proof: Since $\alpha_j \in [0,\, 1)$, all the simple cycles of $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ have cycle gains less than one. The proof is completed using Definition 3. ■

Given a walk $\mathcal{W} = (v_0,\, v_1,\, \ldots)$, define $\mathcal{W}_{[a,\, b]} := (v_a,\, \ldots,\, v_b)$ for $a \le b,\, a, b \in \mathbb{N}$. A decomposition of $\mathcal{W}$ is defined as a sequence of walks $(\mathcal{W}_1,\, \mathcal{W}_2,\, \ldots)$ such that $\mathcal{W}_1 = \mathcal{W}_{[i_1,\, i_2]},\, \mathcal{W}_2 = \mathcal{W}_{[i_2,\, i_3]}, \ldots$ and $0 = i_1 < i_2 < \cdots$. The proof of the main result depends on the following lemma, which establishes the fact that the gain of a walk can be expressed as the product of gains of simple cycles and a path in the given digraph.

Lemma 3: Suppose that $\mathcal{W} = (v_0,\, v_1,\, \ldots,\, v_{t-1})$ is a walk in $G(\mathcal{V},\,\mathcal{E},\,\alpha)$. Then, $g(\mathcal{W})$ can be expressed as

$$g(\mathcal{W}) = g(\mathcal{P}) \prod_{i=1}^{h} g(\mathcal{C}_i),$$

where $h \in \mathbb{N}_+,\, \mathcal{C}_p,\, p \in \{1,\, 2,\, \ldots,\, h\}$, are simple cycles and $\mathcal{P}$ is a path, such that $|\mathcal{W}| = |\mathcal{P}| + \sum_{p=1}^{h} |\mathcal{C}_p|$.

Proof: If $\mathcal{W} = (v_0, v_1, \ldots, v_{t-1}) =: \mathcal{W}^{[1]}$ is not a path, then there exists a simple cycle $\mathcal{C}_1$ in $\mathcal{W}$. Remove the simple cycle $\mathcal{C}_1$ and get a shorter walk $\mathcal{W}^{[2]}$. For $p \in \mathbb{N}_+$, if $\mathcal{W}^{[p]}$ is not a path, then one can remove a simple cycle $\mathcal{C}_p$ and get a new walk $\mathcal{W}^{[p+1]}$. Noting that the initial walk $\mathcal{W}$ is finite and by the induction argument, we obtain a decomposition of $\mathcal{W}$ which consists of a finite sequence of simple cycles (including loops) $\mathcal{C}_p$, $p \in \{1, 2, \ldots, h\}$, and a path $\mathcal{P}$. Therefore, the gain of the walk $g(\mathcal{W})$ can be expressed as a product of the gains of the simple cycles and the path. This completes the proof. ■

In what follows, it will be proved that the GCLF can be used to certify stabilizability. For easy reference, we formally define the state-feedback switching policy, the corresponding walk on the given digraph, and the switching sequence.

Definition 4: Let a weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$, be given. Suppose that $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with $G(\mathcal{V}, \mathcal{E}, \alpha)$. For any $x \in \mathbb{R}^n$ and $j \in \mathcal{V}$, define the sets

$$I(j, x) := \arg\min_{i\in\mathcal{N}_j^+} \min_{A\in\mathcal{A}_{j\to i}} V_i(Ax), \quad \forall j \in \mathcal{V},$$

and

$$\Phi(j, i, x) := \arg\min_{A\in\mathcal{A}_{j\to i}} V_i(Ax), \quad \forall j \in \mathcal{V}, i \in \mathcal{N}_j^+.$$

