Periodic Stabilization of Switched Linear Systems Revisited

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***Abstract*— The goal of this paper is to study the switching stabilization of discrete-time autonomous switched linear systems (SLSs) based on the periodic Lyapunov functions. In recent papers, a convex switching stabilizability condition was presented. We investigate properties of the condition. In particular, it is proved that the condition is only a sufficient condition, and also provide a class of stabilizable SLSs which cannot be identified as stabilizable by the condition. In addition, for two-dimensional SLSs, another sufficient and necessary stabilizability condition is proposed by using the S-procedure and considering sufficiently fine partitions of the state space. Examples are given to show the validity of the proposed approaches.**

I. INTRODUCTION

The switched linear system (SLS) is an important class of hybrid systems that can be used to model more general dynamic systems [1]. The stability analysis and control design of the SLSs are important problems, and previous researches on this topic include approaches based on multiple Lyapunov functions [4]–[8], dynamic programming approaches [9], [10], and the generating function approach [11]. Recent surveys on this issue can be found in [2], [3].

The goal of this paper is to study the switching stabilization problem of the SLSs using the periodic Lyapunov functions. The periodic Lyapunov functions were widely used in systems and control theories, for instance, those for the stability and stabilization problems of linear [16] and nonlinear systems [15]. More general non-monotonic Lyapunov functions were pioneered in [17] for nonlinear and switching systems. In [12], the concept of the periodic Lyapunov function was introduced for the stabilizability of the SLS mainly based on set-theoretic approaches, and this approach was further studied in [13] based on sufficient linear matrix inequality (LMI) and bilinear matrix inequality (BMI) feasibility problems. A particular class of periodic Lyapunov functions was also studied in [14] for the same problem mainly from the viewpoint of the Lyapunov theorem.

In this paper, we revisit the periodic stabilization problem studied in [14], where a linear matrix inequality (LMI) condition associated with a periodic Lyapunov function was developed to check the switching stabilizability. It was proved that, if the LMI condition is feasible, then the associated quadratic function is a periodic Lyapunov function that certifies the switching stabilizability of the SLSs. Moreover, it was revealed that, the conservatism can be reduced by increasing the period h of the Lyapunov function. A natural question is whether the LMI condition in [14] with an arbitrary period h is also a necessary condition. In other words, does the switching stabilizability of a SLS imply the existence of a period h which leads to the feasibility of the LMI condition? The answer was given in [13], where using a counter example, it was proved that the condition is only sufficient. In this paper, we also prove that, even if h can be selected arbitrarily large, there still exists conservatism, and hence, the condition is only sufficient. In comparison with the proof in [13], we provide a more general class of the stabilizable SLSs which cannot be identified as stabilizable. Furthermore, a necessary and sufficient condition is developed to prove the asymptotic stabilizability of two-dimensional SLSs. The developed condition extends the condition in [14] by checking it on some partitions of the state space. It is proved that the conservatism vanishes as the partitions become sufficiently fine.

II. PRELIMINARIES

A. Notation

The adopted notation is as follows: $\mathbb{R}$: set of real numbers; $\mathbb{R}_+$: set of nonnegative real numbers; $\mathbb{R}_{++}$: set of positive real numbers; $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$: set of all $n \times m$ real matrices; A^T: transpose of matrix A; $A \succ 0$ ($A \prec 0$, $A \succeq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix A; I_n: $n \times n$ identity matrix; $||\cdot||$: Euclidean norm of a vector or spectral norm of a matrix; $\mathbb{S}^n$: symmetric $n \times n$ matrices; $\mathbb{S}^n_+$: cone of symmetric $n\times n$ positive semi-definite matrices; $\mathbb{S}^n_{++}$: symmetric $n \times n$ positive definite matrices; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$: minimum and maximum eigenvalues of symmetric matrix A, respectively; $\mathrm{co}\{S\}$: convex hull of a set S; $\rho(A)$: spectral radius of a matrix A; for an $n\times n$ symmetric matrix P, $\mathrm{vech}(P)$ is the *half-vectorization operator* [18], which lexicographically orders the "lower-triangular" portion of a matrix into an $n(n+1)/2$ column vector; for any positive integer k, Δ_k is the unit simplex in $\mathbb{R}^k$: $\Delta_k := \left\{ \alpha \in \mathbb{R}^k : \alpha_i \geq 0, \quad \sum_{i=1}^k \alpha_i = 1 \right\}$.

B. Previous results

Consider the discrete-time (autonomous) SLS

$$x(k+1) = A_{\sigma(k)} x(k), \quad x(0) = z \in \mathbb{R}^n, \qquad (1)$$

where $k \in \{0, 1, \ldots\}$, $x(k) \in \mathbb{R}^n$ is the state, $\sigma(k) \in \mathcal{M} := \{1, 2, \ldots, N\}$ is called the mode, and $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$,

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are the subsystem (dynamics) matrices. We assume that none of the matrices A_i, is Schur stable. Starting from $x(0) = z \in \mathbb{R}^n$ and under the switching sequence $\sigma := \{\sigma(0), \sigma(1), \ldots\}$, the trajectory of the SLS is denoted by $x(k;\, z,\, \sigma)$.

