Graph Control Lyapunov Function for Stabilization of Discrete-Time Switched Linear Systems

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***Abstract*— This paper studies the stabilization problem of discrete-time switched linear systems (SLSs). To analyze the switching stabilizability and to design switching policies, we first introduce the periodic and aperiodic Lyapunov functions, which are then generalized to the graph Lyapunov function recently developed in the literature. Other extensions such as the periodic and aperiodic piecewise quadratic Lyapunov functions are also discussed. Relations between these types of Lyapunov functions are studied in a unified fashion.**

I. Introduction

This paper studies the stabilization of discrete-time autonomous *switched linear systems (SLSs)*. The stabilization problem of the SLSs have been often studied in the context of the Lyapunov approaches. One of the useful classes of Lyapunov functions is the piecewise quadratic Lyapunov functions, for instance, those developed in [1]–[4], [6]. The goal of this paper is to investigate generalized Lyapunov approaches beyond the piecewise Lyapunov methods, such as the periodic and aperiodic Lyapunov functions [7]–[9], and the graph Lyapunov function [14].

Previous results: To the authors' knowledge, the periodic Lyapunov function was pioneered in [10], [11] for robust stability of LTI or nonlinear systems. Specifically, the multi-samples difference of Lyapunov functions which relaxes the restriction of their monotonicity was considered in [10], [11] for linear difference inclusions. A class of non-monotonic Lyapunov functions were investigated in [12], [13] to relax the monotonicity requirement of the classical Lyapunov's theorem and reduce the conservatism for stability analysis of nonlinear and switching systems. Moreover, it was generalized in [14] by introducing the notion of the graph Lyapunov function. For the stabilization of the SLSs, the concept of the periodic Lyapunov function was introduced in [7] to check the stabilizability of the SLS mainly based on set-theoretic approach. It was further studied in [8] based on sufficient linear matrix inequality (LMI) and bilinear matrix inequality (BMI) feasibility problems. Especially, the authors of [8] suggested a general class of the Lyapunov functions, the class of aperiodically piecewise quadratic Lyapunov functions which include several other classes of Lyapunov functions as special cases. A particular class of periodic Lyapunov functions was also studied in [9] for the same problem mainly from the viewpoint of the Lyapunov theorem.

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Proposed results: The main result of this paper is a generalization of the graph Lyapunov function developed in [14] for the stabilization problem of the SLSs. The approach in [14] considered the stability analysis of switching systems with arbitrary switching sequences, while in this paper, the stabilization of the SLSs with controlled switching sequences is investigated. The graph Lyapunov function approach allows us to unify several classes of Lyapunov functions, such as the piecewise quadratic Lyapunov functions [1]–[4], [6], periodic and aperiodic Lyapunov functions [7]–[9], and the aperiodically piecewise quadratic Lyapunov functions [7], [8]. This generalization also provides more flexibility in finding the Lyapunov function and less conservative control designs[1].

II. Preliminaries

The adopted notation is as follows: $\mathbb{N}$ and $\mathbb{N}_+$: sets of nonnegative integers and positive integers, respectively; $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$: sets of real numbers, nonnegative real numbers, and positive real numbers, respectively; $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$: set of all $n \times m$ real matrices; A^T: transpose of matrix A; $A \succ 0$ ($A \prec 0$, $A \succeq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix A; I_n: $n \times n$ identity matrix; $||\cdot||$: Euclidean norm of a vector or spectral norm of a matrix; $\mathbb{S}^n$, $\mathbb{S}^n_+$, and $\mathbb{S}^n_{++}$: sets of symmetric $n \times n$ matrices, positive semi-definite matrices, and positive definite matrices, respectively; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$: minimum and maximum eigenvalues of symmetric matrix A, respectively.