Then, the set defined as

$$\begin{aligned}\sigma(j, i, x) :=\{&(i_0, \ldots, i_{h-1}) \in \mathcal{M}^h : \\ &A_{i_{h-1}} \cdots A_{i_1} A_{i_0} \in \Phi(j, i, x), h \in \mathbb{N}_+\}\end{aligned} \tag{7}$$

is called a state-feedback switching policy associated with the GCLF $\{V_i\}_{i\in\mathcal{V}}$. For any $\xi_0 = z \in \mathbb{R}^n$ and $j_0 \in \mathcal{V}$, if the sequences $\{\xi_t\}_{t=0}^\infty$ and $\{j_t\}_{t=0}^\infty$ are defined by the inclusions

$$\begin{aligned}&j_{t+1} \in I(j_t, \xi_t),\\ &(i_0, \ldots, i_{h-1}) \in \sigma(j_t, j_{t+1}, \xi_t),\\ &\xi_{t+1} = A_{i_{h-1}} \cdots A_{i_1} A_{i_0} \xi_t, \quad t \in \mathbb{N},\end{aligned} \tag{8}$$

respectively, then ξ_t will be called the state corresponding to the node j_t, and the sequence of nodes $\mathcal{W}_\infty = (j_0, j_1, j_2, \ldots)$ represents a walk in $G(\mathcal{V}, \mathcal{E}, \alpha)$ and will be called a walk associated with the switching policy (7). The corresponding switching sequence is

$$\sigma(j_0, z) := (\sigma(j_0, j_1, \xi_0), \sigma(j_1, j_2, \xi_1), \sigma(j_2, j_3, \xi_2), \ldots). \tag{9}$$

In the following theorem, we show that if the SLS (1) admits a GCLF, then the switching sequence (9) exponentially stabilizes the SLS (1).

Theorem 1 (GCLF theorem): If $\{V_i\}_{i\in\mathcal{V}}$ is a GCLF associated with the digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ and the parameters $\alpha \in \mathbb{R}_+^{|\mathcal{V}|}$, then the SLS (1) under the switching sequence (9) is exponentially stable with the parameters

$$K = \tau^L \left(\delta \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right)^{\frac{1}{2}} \gamma^{-\frac{\beta+1}{2\eta}}, \quad \phi = \gamma^{\frac{1}{2\eta L}}, \tag{10}$$

where $\tau := \max_{\mu\in\mathcal{M}} \|A_\mu\|$, $L := \max\{|\mathcal{A}_{j\to i}| : (j, i) \in \mathcal{E}\}$, η and γ are the maximum length and gain of simple cycles, β and δ are the maximum length and gain of paths, respectively. In particular, if $\alpha_i \in [0, 1)$, $i \in \mathcal{V}$, then the SLS (1) under (9) is exponentially stable with the parameters

$$K = \tau^L \left(\frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{(\min_{i\in\mathcal{V}} \underline{\kappa}_i)(\max_{i\in\mathcal{V}} \alpha_i)}\right)^{1/2},$$

$$\phi = \left(\max_{i\in\mathcal{V}} \alpha_i\right)^{1/2L}.$$

Proof: Let $j_0 \in \mathcal{V}$ and $\xi_0 \in \mathbb{R}^n$ be arbitrary. Define the walk $\mathcal{W} = (j_0, j_1, \ldots, j_t)$, and the sequence of times $\{k_t\}_{t=0}^\infty$ by $k_0 = 0$, and $k_{t+1} = k_t + |\Phi(j_t, j_{t+1}, \xi_t)|$ for $t \in \mathbb{N}_+$, where $\{\xi_t\}_{t=0}^\infty$ is the subsequence of the states defined in (8) so that $\xi_t = x(k_t; z, \sigma)$, $\forall t \in \mathbb{N}$. Then, by the definition of the switching policy in (7), the inequalities in (5) are satisfied for all $z = \xi_t$, $t \in \mathbb{N}_+$, and the Lyapunov function value long the trajectory $\{\xi_t\}_{t=0}^\infty$ satisfies $V_{j_t}(\xi_t) \le g(\mathcal{W}) V_{j_0}(\xi_0)$, $\forall t \in \mathbb{N}_+$. By Lemma 3, there exists simple cycles $\mathcal{C}_p$, $p \in \{1, 2, \ldots, h\}$, $h \in \mathbb{N}_+$, and a path $\mathcal{P}$ such that $g(\mathcal{W}) = g(\mathcal{P}) \prod_{p=1}^h g(\mathcal{C}_p)$ and $|\mathcal{W}| = |\mathcal{P}| + \sum_{p=1}^h |\mathcal{C}_p|$. Thus, we have

$$V_{j_t}(\xi_t) \le g(\mathcal{P}) \prod_{p=1}^h g(\mathcal{C}_p) V_{j_0}(\xi_0) \le \delta\gamma^h V_{j_0}(\xi_0),$$

where γ is the maximum gain of simple cycles, and δ is the maximum gain of paths.