Definition 1: The SLS (1) is called

1) asymptotically switching stabilizable if starting from any $x(0) = z \in \mathbb{R}^n$, there exists a switching sequence σ for which the trajectory $x(k;\, z,\, \sigma)$ satisfies $\lim_{k\to\infty} ||x(k;\, z,\, \sigma)|| = 0$.
2) exponentially switching stabilizable (with the parameters κ and r) if there exist $\kappa \geq 1$ and $r \in [0, 1)$ such that starting from any $x(0) = z \in \mathbb{R}^n$, there exists a switching sequence σ for which the trajectory $x(k;\, z,\, \sigma)$ satisfies $||x(k;\, z,\, \sigma)|| \leq \kappa r^k ||z||$, for all $k \in \{0,\, 1,\, \ldots\}$.

From [11, Theorem 1], the asymptotic switching stabilizability and the exponential switching stabilizability of the SLS (1) are equivalent. Thus, we refer to either notion of stabilizability simply as switching stabilizability or stabilizability. The problem addressed in this paper is as follows.

Problem 1: Check whether or not the SLS (1) is switching stabilizable.

To this end, we consider the following periodic state-feedback controller suggested in [14]:

$$\begin{aligned}(\sigma_k(x(k)),\, \ldots,\, \sigma_{k+h-1}(x(k)))\\ = \operatorname*{arg\,min}_{(i_1,\ldots,\, i_h)\in\mathcal{M}^h} V(A_{i_h}\cdots A_{i_1}x(k)),\end{aligned} \tag{2}$$

where $h \in \{1,\, 2, \ldots\}$, $k \in \{0,\, h,\, 2h, \ldots\}$, $\sigma_j(x(i))$, $j \geq i$ denotes the mode at time j determined based on the state $x(i)$ at time i, and $V(x) = x^T Px > 0$, $\forall x \in \mathbb{R}^n \backslash \{0_n\}$ is the (quadratic) periodic control Lyapunov function. The controller is designed in such a way that, every h time steps, the controller generates the current and future sequence of modes of length h selected so that the Lyapunov function's value after h steps is minimized. Then, the switching sequence of length h is applied to the system, and the same process is repeated after h steps. Under (2), the corresponding Lyapunov inequality is

$$\min_{(i_1,\ldots,\, i_h)\in\mathcal{M}^h} V(A_{i_h}\cdots A_{i_1}x) - V(x) < 0,\, \forall x \in \mathbb{R}^n\backslash\{0_n\}. \tag{3}$$

Remark 1: The periodic Lyapunov inequality (3) was introduced in [14] and earlier in [13].

The Lyapunov inequality (3) implies that the quadratic Lyapunov function does not need to decrease at each time step k; it only needs to decrease every h time steps. Then, the switching stabilizability of the SLS (1) can be proved in terms of the Lyapunov inequality (3).

Lemma 1: ([14, Proposition 1]) The following statements are equivalent:

1) The SLS is switching stabilizable;
2) There exist a matrix $P \in \mathbb{S}^n_{++}$ and a positive integer h such that the condition (3) holds;
3) For any given matrix $P \in \mathbb{S}^n_{++}$, there exists a sufficiently large positive integer h such that the condition (3) holds.

The statement 3) of the above lemma means that for any fixed matrix $P \in \mathbb{S}^n_{++}$, e.g., $P = I_n$, the condition (3) with a sufficiently large h can be a necessary and sufficient condition for the switching stabilizability of the SLS (1). Therefore, Problem 1 reduces to the following problem.

Problem 2: For $P = I_n$, check if there exists a sufficiently large positive integer h such that the condition (3) holds.

We first introduce the following sufficient linear matrix inequality (LMI) feasibility problem to check the condition (3) with $P = I_n$.

Problem 3: ([14, Problem 1]) Let $P = I_n$. For a positive integer h, find scalars $\alpha_{(i_1,\, i_2,\ldots,\, i_h)} \in \mathbb{R}$ such that

$$\begin{aligned}&\sum_{(i_1,\, i_2,\ldots,\, i_h)\in\mathcal{M}^h} \alpha_{(i_1,\, i_2,\ldots,\, i_h)}(A_{i_h}\cdots A_{i_1})^T P(A_{i_h}\cdots A_{i_1})\\ &- P \prec 0, \quad \alpha \in \Delta_{N^h},\end{aligned} \tag{4}$$

where α is a vector whose elements are $\{\alpha_{(i_1,\, i_2,\ldots,\, i_h)}\}_{(i_1,\, i_2,\ldots,\, i_h)\in\mathcal{M}^h}$.

Lemma 2: ([14, Proposition 2]) Let $P = I_n$. Suppose that Problem 3 is feasible for a positive integer h. Then, the state-feedback switching policy (2) will stabilize the SLS (1) (the SLS (1) is switching stabilizable).