Consider the discrete-time (autonomous) switched linear system (SLS)

$$x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = z \in \mathbb{R}^n, \tag{1}$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state, $\sigma(k) \in \mathcal{M} := \{1, 2, \ldots, N\}$ is called the mode, and A_i, $i \in \mathcal{M}$, are the subsystem matrices. We assume that none of the matrices $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$, is Schur stable. A switching sequence over the entire time steps will be denoted by $\sigma_\infty := \{\sigma(0), \sigma(1), \ldots\}$.

Definition 1: ([6, Definition 1]) The SLS (1) is called

1) asymptotically switching stabilizable if starting from any initial state $x(0) = z \in \mathbb{R}^n$, there exists a switching sequence σ_∞ for which the trajectory $x(k)$ satisfies $\lim_{k\to\infty} ||x(k)|| = 0$.

[1]Due to the space limitation, throughout this article, proofs that are intuitively straightforward will be omitted.

2) exponentially switching stabilizable (with the parameters κ and r) if there exist $\kappa \geq 1$ and $r \in [0, 1)$ such that starting from any initial state $x(0) = z \in \mathbb{R}^n$, there exists a switching sequence σ_∞ for which the trajectory $x(k)$ satisfies $||x(k)|| \leq \kappa r^k ||z||$, for all $k \in \mathbb{N}$.

From [6, Theorem 1], the asymptotic switching stabilizability and the exponential switching stabilizability of the SLS (1) are equivalent, and hence, we will refer to either notions simply as stabilizability. Finally, the problem addressed in this paper is stated as follows.

Problem 1: Identify the stabilizability of the SLS (1), and find, if possible, a (state-feedback) switching policy σ_∞ under which the SLS (1) is asymptotically stable.

III. Main Results

A. Periodically quadratic Lyapunov function [7]–[9]

In this subsection, we briefly review the *periodically quadratic Lyapunov function* $V(x) = x^T Px$, $P \in \mathbb{S}^n_{++}$ studied in [7]–[9], which satisfies the *periodic Lyapunov inequality*

$$\min_{(i_1, i_2, \ldots, i_h) \in \mathcal{M}^h} V(A_{i_h} \cdots A_{i_1} x) - V(x) < 0 \qquad (2)$$

for all $x \in \mathbb{R}^n \backslash \{0_n\}$. If $V(x)$ satisfies the above condition for some $h \in \mathbb{N}_+$, then for any initial state $z \in \mathbb{R}^n$, it is a control Lyapunov function for the following state-feedback switching policy:

$$\sigma_\infty(z) := (\sigma_h(z),\, \sigma_h(x(h)),\, \sigma_h(x(2h)), \ldots) \qquad (3)$$

obtained by concatenating the finite-horizon switching policy $\sigma_h(z) := \underset{(i_1, \ldots, i_h) \in \mathcal{M}^h}{\arg\min}\, V(A_{i_h} \cdots A_{i_1} z)$. This controller will be called the periodic controller. Note that the periodically quadratic Lyapunov function that considers h-samples difference is a generalization of the classical quadratic Lyapunov function which considers the one-sample difference $\min_{i \in \mathcal{M}} x^T A_i^T P A_i x - x^T P x < 0$, $\forall x \in \mathbb{R}^n \backslash \{0_n\}$. Let $\mathcal{P}_k$, $k \in \{0, 1, \ldots, h\}$ be a sequence of sets of positive semi-definite matrices defined by: $\mathcal{P}_0 = \mathcal{F} \subset \mathbb{S}^n_{++}$, and for $k \in \{1, 2, \ldots, h\}$,

$$\mathcal{P}_k = \rho_{\mathcal{M}}(\mathcal{P}_{k-1}) := \{A_i^T H A_i : H \in \mathcal{P}_{k-1},\, i \in \mathcal{M}\}, \quad (4)$$

which will be called the *Switched Riccati Set (SRS)* following the term in [4].

Remark 1: Notice that for a given SLS (1), the set $\mathcal{P}_k$ is dependent on the initial selection of the set $\mathcal{F} \subset \mathbb{S}^n_{++}$. If necessary, the dependence of $\mathcal{P}_k$ on $\mathcal{F}$ will be explicitly expressed as $\mathcal{P}_k(\mathcal{F})$. However, if $\mathcal{P}_0 = \mathcal{F}$ is explicitly stated before the notation $\mathcal{P}_k$, the dependence will be dropped for notational simplicity.