Noting that $V_{j_0}(\xi_0) \le \bar{\kappa}_{j_0} \|\xi_0\|^2$ and $V_{j_t}(\xi_t) \ge \min_{i\in\mathcal{V}} \underline{\kappa}_i \|\xi_t\|^2$ and combining them, we have

$$\|\xi_t\|^2 \le \delta \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i} \gamma^h \|\xi_0\|^2.$$

Since $h \ge \frac{t-\beta}{\eta}$ and $\gamma < 1$, it follows that

$$\|\xi_t\|^2 \le \delta \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i} \gamma^{-\frac{\beta}{\eta}} \gamma^{\frac{t}{\eta}} \|\xi_0\|^2,$$

which gives $\|\xi_t\| \le r c^t \|z\|$, where

$$r = \left(\delta \frac{\max_{i\in\mathcal{V}} \bar{\kappa}_i}{\min_{i\in\mathcal{V}} \underline{\kappa}_i}\right)^{\frac{1}{2}} \gamma^{-\frac{\beta}{2\eta}}, \quad c = \gamma^{\frac{1}{2\eta}}.$$

To obtain an exponential convergence rate of the SLS, for any $k \in \mathbb{N}$, select $t \in \mathbb{N}$ such that $k \in [k_t, k_{t+1})$. Then, we have

$$\begin{aligned}\|x(k; z, \sigma)\| &= \|x(k_t + k - k_t; z, \sigma)\|\\ &\le \tau^{(k-k_t)} \|x(k_t; z, \sigma)\|\\ &\le \tau^L \|\xi_t\|\\ &\le \tau^L r c^t \|z\|,\end{aligned}$$

where we have used $\tau \ge 1$ in (3). Again, as $t \ge (k/L) - 1$ and $c < 1$, we obtain

$$\|x(k; z, \sigma)\| \le \tau^L r c^{\left(\frac{k}{L}-1\right)} \|z\| = \frac{\tau^L r}{c} c^{k/L} \|z\|.$$

Therefore, the SLS (1) under the switching sequence (9) is exponentially stable with the parameters in (10). The proof for the case $\alpha_i \in [0, 1)$, $i \in \mathcal{V}$, is similar, so omitted for brevity. ■

The result proves that the existence of a GCLF is sufficient condition for the stabilizability of the SLS (1). Next, we prove that it is also a necessary condition.

Theorem 2 (Converse GCLF theorem): Suppose that the SLS (1) is exponentially switching stabilizable. Let a digraph $G(\mathcal{V},\,\mathcal{E})$ with no sink and the positive definite matrices $P_i \succ 0,\, i \in \{1, 2, \dots, |\mathcal{V}|\}$, be arbitrarily given. Then, the set of quadratic functions $V_i(x) = x^T P_i x,\, i \in \mathcal{V}$, is a QGCLF associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ with some parameters $\alpha_i \in \mathbb{R}_+,\, i \in \mathcal{V}$. In other words, there exist $\mathcal{A}_{j\to i} \subset \mathcal{A}^*,\, (j,\, i) \in \mathcal{E}$, such that the inequalities (5) associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ is satisfied, and all the simple cycles of $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ have cycle gains less than one.