An intuitive insight on Problem 3 is that if it admits a feasible solution $\alpha^*_{(i_1,\, i_2,\ldots,\, i_h)}$, $(i_1,\, i_2, \ldots,\, i_h) \in \mathcal{M}^h$, then we can find a quadratic over approximation of $\min_{(i_1,\ldots,\, i_h)\in\mathcal{M}^h} V(A_{i_h}\cdots A_{i_1}x)$ satisfying

$$\begin{aligned}&\min_{(i_1,\ldots,\, i_h)\in\mathcal{M}^h} x^T(A_{i_h}\cdots A_{i_1})^T P(A_{i_h}\cdots A_{i_1})x\\ &\leq \sum_{(i_1,\ldots,\, i_h)\in\mathcal{M}^h} \alpha^*_{(i_1,\ldots,\, i_h)} x^T(A_{i_h}\cdots A_{i_1})^T P(A_{i_h}\cdots A_{i_1})x,\end{aligned}$$

which is less than $x^T Px$. In other words, the right-hand side of the above inequality is a quadratic function that separates $\min_{(i_1,\ldots,\, i_h)\in\mathcal{M}^h} V(A_{i_h}\cdots A_{i_1}x)$ and $V(x) = x^T Px$. From Lemma 2, we know that the LMI test is a sufficient certificate for the switching stabilizability of the SLS (1). A natural question is whether or not the LMI test with the period h being a decision variable is also necessary, i.e., if the SLS (1) is switching stabilizable, then does there exist a period h such that Problem 3 has a feasible solution?

Question 1: Let $P = I_n$. If the SLS (1) is switching stabilizable, then does there exist a period h such that Problem 3 has a feasible solution?

First of all, we introduce the following non-convex bilinear matrix inequality (BMI) feasibility problem to present a partial answer to the question.

Problem 4: For a positive integer h, find scalars $\alpha_{(i_1,\, i_2,\ldots,\, i_h)} \in \mathbb{R}$ and a matrix $P \in \mathbb{S}^n_{++}$ such that (4) holds, where α is a vector whose elements are $\{\alpha_{(i_1,\, i_2,\ldots,\, i_h)}\}_{(i_1,\, i_2,\ldots,\, i_h)\in\mathcal{M}^h}$.

Lemma 3: ([14, Proposition 6] and [13, Theorem 14]) If the BMI condition (4) in Problem 4 admits a feasible solution, then there exists a sufficiently large integer h such that (4) is feasible with $P = I_n$.

However, the above lemma does not provide an answer to Question 1. In the next section, we prove that the LMI condition in Problem 3 with h undecided is only sufficient.

C. Main result

In this section, we will prove the following argument.

Proposition 1: Suppose that the SLS (1) is switching stabilizable. There is a class of SLSs such that Problem 3 is not feasible for any $P \in \mathbb{S}^n_{++}$ and any positive integer h.

In order to prove Proposition 1, we will show that, for some system matrices $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$, the convex hull of the following set of symmetric matrices:

$$L_h(P) := \{P - (A_{i_h}\cdots A_{i_1})^T P (A_{i_h}\cdots A_{i_1}) \in \mathbb{S}^n : (i_1, \ldots, i_h) \in \mathcal{M}^h\}$$

does not intersect the positive definite cone for all positive integer h and all $P \in \mathbb{S}^n_{++}$, i.e., $\mathrm{co}\{L_h(P)\} \cap \mathbb{S}^n_{++} = \emptyset$, $\forall h \in \{1, 2, \ldots\}$, $P \in \mathbb{S}^n_{++}$. This can be equivalently expressed as $\mathrm{vech}\{\mathrm{co}\{L_h(P)\}\} \cap \mathrm{vech}\{\mathbb{S}^n_{++}\} = \emptyset$, $\forall h \in \{1, 2, \ldots\}$, $P \in \mathbb{S}^n_{++}$. The following two preliminary results are established first.

Theorem 1: Consider the SLS (1) with the system matrices A_i, $i \in \mathcal{M}$, satisfying $\det(A_i) \geq 1$, $\forall i \in \mathcal{M}$. Then, $\mathrm{co}\{L_h(P)\} \cap \mathbb{S}^n_{++} = \emptyset$ holds for all $h \in \{1, 2, \ldots\}$ and all $P \in \mathbb{S}^n_{++}$.

Proof: The statement $\mathrm{co}\{L_h(P)\} \cap \mathbb{S}^n_{++} = \emptyset$, $\forall P \in \mathbb{S}^n_{++}$, $h \in \{1, 2, \ldots\}$ can be expressed as $P - \sum_{i\in\mathcal{I}} \alpha_i \bar{A}_i^T P \bar{A}_i \notin \mathbb{S}^n_{++}$ for all $\alpha \in \Delta_{N^h}$, $P \in \mathbb{S}^n_{++}$, $h \in \{1, 2, \ldots\}$, where $\alpha \in \Delta_{N^h}$ is a vector whose elements are an appropriate enumeration of all $\alpha_{(i_1, i_2, \ldots, i_h)}$, $\forall (i_1, i_2, \ldots, i_h) \in \mathcal{M}^h$, $\mathcal{I} := \{1, 2, \ldots, N^h\}$, and $\bar{A}_i$, $i \in \mathcal{I}$, is an enumeration of the elements of the set $\{A_{i_h}\cdots A_{i_1} \in \mathbb{R}^{n\times n} : (i_1, i_2, \ldots, i_h) \in \mathcal{M}^h\}$. For convenience, it is reformulated as

$$I_n - \sum_{i\in\mathcal{I}} \alpha_i P^{-1/2} \bar{A}_i^T P \bar{A}_i P^{-1/2} \notin \mathbb{S}^n_{++}, \quad \forall \alpha \in \Delta_{N^h}, P \in \mathbb{S}^n_{++}, h \in \{1, 2, \ldots\}. \tag{5}$$