In addition, the mapping $\rho_{\mathcal{M}}(\cdot)$ will be called the *Switched Riccati Mapping (SRM)*. For any finite subset $\mathcal{F}$ of $\mathbb{S}^n_+$, define $L(\mathcal{H}, P) := \max_{z \in \mathbb{R}^n, \|z\|=1} \min_{H \in \mathcal{H}} z^T (H - P) z$. Then, the periodic Lyapunov inequality (2) is equivalent to $L(\mathcal{P}_h(\{P\}), P) < 0$. Based on this observation and the ideas given in [4], [6], for any given $P \in \mathbb{S}^n_{++}$ and $h \in \mathbb{N}_+$, a convex test to check (2) can be obtained as follows.

Problem 2: ([9, Problem 1]) Let $\mathcal{F} \subset \mathbb{S}^n_+$ be any finite subset of the positive semi-definite matrices and let $P \in \mathbb{S}^n_{++}$. Solve the convex optimization problem

$$c(\mathcal{F}, P) := \min_{\alpha \in \Delta_k,\, c \in \mathbb{R}} c \quad \text{subject to} \quad \sum_{i=1}^k \alpha_i F^{(i)} - P \preceq c I_n$$

where Δ_k is the unit simplex $\Delta_k := \left\{\alpha \in \mathbb{R}^k : \sum_{i=1}^k \alpha_i = 1,\, \alpha_i \geq 0\right\}$, $k = |\mathcal{F}|$, and $\{F^{(i)}\}_{i=1}^k$ is an enumeration of $\mathcal{F}$.

Remark 2: If the Lyapunov matrix P is given, then Problem 2 is a linear matrix inequality (LMI) feasibility problem easily checked by using semidefinite programming (SDP) solvers. If P should be determined simultaneously, then Problem 2 is a non-convex bilinear matrix inequality feasibility problem.

Proposition 1: ([9, Proposition 1]) Let $P \in \mathbb{S}^n_{++}$ and $h \in \mathbb{N}_+$ be given. If $c(\mathcal{P}_h(\{P\}), P) < 0$, then the SLS (1) is stabilizable, and the switching policy (3) will stabilize the SLS (1).

Example 1: Consider the SLS (1) with $A_1 = \begin{bmatrix} 0.9995 & 0.0656 \\ 0.1323 & 0.4089 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0.9788 & 0.1527 \\ -0.3030 & 2.1905 \end{bmatrix}$. By solving Problem 2, one finds that $c(\mathcal{P}_h(\{I_n\}), I_n) < 0$ is infeasible for $h \in \{1, \ldots, 7\}$, while feasible for $h = 8$. If $P = \begin{bmatrix} 0.7747 & 0.3301 \\ 0.3301 & 1.2250 \end{bmatrix}$, then one has $c(\mathcal{P}_h(\{P\}), P) < 0$ with $h = 4$.

B. Aperiodically quadratic Lyapunov function [7], [8]

As a next step, a more general Lyapunov function, called the *aperiodically quadratic Lyapunov function* proposed in [7], [8], will be discussed. Consider the *aperiodic Lyapunov inequality*

$$\min_{\substack{(i_1, i_2, \ldots, i_j) \in \mathcal{M}^j, \\ j \in \{1, 2, \ldots, h\}}} V(A_{i_j} \cdots A_{i_1} x) - V(x) < 0, \qquad (5)$$

for all $x \in \mathbb{R}^n \backslash \{0_n\}$, where $V(x) = x^T P x$, $P \in \mathbb{S}^n_{++}$, is the aperiodically quadratic Lyapunov function. In contrast to the periodic Lyapunov function, the period of the Lyapunov inequality varies from 1 to h.