Proof: Consider the set of words $\mathcal{A}_{j\to i} = \mathcal{A}^{h_j}$ with $h_j \in \mathbb{N}_+$ for all $(j,\, i) \in \mathcal{E}$. Since the SLS (1) is exponentially stabilizable, there exist $K \in [0,\, \infty)$ and $\phi \in [0,\, 1)$ such that (2) holds. Thus, we have

$$
\begin{aligned}
&\min_{i\in\mathcal{N}_j^+} \min_{(\mu_0,\cdots,\,\mu_{h_j-1})\in\mathcal{M}^{h_j}} V_i(A_{\mu_{h_j-1}}\cdots A_{\mu_0} z)\\
&\le \min_{i\in\mathcal{N}_j^+} \min_{(\mu_0,\cdots,\,\mu_{h_j-1})\in\mathcal{M}^{h_j}} \|A_{\mu_{h_j-1}}\cdots A_{\mu_0} z\|^2 \lambda_{\max}(P_i)\\
&\le \min_{i\in\mathcal{N}_j^+} \lambda_{\max}(P_i) K^2 \phi^{2h_j}\|z\|^2\\
&\le \alpha_j V_j(z), \quad \forall j \in \mathcal{V},
\end{aligned}
$$

where

$$
\alpha_j = \frac{\min_{i\in\mathcal{N}_j^+}\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)} K^2 \phi^{2h_j}.
$$

By increasing h_j, we can make $\alpha_j < 1$ for all $j \in \mathcal{V}$. Therefore, $\{V_i\}_{i\in\mathcal{V}}$ is a QGCLF. ■

The following result proves that the existence of a QGCLF for the SLS (1) ensures the existence of a piecewise quadratic GCLF.

Proposition 3: If the SLS (1) admits a QGCLF $\{V_i\}_{i\in\mathcal{V}}$ associated with $G(\mathcal{V},\,\mathcal{E},\,\alpha)$ with some parameters $\alpha_j \in \mathbb{R}_+,\, j \in \mathcal{V}$, then there exists a piecewise quadratic GCLF.

Proof: Consider a subgraph $G(\tilde{\mathcal{V}},\,\tilde{\mathcal{E}})$ of $G(\mathcal{V},\,\mathcal{E})$, which is obtained by eliminating nodes that are sources. Then, each node of $G(\tilde{\mathcal{V}},\,\tilde{\mathcal{E}})$ is neither a sink nor a source. The inequalities (5) in Definition 3 leads to

$$
\begin{aligned}
&\min_{i\in\mathcal{N}_j^+}\min_{A\in\mathcal{A}_{j\to i}} V_i(Az) \le \left(\max_{i\in\tilde{\mathcal{V}}}\alpha_i\right) V_j(z),\\
&\forall z\in\mathbb{R}^n\backslash\{0_n\},\, j\in\tilde{\mathcal{V}}\\
&\Rightarrow \min_{(i,\,l)\in\tilde{\mathcal{E}}}\min_{A\in\mathcal{A}_{l\to i}} V_i(Az) \le \left(\max_{i\in\tilde{\mathcal{V}}}\alpha_i\right) V_j(z),\\
&\forall z\in\mathbb{R}^n\backslash\{0_n\},\, j\in\tilde{\mathcal{V}}.
\end{aligned}
$$

Thus, the piecewise quadratic function $V_{\min}(z) := \min_{i\in\tilde{\mathcal{V}}} V_i(z)$ satisfies

$$
\begin{aligned}
&\min_{(i,\,j)\in\tilde{\mathcal{E}}}\min_{A\in\mathcal{A}_{j\to i}} V_i(Az) \le \left(\max_{i\in\tilde{\mathcal{V}}}\alpha_i\right) V_{\min}(z),\\
&\forall z\in\mathbb{R}^n\backslash\{0_n\}.
\end{aligned}
$$

Since $\min_{(i,\,j)\in\tilde{\mathcal{E}}}\min_{A\in\mathcal{A}_{j\to i}} V_i(Az) \ge \min_{(i,\,j)\in\tilde{\mathcal{E}}}\min_{A\in\mathcal{A}_{j\to i}} V_{\min}(Az)$, we have

$$
\begin{aligned}
&\min_{A\in\mathcal{A}_{j\to i},\,(i,\,j)\in\tilde{\mathcal{E}}} V_{\min}(Az) \le \left(\max_{i\in\tilde{\mathcal{V}}}\alpha_i\right) V_{\min}(z),\\
&\forall z\in\mathbb{R}^n\backslash\{0_n\}.
\end{aligned}
$$