To prove (5), we will use the fact that, for any $P \in \mathbb{S}^n$, a sufficient condition for $P \notin \mathbb{S}^n_{++}$ is that P lies within the half-plane $\mathcal{P}_n := \{S \in \mathbb{S}^n : \mathrm{trace}(S) = \mathrm{vech}(S)^T \mathrm{vech}(I_n) \leq 0\}$, which satisfies $\mathcal{P}_n \cap \mathbb{S}^n_{++} = \emptyset$. Thus, a sufficient condition for (5) is

$$\begin{aligned} & I_n - \sum_{i\in\mathcal{I}} \alpha_i P^{-1/2} \bar{A}_i^T P \bar{A}_i P^{-1/2} \in \mathcal{P}_n \\ \Leftrightarrow\ & n - \sum_{i\in\mathcal{I}} \alpha_i \mathrm{trace}(P^{-1/2} \bar{A}_i^T P \bar{A}_i P^{-1/2}) \\ & \leq 0, \quad \forall \alpha_{N^h} \in \Delta_{N^h}, P \in \mathbb{S}^n_{++}, h \in \{1, 2, \ldots\}. \end{aligned}$$

Next, since $\det(A_i) \geq 1, \forall i \in \mathcal{M}$, we have $\det(P^{-1/2} \bar{A}_i^T P \bar{A}_i P^{-1/2}) = \det(P^{-1}) \det(P) \det(\bar{A}_i)^2 = \det(\bar{A}_i)^2 \geq 1, \forall i \in \mathcal{I}$, which implies that $\lambda_1^{(i)} \times \lambda_2^{(i)} \times \cdots \times \lambda_n^{(i)} \geq 1$, where $\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_n^{(i)}$ stand for all the eigenvalues of $P^{-1/2} \bar{A}_i^T P \bar{A}_i P^{-1/2}$, $i \in \mathcal{I}$. Then, using the inequality of arithmetic and geometric means, we have $\mathrm{trace}(P^{-1/2} \bar{A}_i^T P \bar{A}_i P^{-1/2})/n = (\lambda_1^{(i)} + \cdots + \lambda_n^{(i)})/n \geq \sqrt[n]{\lambda_1^{(i)} \times \cdots \times \lambda_n^{(i)}} \geq 1$. As a result, one gets $\mathrm{trace}(I_n - \sum_{i\in\mathcal{I}} \alpha_i P^{-1/2} \bar{A}_i^T P \bar{A}_i P^{-1/2}) = n - \sum_{i\in\mathcal{I}} \alpha_i \mathrm{trace}(P^{-1/2} \bar{A}_i^T P \bar{A}_i P^{-1/2}) \leq 0$ for all $\alpha \in \Delta_{N^h}$, $P \in \mathbb{S}^n_{++}$, and $h \in \{1, 2, \ldots\}$. Therefore, (5) is not positive definite for all $\alpha \in \Delta_{N^h}$, $P \in \mathbb{S}^n_{++}$, and all $h \in \{1, 2, \ldots\}$. This completes the proof. ■

Lemma 4: There is a class of switching stabilizable SLS (1) with the system matrices A_i, $i \in \mathcal{M}$ satisfying $\det(A_i) \geq 1$, $\forall i \in \mathcal{M}$.

Proof: It is sufficient to provide a counter example in [11, section IV]. ■

We are now in position to prove Proposition 1.

Proof: [Proof of Proposition 1] The proof can be completed by combining Theorem 1 and Lemma 4. ■

Remark 2: The same claim was proved in [13] through [13, Example 13].

III. Example

Consider the SLS (1) with

$$A_1 = \begin{bmatrix} 1/a & 0 \\ 0 & a \end{bmatrix}, \quad A_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{6}$$

where a is a real number such that $a > 1$ and $0 < \theta < \pi/2$.

Remark 3: A similar system was used in [13, Example 13].

To stabilize this SLS, consider the following state-feedback switching policy $\sigma(x) = \begin{cases} 1, & x \in U \\ 2, & x \in U^c \end{cases}$, where $U := \{x \in \mathbb{R}^2 : 2\pi - \phi \leq \Theta(x) \leq 2\pi\}$, U^c is the complement of U, $\pi/2 > \phi > \theta$, and $\Theta(x)$ is the phase angle of the 2D vector x as a complex number. Next, it can be proved that the SLS (1) with (6) under the policy $\sigma(x)$ is asymptotically stable under a certain condition.

Proposition 2: Suppose that the following condition is satisfied:

$$\cos(\phi)^2 > \frac{a^2 - 1}{a^2 - a^{-2}}, \tag{7}$$

where ϕ is the design parameter of the policy $\sigma(x)$. Then, the SLS (1) with (6) under the policy $\sigma(x)$ is asymptotically stable. In other words, the SLS is switching stabilizable.

Proof: We consider two cases. 1) $x \in U^c$: It is straightforward to show that $\|x^+\| = \|x\|$ is always satisfied. 2) $x \in U$: Any $x \in \mathbb{R}^2$ can be expressed as $x = [||x|| \cos(\Theta(x)),\, ||x|| \sin(\Theta(x))]^T$. Since $x \in U$, the state at the next time instant is $x^+ = A_1 x = [a^{-1}||x|| \cos(\Theta(x)),\, a||x|| \sin(\Theta(x))]^T$ and $||x^+|| = ||x|| \sqrt{a^{-2}\cos(\Theta(x))^2 + a^2 \sin(\Theta(x))^2}$. On the other hand, if (7) holds, then by using elementary algebraic manipulations and $\cos(2\pi - \phi) = \cos(\phi)$, we have $(a^{-2} - a^2)\cos(2\pi - \phi)^2 + a^2 < 1$. Since x lies in the forth quadrant, $\cos(2\pi - \phi)^2 = \cos(\phi)^2$ is strictly increasing in $\phi \in [(3/2)\pi,\, 2\pi]$, and hence, $(a^{-2} - a^2)\cos(\Theta(x))^2 + a^2 < 1$ for all $\Theta(x) \in [2\pi - \phi,\, 2\pi]$. Using $\cos(\Theta(x))^2 = 1 - \sin(\Theta(x))^2$ and rearranging the last inequality yield $a^{-2}\cos(\Theta(x))^2 + a^2 \sin(\Theta(x))^2 < 1$