Aperiodic controller: According to the aperiodic Lyapunov inequality (5), the corresponding state-feedback switching policy can be defined as

$$\sigma_\infty(z) := (\sigma(v(0)),\, \sigma(v(1)),\, \sigma(v(2)), \ldots) \qquad (6)$$

obtained by concatenating the finite-horizon switching sequence of varying lengths generated by the *aperiodic controller*

$$\sigma(z) := \underset{(i_1, \ldots, i_{j(z)}) \in \mathcal{M}^{j(z)}}{\arg\min} V(A_{i_{j(z)}} \cdots A_{i_1} z)$$

$$j(z) := \underset{j \in \{1, \ldots, h\}}{\arg\min} \min_{(i_1, \ldots, i_j) \in \mathcal{M}^j} V(A_{i_j} \cdots A_{i_1} z)$$

$$v(k+1) = (A_{i_{j(v(k))}} \cdots A_{i_1}) v(k), \quad v(0) = z$$

Remark 3: The switching policy in (3) and (6) can be calculated by solving the minimization problems at the end of all periods. Alternatively, they can be obtained using the dynamic programming approach given in [5].

For any $\mathcal{F} \subset \mathbb{S}_{++}^n$, define the set of matrices

$$\mathcal{H}_h(\mathcal{F}) := \bigcup_{k=1}^{h} \mathcal{P}_k(\mathcal{F}), \tag{7}$$

where $\{\mathcal{P}_k(\mathcal{F})\}_{k=1}^h$ is the sequence of the SRS defined in (4). Then, the aperiodic Lyapunov inequality (5) can be written as $L(\mathcal{H}_h(\{P\}), P) < 0$, and it can be checked by solving Problem 2 with $\mathcal{F} = \mathcal{H}_h(\{P\})$. The following result can be easily proved.

Proposition 2: Let $P \in \mathbb{S}_{++}^n$ and $h \in \mathbb{N}_+$ be given. If $c(\mathcal{H}_h(\{P\}), P) < 0$, then the SLS (1) is stabilizable, and the switching policy (6) will stabilize the SLS.

There are some useful features of this generalization.

Proposition 3: The following statements are true:

1) $\mathcal{P}_0 \subseteq \mathcal{H}_0, \mathcal{P}_1 \subseteq \mathcal{H}_1, \cdots$;
2) $\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots$;
3) $L(\mathcal{H}_h(\{P\}), P) \le L(\mathcal{P}_h(\{P\}), P)$ and $c(\mathcal{H}_h(\{P\}), P) \le c(\mathcal{P}_h(\{P\}), P)$;
4) (Monotonicity): $L(\mathcal{H}_0(\{P\}), P) \ge L(\mathcal{H}_1(\{P\}), P) \ge L(\mathcal{H}_2(\{P\}), P) \ge \cdots$ and $c(\mathcal{H}_0(\{P\}), P) \ge c(\mathcal{H}_1(\{P\}), P) \ge c(\mathcal{H}_2(\{P\}), P) \ge \cdots$.

From Proposition 3, a benefit of using the aperiodic Lyapunov function is that, as the maximum period h gets larger, both $L(\mathcal{H}_h(\{P\}), P)$ and $c(\mathcal{H}_h(\{P\}), P)$ are non-increasing. Therefore, this approach can be less conservative than the periodic Lyapunov function approach.

Example 2: Consider the SLS given in Example 1 again. For this system, $c(\mathcal{P}_h(\{I_n\}), I_n) < 0$ is infeasible for $h \in \{1, \dots, 7\}$, while $c(\mathcal{H}_h(\{I_n\}), I_n) < 0$ is achieved for $h = 7$.

Remark 4: The number of elements of $\mathcal{P}_h$ exponentially increases as h increases. For the aperiodic Lyapunov approach, $|\mathcal{H}_h|$ increases even more quickly. To alleviate the computational complexity, the numerical relaxation technique employed in [4], [6] can be adopted.