Thus, $V_{\min}(z)$ is a piecewise quadratic GCLF associated with $G(\bar{\mathcal{V}},\,\bar{\mathcal{E}})$ with the parameter $\left(\max_{i\in\tilde{\mathcal{V}}}\alpha_i\right)$, where $\bar{\mathcal{V}} = \{1\}$ and $\bar{\mathcal{E}} = \{(1,\,1)\}$. ■

IV. Numerical Computation

Consider the digraph $G(\mathcal{V},\,\mathcal{E})$, the set of matrices $\{P_i\}_{i\in\mathcal{V}} \subset \mathbb{S}^n_{++}$, and the set of quadratic functions $\{V_i\}_{i\in\mathcal{V}}$ with $V_i(z) := z^T P_i z,\, i \in \mathcal{V}$. Define

$$
\begin{aligned}
&\alpha_{\min,\,j} := \min \alpha \quad \text{subject to}\\
&\min_{i\in\mathcal{N}_j^+}\min_{A\in\mathcal{A}_{j\to i}} z^T A^T P_i A z \le \alpha z^T P_j z, \quad \forall z\in\mathbb{R}^n.
\end{aligned} \tag{11}
$$

To compute an over approximation of (11), we consider the following problem.

Problem 3: Let $G(\mathcal{V},\,\mathcal{E})$, $\{P_i\}_{i\in\mathcal{V}} \subset \mathbb{S}^n_{++}$, and $\mathcal{A}_{j\to i} \subset \mathcal{A}^*,\, (i,\, j) \in \mathcal{E}$ be given. For $j \in \mathcal{V}$, solve the semidefinite programming (SDP) problem associated with $G(\mathcal{V},\,\mathcal{E})$

$$
\begin{aligned}
&\tilde{\alpha}_{\min,\,j} := \min_{\lambda_{(A,\,i,\,j)}\in\mathbb{R},\,\alpha_j\in\mathbb{R}} \alpha_j \quad \text{subject to}\\
&\sum_{A\in\mathcal{A}_{j\to i},\,i\in\mathcal{N}_j^+} \lambda_{(A,\,i,\,j)} A^T P_i A \preceq \alpha_j P_j,\\
&\sum_{A\in\mathcal{A}_{j\to i},\,i\in\mathcal{N}_j^+} \lambda_{(A,\,i,\,j)} = 1, \quad \lambda_{(A,\,i,\,j)} \ge 0,\\
&\forall A\in\mathcal{A}_{j\to i},\, i\in\mathcal{N}_j^+.
\end{aligned} \tag{12}
$$

Proposition 4: $\alpha_{\min,\,j} \le \tilde{\alpha}_{\min,\,j}$ for all $j \in \mathcal{V}$.

Proof: The proof is completed by showing

$$
\begin{aligned}
\alpha_{\min,\,j} &= \sup_{z\in\mathbb{R}^n,\,V_j(z)=1}\min_{i\in\mathcal{N}_j^+}\min_{A\in\mathcal{A}_{j\to i}} V_i(Az)\\
&= \sup_{z\in\mathbb{R}^n,\,z^TP_jz=1}\min_{\lambda\in\Delta}\sum_{A\in\mathcal{A}_{j\to i},\,i\in\mathcal{N}_j^+}\lambda_{(A,\,i,\,j)} z^T A^T P_i A z\\
&\le \min_{\lambda\in\Delta}\sup_{z\in\mathbb{R}^n,\,z^TP_jz=1}\sum_{A\in\mathcal{A}_{j\to i},\,i\in\mathcal{N}_j^+}\lambda_{(A,\,i,\,j)} z^T A^T P_i A z\\
&= \min_{\lambda\in\Delta}\sup_{z\in\mathbb{R}^n,\,z^TP_jz=1} c \quad \text{subject to}\\
&\quad \sum_{A\in\mathcal{A}_{j\to i},\,i\in\mathcal{N}_j^+}\lambda_{(A,\,i,\,j)} z^T A^T P_i A z \le c\\
&= \min_{\lambda\in\Delta} c \quad \text{subject to}\\
&\quad \sum_{A\in\mathcal{A}_{j\to i},\,i\in\mathcal{N}_j^+}\lambda_{(A,\,i,\,j)} A^T P_i A \preceq c P_j\\
&= \tilde{\alpha}_{\min,\,j},
\end{aligned}
$$