for all $\forall\Theta(x) \in [2\pi - \phi, 2\pi]$. This implies $||x^+|| < ||x||$. Finally, from the assumption $\phi > \theta$, it can be easily seen that any $x \in U^c$ will reach the region U in finite time. This concludes the proof. ■

Now, consider the SLS (1) with (6) and $a = 1.5$, $\theta = \pi/36$. Then, it can be numerically proved that the SLS under the policy $\sigma(x)$ with $\phi = \pi/20$ is asymptotically stable (the condition (7) is satisfied since $0.9755 = \cos(\phi)^2 > \frac{a^{-2}-1}{a^2-a^{-2}} = 0.6923$). In other words, the SLS is switching stabilizable. The corresponding state trajectory is illustrated in Fig. 1. However, since $\det(A_1) = 1$ and $\det(A_2) =$

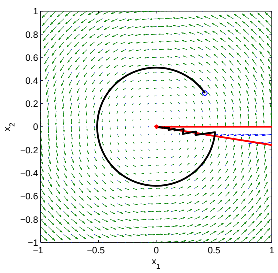

Fig. 1. State trajectory of the SLS (1) with (6) under the policy $\sigma(x)$.

$b^2 \geq 1$, by Theorem 1, $\text{co}\{L_h(P)\} \cap \mathbb{S}^n_{++} = \emptyset$ holds for all $h \in \{1, 2, \ldots\}$ and all $P \in \mathbb{S}^n_{++}$. This means that, for all $h \in \{1, 2, \ldots\}$ and all $P \in \mathbb{S}^n_{++}$, Problem 3 is not feasible. For $h = 3$, Figure 2 shows the boundaries of $\text{vech}\{\text{co}\{L_h(P)\}\}$ and $\text{vech}\{\mathbb{S}^n_{++}\}$, and also visually verifies that the two sets do not intersect each other. By increasing h, it can be observed that the distance between the two sets gets larger.

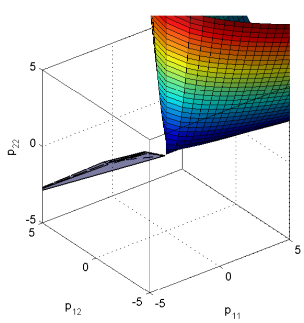

Fig. 2. The boundaries of $\text{vec}\{\text{co}\{L_h(P)\}\}$ and $\text{vec}\{\mathbb{S}^n_{++}\}$ for $h = 3$.

IV. A Necessary Condition

In this section, we study a necessary condition for the feasibility of Problem 3.

Proposition 3: Define $\mathcal{A}_h := \{A_{i_h} \cdots A_{i_1} \in \mathbb{R}^{n\times n} : (i_1, i_2, \ldots, i_h) \in \mathcal{M}^h\}$. If Problem 3 is feasible, then there exists a matrix $A \in \text{co}\{\mathcal{A}_h\}$ whose spectral radius $\rho(A)$ is less than one.

Proof: Assume that Problem 3 is feasible. This means that there is a feasible solution $\alpha^* \in \Delta_{N^h}$ so that $\sum_{i=1}^{N^h} \alpha_i^* \bar{A}_i^T P \bar{A}_i - P \prec 0$ holds. Using [21, Lemma 2], we find $\left(\sum_{i=1}^{N^h} \alpha_i^* \bar{A}_i\right)^T P \left(\sum_{i=1}^{N^h} \alpha_i^* \bar{A}_i\right) - P \preceq \sum_{i=1}^{N^h} \alpha_i^* \bar{A}_i^T P \bar{A}_i - P$. This means that there is a Scuhr stable matrix $A \in \text{co}\{\mathcal{A}_h\}$. This concludes the proof. ■

The following result shows that, if $N = 2$ and $h = 1$, the BMI feasibility of Problem 4 is a necessary and sufficient condition for (3).

Proposition 4: There exists $P \in \mathbb{S}^n_{++}$ such that

$$\min_{i\in\{1,2\}} V(A_i x(k)) < V(x), \quad \forall x \in \mathbb{R}^n \backslash \{0_n\}, \tag{8}$$

holds, where $V(x) = x^T P x$, *if and only if* there exist $P \in \mathbb{S}^n_{++}$ and $\alpha \in \Delta_2$ such that

$$\sum_{i=1}^{2} \alpha_i A_i^T P A_i - P \prec 0. \tag{9}$$

Proof: (Sufficiency): Suppose that the BMI problem (9) is feasible with the feasible solution $P = P^*$ and $\alpha = \alpha^*$. Then, we have $\min_{i\in\{1,2\}} x^T A_i^T P^* A_i x \leq \sum_{i=1}^{2} \alpha_i^* x^T A_i^T P^* A_i x < x^T P^* x$ for all $x \in \mathbb{R}^n \backslash \{0_n\}$ since, for any fixed $x \in \mathbb{R}^n \backslash \{0_n\}$, a convex combination of two numbers $x^T A_i^T P^* A_i x$, $i \in \{1, 2\}$ always lies between them. Therefore, (8) is fulfilled.