C. Quadratic graph Lyapunov function [14]

In this subsection, we will generalize the graph-theoretic Lyapunov theory in [14] to the stabilization problem. Accordingly, the Lyapunov theorem in [14] should be modified. To proceed, some notions in [14] will be briefly reviewed. Hereafter, we will think of the set of system submatrices $\mathcal{A} := \{A_1, \dots, A_N\}$ as a finite *alphabet* and we will refer to a finite product of matrices from this set as a *word*. The set of all words $A_{i_h} \cdots A_{i_1}$ of length h is denoted by $\mathcal{A}^h$. Consider m quadratic functions $V_1(x) = x^T P_1 x, \dots, V_m(x) = x^T P_m x$ characterized by the set $\{P_1, \dots, P_m\} \subset \mathbb{S}_{++}^n$. In [14], a set of Lyapunov inequalities is represented by a labelled *directed graph (or digraph)* $G(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \dots, m\}$ the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges (a set of ordered pairs of nodes). Each node of this digraph is assigned to a Lyapunov function V_i, $i \in \mathcal{V}$, and each edge is labelled by a finite product of matrices, i.e., a word from the set $\mathcal{A}^h$. In [14], it was proved that, if the digraph $G(\mathcal{V}, \mathcal{E})$ associated with the set of Lyapunov function candidates satisfies a certain condition, then the *switching system (arbitrarily switched linear system)* is asymptotically stable. Some standard definitions in graph theory are presented below.

Definition 2: For a given node $i \in \mathcal{V}$ of a digraph, $\mathcal{N}_i^- := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ is called the *in-neighbor* of the node i, $\mathcal{N}_i^+ := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ is called the *out-neighbor* of the node i, $|\mathcal{N}_i^-|$ is the number of incoming edges adjacent to i and is called the *indegree* of the node i. $|\mathcal{N}_i^+|$ is the number of outgoing edges adjacent to it is called its *outdegree* of the node i. A node of a digraph with zero indegree is called a *source* and a node with zero outdegree is called a *sink*.

For a subset $\mathcal{A}_{j\to i} \subseteq \bigcup_{k=1}^h \mathcal{A}^k$, consider the Lyapunov inequality

$$\min_{A \in \mathcal{A}_{j\to i}} V_i(Az) - V_j(z) < 0, \quad \forall z \in \mathbb{R}^n \backslash \{0_n\}. \tag{8}$$

This Lyapunov inequality is expressed as a labelled digraph $G(\mathcal{V}, \mathcal{E})$ that consists of two nodes $\{i, j\}$ and a directed edge (j, i) from node j to node i which is labelled by the set of words $\mathcal{A}_{j\to i}$. For instance, consider the SLS (1) with two subsystems. If the Lyapunov inequality (8) holds for $\mathcal{A}_{j\to i} = \{(A_1A_2), (A_1A_1), (A_2A_1A_1)\}$, then it can be represented by the labelled digraph shown in Fig. 1.

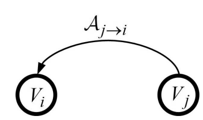

Fig. 1. Graphical representation of Lyapunov inequalities developed in [14]. The digraph above corresponds to the Lyapunov inequality $\min_{A \in \mathcal{A}_{j\to i}} V_i(Az) - V_j(z) < 0, \forall z \in \mathbb{R}^n \backslash \{0_n\}$, where $\mathcal{A}_{j\to i} = \{(A_1A_2), (A_1A_1), (A_2A_1A_1)\}$.

In general, denote by $\mathcal{J} := \{j \in \mathcal{V} : |\mathcal{N}_i^+| > 0\}$ a subset of $\mathcal{V}$ that have outgoing edges (a set of nodes that are not sinks) and denote by

$$\mathcal{A}_{j\to i} \subseteq \bigcup_{k=1}^{h} \mathcal{A}^k \tag{9}$$

the set of words that corresponds to the edge from the node j to node i for all $j \in \mathcal{J}$, $i \in \mathcal{N}_j^+$.