where λ is the vector whose elements are $\lambda_{(A,\,i,\,j)}$ for all $A \in \mathcal{A}_{j\to i},\, i \in \mathcal{N}_j^+$, and Δ is the unit simplex of dimension $\sum_{i\in\mathcal{N}_j^+}|\mathcal{A}_{j\to i}|$. ■

Remark 3: Computational complexities of Problem 3 grows quickly with the problem size, i.e., the number of modes, dimensions of SLSs, and the size of weighted digraphs. However, Problem 3 has a structure that can be computationally parallelized by the existing multi-agent optimization techniques, for example, the distributed optimization in [28], and this can be a possible subject of the future research.

Example 2: Consider the SLS (1) with

$$A_1 = \begin{bmatrix} 0.5923 & 0.5283 & 0.7565 \\ 1.5375 & 0.7170 & 1.0567 \\ 0.9333 & 0.2953 & -0.0096 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.1003 & 0.0578 & 0.2078 \\ 1.1190 & -0.6934 & 0.6320 \\ 1.3056 & 0.0255 & -3.2854 \end{bmatrix}.$$

We generate the quadratic functions $V_i(z) = z^T P_i z,\ i \in \mathcal{V}$, where $P_i,\ i \in \mathcal{V}$, are randomly chosen from the set of matrices $\mathcal{H}_4$ obtained by the iteration $\mathcal{H}_0 := \{I_n\}$ and $\mathcal{H}_k := \{A_i^T H A_i + I_n : H \in \mathcal{H}_{k-1},\ i \in \mathcal{M}\}$ for $k \in \{1, 2, \ldots, 4\}$. We also randomly generate a graph $G(\mathcal{V}, \mathcal{E})$ with the adjacency matrix

$$E = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

With $\mathcal{A}_{j\to i} = \mathcal{A}^{[1,6]},\ \forall (i,j) \in \mathcal{E}$, we obtain $\tilde{\alpha}_{\min,1} = 1.9758$, $\tilde{\alpha}_{\min,2} = 0.7905$, $\tilde{\alpha}_{\min,3} = 0.7904$, $\tilde{\alpha}_{\min,4} = 0.4389$, $\tilde{\alpha}_{\min,5} = 1.7832$, $\tilde{\alpha}_{\min,6} = 0.4415$, and $\tilde{\alpha}_{\min,7} = 0.3820$. Then, consider the weighted digraph $G(\mathcal{V}, \mathcal{E}, \alpha)$ with $\alpha = \begin{bmatrix} \tilde{\alpha}_{\min,1} & \tilde{\alpha}_{\min,2} & \cdots & \tilde{\alpha}_{\min,7} \end{bmatrix}^T$. The weighted digraph has the simple cycles $\mathcal{C}_1 = (1, 7, 2, 1)$, $\mathcal{C}_2 = (1, 4, 1)$, $\mathcal{C}_3 = (1, 7, 4, 1)$, $\mathcal{C}_4 = (1, 7, 1)$, and the corresponding cycle gains are $g(\mathcal{C}_1) = 0.5967$, $g(\mathcal{C}_2) = 0.8671$, $g(\mathcal{C}_3) = 0.3313$, $g(\mathcal{C}_4) = 0.7548$, respectively. Since all the cycle gains are less than one, by Theorem 1, the SLS is stabilizable, and the exponential convergence rate is $\rho = 0.9960$.

V. Conclusion

In this paper, we have extended the work in [20] to deal with stabilization problems of SLSs. The notion of GCLFs has been proposed, and we have proved that the existence of a GCLF is necessary and sufficient for stabilizability of SLSs. A computational algorithm based on convex optimizations has been developed, and a numerical example has been given to illustrate the proposed algorithm and demonstrate the potential advantage of the GCLF approach.

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