(Necessity): Suppose that there exists $P^* \in \mathbb{S}^n_{++}$ such that (9) holds. This means $\bigcup_{i=1}^{2} \{x \in \mathbb{R}^2 : x^T(A_i^T P A_i - P)x < 0\} = \mathbb{R}^2$. By using the covering lemma in [22, equation (6)], it holds if and only if there exist $\theta_1, \theta_2 \geq 0$ such that $\theta_1(A_1^T P A_1 - P) + \theta_2(A_2^T P A_2 - P) \prec 0$, which is equivalent to (9) through $\alpha_1 = \theta_1/(\theta_1 + \theta_2)$, $\alpha_2 = \theta_2/(\theta_1 + \theta_2)$. ■

Combining Propositions 3 and 4, we obtain the following result.

Corollary 1: Consider the SLS (1) with two subsystems. If the spectral radius of A is equal to or larger than one for all $A \in \text{co}\{A_1, A_2\}$, then the SLS does not admit a quadratic control Lyapunov function that satisfies (8).

Example 1: Consider the SLS (1) with $A_1 = \begin{bmatrix} 1.6 & 0.1 \\ -1 & -0.9 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0.5 & 1 \\ 1 & 2 \end{bmatrix}$. The SLS is switching stabilizable because Problem 3 with $P = I_n$ is feasible with $h = 7$. However, it can be numerically shown using fine grid points that the minimum spectral radius $\rho(A(\lambda))$ of matrix $A(\lambda) := \lambda A_1 + (1 - \lambda)A_2$ is 1.0233 for all $\lambda \in [0, 1]$. Therefore, by Corollary 4, the BMI condition (9) is not feasible.

V. Tests for 2D SLSs with Reduced Conservatism

In the previous section, it was proved that the converse argument of Lemma 2 is not true. Therefore, the stabilizability test provided in Problem 3 has some degree of conservatism. In this section, we will show how the conservatism can be reduced. The basic idea is to divide the state space into several partitions, and then using the S-procedure [19, Chapter 2.6.3], to solve a version of Problem 3 assigned to each partition. It can be proved that as the number of partitions increases, the conservatism asymptotically vanishes. For simplicity, this approach is presented only for the SLS (1) with

$n = 2$. An extension of the problem to general cases can be more complicated, and will be investigated in the future. Define the half-planes $P_-(\theta) := \{x \in \mathbb{R}^2 : [\cos(\theta), \sin(\theta)]x \leq 0\}$ and $P_+(\theta) := \{x \in \mathbb{R}^2 : [\cos(\theta), \sin(\theta)]x \geq 0\}$. Consider the set $(P_-(\theta) \cap P_+(\theta + \Delta\theta)) \cup (P_+(\theta) \cap P_-(\theta + \Delta\theta))$, which can be expressed as $\Pi(\theta, \Delta\theta) := \{x \in \mathbb{R}^2 : x^T Q(\theta, \Delta\theta)x \leq 0\}$, where $Q(\theta, \Delta\theta) := \frac{1}{2}\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}\begin{bmatrix}\cos(\theta+\Delta\theta)\\ \sin(\theta+\Delta\theta)\end{bmatrix}^T + \frac{1}{2}\begin{bmatrix}\cos(\theta+\Delta\theta)\\ \sin(\theta+\Delta\theta)\end{bmatrix}\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}^T$. The set $\Pi(\theta, \Delta\theta)$ is illustrated in Fig. 3. Consider the

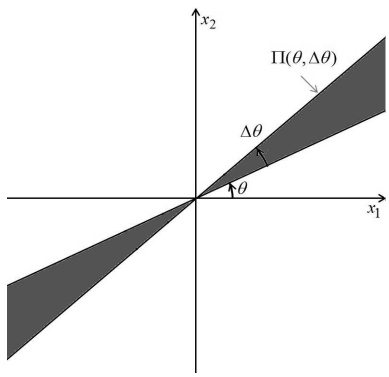

Fig. 3. Partitioning region $\Pi(\theta, \Delta\theta)$.

partitions $\Pi(i\Delta\theta, \Delta\theta)$, $i \in \{0, 1, \ldots, M-1\}$ of the state space, where $\Delta\theta = 2\pi/M$, that divide $\mathbb{R}^2$ into M parts, i.e., $\bigcup_{i=0}^{M-1} \Pi(i\Delta\theta, \Delta\theta) = \mathbb{R}^2$. Based on the definitions, we introduce the following problem.

Problem 5: Let $P \in \mathbb{S}^n_{++}$ and $Q \in \mathbb{S}^n$ be given. For a positive integer h, find scalars $\alpha_{(i_1, i_2, \ldots, i_h)} \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$\sum_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} \alpha_{(i_1, i_2, \ldots, i_h)} (A_{i_h} \cdots A_{i_1})^T P (A_{i_h} \cdots A_{i_1}) - P \prec \lambda Q,$$

$\alpha_{(i_1, i_2, \ldots, i_h)} \geq 0$, $\lambda \geq 0$, and $\sum_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} \alpha_{(i_1, i_2, \ldots, i_h)} = 1$.