Definition 3 (Graph Lyapunov inequality): Given a labelled digraph $G(\mathcal{V}, \mathcal{E})$ labelled by the set of words $\mathcal{A}_{j\to i}$, the set of Lyapunov inequalities

$$\min_{\substack{A \in \mathcal{A}_{j\to i} \\ i \in \mathcal{N}_j^+}} V_i(Az) - V_j(z) < 0, \quad \forall z \in \mathbb{R}^n \backslash \{0_n\}, j \in \mathcal{J} \tag{10}$$

is called the *graph Lyapunov inequality* associated with the labelled digraph $G(\mathcal{V}, \mathcal{E})$.

The graph Lyapunov inequality for $j \ in \mathcal{V}$ associated with $G(\mathcal{V}, \mathcal{E})$ corresponds to a directed rooted tree, where every directed edges have orientations away from the node j. Based on the notions, the *graph Lyapunov function* is defined as follows.

Definition 4 (Graph Lyapunov function): If there exist a set of the quadratic functions $\{V_1, \dots, V_m\}$, a labelled digraph $G(\mathcal{V}, \mathcal{E})$, and associated labels $\mathcal{A}_{j\to i}$, $\forall j \in \mathcal{J}$, $i \in \mathcal{N}_j^+$ defined in (9) such that

1) $G(\mathcal{V}, \mathcal{E})$ has no sink;
2) the graph Lyapunov inequality (10) associated with $G(\mathcal{V}, \mathcal{E})$ is satisfied,

then the quadratic functions $\{V_1, \dots, V_m\}$ is called the *(quadratic) graph Lyapunov functions* of the SLS (1).

The stabilizability of the SLS (1) can be proved using the graph Lyapunov functions.

Proposition 4 (Graph Lyapunov theorem): If the SLS (1) admits the graph Lyapunov functions $\{V_1, \dots, V_m\}$, then the SLS (1) is stabilizable.

Proof: The proof follows the idea of the proof of [14, Theorem 2.4]. By the definition of the graph Lyapunov function in Definition 4, any node $j \in \mathcal{J}$ has a path (edge) to another node i because the associated graph has no sink. Therefore, if one defines $(A_j^*, i^*) := \underset{A\in\mathcal{A}_{j\to i},\, i\in\mathcal{N}_j^+}{\arg\min} V_i(Az)$, then there exist a constant $\kappa_{j\to i^*} \in \mathbb{R}_{++}$ such that $V_{i^*}(A_j^* z) - V_j(z) \le -\kappa_{j\to i^*}\|z\|^2$ for all $z \in \mathbb{R}^n\backslash\{0_n\}$. Consider the sequence of the nodes $(j_0, j_1, \dots)$ and the sequence of the words $(A_{j_0}^*, A_{j_1}^*, \dots)$ driven by $(A_{j_k}^*, j_{k+1}) := \underset{A\in\mathcal{A}_{j_k\to i},\, i\in\mathcal{N}_{j_k}^+}{\arg\min} V_i(Ax(k))$ and $x(k+1) = A_{j_k}^* x(k)$, where $x(0) \in \mathbb{R}^n$ is the initial state, the initial node $j_0 \in \mathcal{J}$ can be arbitrarily selected, and k is not the time but the aperiodically sampled time. Then, the last inequality can be written by $V_{j_{k+1}}(A_{j_k}^* x(k)) - V_{j_k}(x(k)) \le -\kappa_{j_k\to j_{k+1}}\|x(k)\|^2$, which implies

$$\left(\frac{1}{1+\frac{\kappa_{j_k\to j_{k+1}}}{\lambda_{\max}(P_{j_k})}}\right) V_{j_k}(x(k)) \ge V_{j_{k+1}}(A_{j_k}^* x(k)).$$

Letting $\gamma := \max_{j\in\mathcal{J},\, i\in\mathcal{N}_j^+}\left(\frac{1}{1+\frac{\kappa_{j\to i}}{\lambda_{\max}(P_j)}}\right) < 1$, one gets $\gamma^{k+1}V_{j_0}(z) \ge V_{j_{k+1}}(A_{j_k}^* \cdots A_{j_0}^* z)$, from which it follows that $\gamma^k(p_{\max}/p_{\min})\|z\|^2 \ge \|x(k)\|^2$, where $p_{\max} := \max_{i\in\mathcal{J}} \lambda_{\max}(P_i)$ and $p_{\min} := \min_{i\in\mathcal{J}} \lambda_{\min}(P_i)$. Therefore, $\|x(k)\|^2 \to 0$ as $k \to \infty$. This means that the state trajectory under the switching policy corresponding to the word $A_{j_k}^* \cdots A_{j_0}^*$ converges to zero. ■

Remark 5: The no sink assumption of the graph Lyapunov function in Definition 4 is used in the proof of Proposition 4 so that any node $j \in \mathcal{J}$ has a path (edge) to another node i in the out-neighbor of the node j.

Interpretation: From Proposition 4, the *periodically and aperiodically quadratic Lyapunov functions* can be interpreted as special cases of the graph Lyapunov function. The periodically quadratic Lyapunov function is a quadratic function $V_1(z)$ that satisfies the graph Lyapunov inequalities in (10) with $\mathcal{A}_{j\to i} = \mathcal{A}^h$, $\mathcal{J} = \{1\}$, $\mathcal{N}_1^+ = \{1\}$. The aperiodically quadratic Lyapunov function is a quadratic function $V_1(z)$ that satisfies the graph Lyapunov inequalities in (10) with $\mathcal{A}_{j\to i} = \bigcup_{k=1}^h \mathcal{A}^k$, $\mathcal{J} = \{1\}$, $\mathcal{N}_1^+ = \{1\}$. For example, for the SLS (1) with two subsystems, the labelled digraphs corresponding to the periodically and aperiodically quadratic Lyapunov functions with $h = 2$ are illustrated in Figs. 2 and 3, respectively. Overall, the inclusion relations are visualized in Fig. 4.

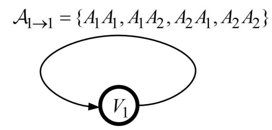

Fig. 2. Graphical representation of the periodic Lyapunov inequality with $h = 2$ for the SLS (1) with two subsystems.

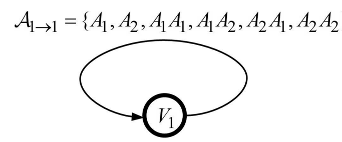

Fig. 3. Graphical representation of the aperiodic Lyapunov inequality with $h = 2$ for the SLS (1) with two subsystems.

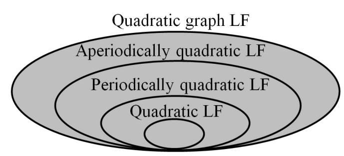

Fig. 4. Inclusion relations of several Lyapunov functions (LFs).

Another interpretation is that finding the graph Lyapunov function can be viewed as finding an aperiodic Lyapunov function, the length of the period of which varies from a finite number to infinite. To explain this, let us consider a traveller who travels from one node to another of the digraph $G(\mathcal{V}, \mathcal{E})$. Since the digraph has no sink, the tour will not be finished. In addition, since the number of nodes is finite, at least one node will be visited infinitely many times. Therefore, the quadratic Lyapunov function that corresponds to the node is an aperiodic Lyapunov function whose range of periods can be arbitrarily large. Therefore, finding a graph Lyapunov function provides a computationally efficient way to find an aperiodic Lyapunov function with longer periods in the sense that we do not need to check the Lyapunov inequalities with infinite periods.