Proposition 5: Let $P \in \mathbb{S}^n_{++}$ and a positive integer h be given. Suppose that Problem 5 with $Q = Q(i\Delta\theta, \Delta\theta)$ is feasible for all $i \in \{0, 1, \ldots, M-1\}$, where $\Delta\theta = 2\pi/M$. Then, the state-feedback switching policy (2) will stabilize the SLS (1).

Proof: Let $i^* \in \{0, 1, \ldots, M-1\}$ and suppose that Problem 5 with $Q = Q(i^*\Delta\theta, \Delta\theta)$ admits a feasible solution $\alpha^*_{(i_1, i_2, \ldots, i_h)}$, $(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h$. Then, $x^T R x < \lambda x^T Q x$, $\forall x \in \mathbb{R}^2 \backslash \{0_n\}$ holds, where

$$R := \sum_{(i_1, \ldots, i_h) \in \mathcal{M}^h} \alpha^*_{(i_1, \ldots, i_h)} (A_{i_h} \cdots A_{i_1})^T P (A_{i_h} \cdots A_{i_1}) - P.$$

By using the S-procedure [19, Chapter 2.6.3] and recalling the definition of the set $\Pi(\theta, \Delta\theta)$, it can be proved that the last inequality holds if and only if $x^T R x < 0$ holds for all $x \in \Pi(i^*\Delta\theta, \Delta\theta)\backslash\{0_n\}$. Since $\min_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} V(A_{i_h} \cdots A_{i_1} x) - V(x) \leq x^T R x$, $\forall x \in \mathbb{R}^2$, we have $\min_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} V(A_{i_h} \cdots A_{i_1} x) - V(x) < 0$, $\forall x \in \Pi(i^*\Delta\theta, \Delta\theta)\backslash\{0_n\}$. Lastly, since Problem 5 is feasible for all $Q = Q(i\Delta\theta, \Delta\theta)$, $i \in \{0, 1, \ldots, M-1\}$, the last inequality is satisfied for all $x \in \bigcup_{i=0}^{M-1} \Pi(i\Delta\theta, \Delta\theta)\backslash\{0_n\} = \mathbb{R}^2\backslash\{0_n\}$. This concludes the proof. ■

Lastly, it is proved that the conservatism of Problem 5 vanishes as the number of partitions M increases.

Proposition 6: Let $P \in \mathbb{S}^n_{++}$ and a positive integer h be given. The condition (3) holds if and only if there exists a sufficiently large positive integer M such that Problem 5 with $Q = Q(i\Delta\theta, \Delta\theta)$ is feasible for all $i \in \{0, 1, \ldots, M-1\}$, where $\Delta\theta = 2\pi/M$.

Proof: The sufficiency part has been proved in Proposition 5. To prove the necessity, suppose that the condition (3) holds. Then, we can find functions $\alpha_{(i_1, i_2, \ldots, i_h)}(x) \in \mathbb{R}$, $(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h$ that depend on the state $x \in \mathbb{R}^2$ such that $\alpha_{(i_1, i_2, \ldots, i_h)}(x) \geq 0$, $\sum_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} \alpha_{(i_1, i_2, \ldots, i_h)}(x) = 1$, and

$$\min_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} V(A_{i_h} \cdots A_{i_1} x) = \sum_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} \alpha_{(i_1, i_2, \ldots, i_h)}(x) V(A_{i_h} \cdots A_{i_1} x)$$

for all $x \in \mathbb{R}^2$. Since both $\min_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} V(A_{i_h} \cdots A_{i_1} x)$ and $V(x)$ are homogeneous, we only need to consider x on the unit circle in $\mathbb{R}^2$ to verify (3). Therefore, letting $x(\theta) = [\cos(\theta), \sin(\theta)]^T$, the condition (3) can be represented by

$$\sum_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} \alpha_{(i_1, i_2, \ldots, i_h)}(x(\theta)) V(A_{i_h} \cdots A_{i_1} x(\theta)) - V(x(\theta)) < 0, \quad \forall \theta \in [0, 2\pi]. \tag{10}$$

Using the continuity of quadratic functions, for each fixed $\theta \in [0, 2\pi]$, we can find $\bar{\Phi}(\theta) \in [0, 2\pi]$ so that

$$\sum_{(i_1, \ldots, i_h) \in \mathcal{M}^h} \alpha_{(i_1, \ldots, i_h)}(x(\theta)) V(A_{i_h} \cdots A_{i_1} x(\theta + \Delta\theta)) - V(x(\theta + \Delta\theta)) < 0, \tag{11}$$

for all $\Delta\theta \in [0, \bar{\Phi}(\theta)]$, $\theta \in [0, 2\pi]$. Next, we will show that there exists a positive constant $\phi \in \mathbb{R}_{++}$ such that (11) holds with $\bar{\Phi}(\theta) \equiv \phi$. By the continuity, we have

$$\sum_{(i_1, \ldots, i_h) \in \mathcal{M}^h} \alpha_{(i_1, \ldots, i_h)}(x(\theta)) V(A_{i_h} \cdots A_{i_1} x(\bar{\theta})) - V(x(\bar{\theta})) < 0, \quad \forall \bar{\theta} \in N_{r(\theta)}(\theta), \tag{12}$$

where $N_{r(\theta)}(\theta)$ is a neighborhood of θ consisting of all points $\bar{\theta} \in [0, 2\pi]$ such that $|\theta - \bar{\theta}| < r(\theta)$. Then, $\bigcup_{\theta \in [0, 2\pi]} N_{r(\theta)}(\theta)$ is an open cover of the compact set $[0, 2\pi]$. By the definition of the compact set, there is a finite subcover $N_{r(\theta_1)}(\theta_1), \ldots, N_{r(\theta_L)}(\theta_L)$, $L \in \{1, 2, \ldots\}$, such that $\bigcup_{i \in \{1, \ldots, L\}} N_{r(\theta_i)}(\theta_i) = [0, 2\pi]$, where $\{\theta_i\}_{i=1}^L$ is a strictly monotonically increasing sequence within $[0,\,2\pi]$. Next, consider the nonempty intersection of two neighboring sets $N_{r(\theta_i)}(\theta_i)\cap N_{r(\theta_{i+1})}(\theta_{i+1})$, $\forall i\in\{1,\ldots,L-1\}$, which are open intervals $(\theta_{i+1}-r(\theta_{i+1}),\,\theta_i+r(\theta_i))$, $\forall i\in\{1,\ldots,L-1\}$. If we define $\bar\theta_i:=(\theta_i+r(\theta_i)+\theta_{i+1}-r(\theta_{i+1}))/2$ and $\bar r_i:=(\theta_i+r(\theta_i)-\theta_{i+1}+r(\theta_{i+1}))/4$, then the closed interval $[\bar\theta_i-\bar r_i,\,\bar\theta_i+\bar r_i]$ is a proper subset of each open interval $(\theta_{i+1}-r(\theta_{i+1}),\,\theta_i+r(\theta_i))$ for all $i\in\{1,\ldots,L-1\}$. By setting $i^*:=\arg\min_{i\in\{1,\ldots,L-1\}}\bar r_i$, it can be proved that, for any $\theta\in[0,\,2\pi]$, $[\theta-\bar r_{i^*},\,\theta+\bar r_{i^*}]\subset N_{r(\theta_j)}(\theta_j)$ for some $j\in\{1,\ldots,L-1\}$. This means that (12) holds for all $\theta\in[0,\,2\pi]$ and $\bar\theta\in N_{r(\theta)}(\theta)$ with a fixed radius $r(\theta)\equiv\bar r_{i^*}$.

Set $M=\lceil 2\pi/\bar r_{i^*}\rceil$[1] and define $\bar\theta:=2\pi/M$. Since $\bar\theta\le\bar r_{i^*}$, it can be seen that (11) still holds for all $\Delta\theta\in[0,\,\bar\theta]$, $\theta=i\bar\theta$, $i\in\{0,1,\ldots,M-1\}$. By the S-procedure [19, Chapter 2.6.3], (11) is satisfied for all $\Delta\theta\in[0,\,\bar\theta]$, $\theta=i\bar\theta$, $i\in\{0,\ldots,M-1\}$ if and only if there exists $\lambda_i\in\mathbb{R}_+$ for each $i\in\{0,1,\ldots,M-1\}$ such that

$$\sum_{(i_1,\ldots,i_h)\in\mathcal{M}^h}\alpha_{(i_1,\ldots,i_h)}(x(i\bar\theta))(A_{i_h}\cdots A_{i_1})^T P(A_{i_h}\cdots A_{i_1})-P\prec\lambda_i Q(i\bar\theta,\,\bar\theta),$$

which ensures the feasibility of Problem 5 with $Q=Q(i\bar\theta,\,\bar\theta)$ for all $i\in\{0,1,\ldots,M-1\}$. This completes the proof. ■

We can summarize the results in the following statement, which is given without the proof.

Corollary 2: Consider any given $P\in\mathbb{S}^n_{++}$. A two-dimensional SLS is switching stabilizable if and only if there exist sufficiently large positive integers M and h such that Problem 5 with $Q=Q(i\Delta\theta,\,\Delta\theta)$ is feasible for all $i\in\{0,1,\ldots,M-1\}$, where $\Delta\theta=2\pi/M$.

Example 2: Consider the SLS (1) with (6), $a=1.3$, $\theta=10\pi/36$, and A_2 multiplied by 1.1. As shown in Proposition 1, the LMI condition (3) with any $P\in\mathbb{S}^n_{++}$ is not feasible for any positive integer h since the determinants of both A_1 and A_2 are greater than one. It can be also demonstrated by experiments that [8, Corollary 1] cannot identify the switching stabilizability of the SLS at a reasonable computational cost. On the other hand, the LMI condition of Problem 5 with $P=I_n$, $h=5$, $M=200$, and $Q=Q(i\Delta\theta,\,\Delta\theta)$ is feasible for all $i\in\{0,1,\ldots,M-1\}$, where $\Delta\theta=2\pi/M$. Therefore, the periodic state-feedback policy (2) with $P=I_n$ and $h=5$ can asymptotically stabilize the SLS (1) with (6).

VI. Conclusion

In this paper, it is proved that the conditions given in [14, Proposition 2] and [14, Proposition 4] are only sufficient. Another sufficient and necessary switching stabilizability condition is developed to check the Lyapunov inequalities using fine partitions of the two-dimensional state space.

[1] For any $x\in\mathbb{R}$, $\lceil x\rceil$ stands for the minimum integer greater than x.

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