D. Computation of the graph Lyapunov function

In this subsection, an algorithm to find a graph Lyapunov function is suggested. First of all, a notion presented in [4] is briefly reviewed. For any $\mathcal{F} \subset \mathbb{S}_+^n$, a subset $\mathcal{F}^\varepsilon \subseteq \mathcal{F}$ is called an ε-equivalent set of $\mathcal{F}$ or an ε-ES of $\mathcal{F}$ for some $\varepsilon > 0$ if

$$\min_{H\in\mathcal{F}} z^T Hz \le \min_{H\in\mathcal{F}^\varepsilon} z^T Hz \le \min_{H\in\mathcal{F}} z^T Hz + \varepsilon\|z\|^2 \quad (11)$$

for all $z \in \mathbb{R}^2$. As stated in [4], to compute an ε-ES of $\mathcal{F}$, one needs to remove the matrices $F \in \mathcal{F}$ that satisfy the condition (11). A convex test to identify a matrix satisfying

([11](#)) was developed in [4, Lemma 5]. Using this condition, an efficient algorithm [4, Algorithm] was developed to obtain an ε-ES of $\mathcal{F}$. In this paper, we will denote the algorithm by $\mathcal{F}^{\varepsilon} = ES_{\varepsilon}(\mathcal{F})$. Assume that the set $\mathcal{F} := \{P_1, P_2, \ldots, P_m\}$ of Lyapunov matrices is given. The Lyapunov inequalities in (10) can be written by

$$L(\mathcal{H}_j, P_j) < 0, \forall j \in \mathcal{J}, \tag{12}$$

where $\mathcal{H}_j := \bigcup_{i \in \mathcal{N}_j^+} \{A^T P_i A \in \mathbb{S}_+^n : A \in \mathcal{A}_{j \to i}\}$. Motivated by this, it is possible to develop algorithms based on the SRS and the ε-relaxed SRS. Firstly, Algorithm 1, denoted by $\mathbf{Tree}_j(\cdot)$, is introduced.

Algorithm 1 $[\mathbf{Tree}_j(\cdot)]$

Input: $\mathcal{F} \subset \mathbb{S}_{++}^n$, $\varepsilon \in \mathbb{R}_+$, $h_{\max} \in \mathbb{N}_+$
Output: $\mathcal{H} \subset \mathbb{S}_{++}^n$

Set $h = 0$, $\mathcal{P}_0^{\varepsilon} = \mathcal{F}$, $\mathcal{H} = \emptyset$
repeat
  $h \leftarrow h + 1$
  $\mathcal{P}_h^{\varepsilon} = ES_{\varepsilon}(\rho_{\mathcal{M}}(\mathcal{P}_{h-1}^{\varepsilon}))$
  $\mathcal{H}_h^{\varepsilon} := \bigcup_{i=1}^{h} \mathcal{P}_i^{\varepsilon}$
  if $c(\mathcal{H}_h^{\varepsilon}, P) < 0$ **then**
    $\mathcal{H} \leftarrow \mathcal{H}_h^{\varepsilon}$
  end if
until $c(\mathcal{H}_h^{\varepsilon}, P) < 0$ or $h \geq h_{\max}$
return $\mathcal{H}$

Let $\mathcal{H}_j = \mathbf{Tree}_j(\mathcal{F}, \varepsilon, h_{\max})$. If $\mathcal{H}_j = \emptyset$, then Algorithm 1 cannot find the sets of words $\mathcal{A}_{j \to i} \subseteq \bigcup_{k=1}^{h} \mathcal{A}^k$, $i \in \{1, \ldots, m\}$ corresponding to the edges from the node j to others. If $\mathcal{H}_j \neq \emptyset$, then (12) holds with $\mathcal{H}_j = \mathbf{Tree}_j(\mathcal{F}, \varepsilon, h_{\max})$. The overall algorithm to find the graph Lyapunov function is presented in Algorithm 2. For a given $\mathcal{F} \subset \mathbb{S}_{++}^n$, let $(\mathcal{J}, \{\mathcal{H}_j\}_{j \in \mathcal{J}}) = \mathbf{Algo}(\mathcal{F})$. Since the output of Algorithm 2 satisfies (12), $\{V_j\}_{j \in \mathcal{J}}$ can be a possible candidate of the graph Lyapunov function, where $\mathcal{J}$ is a set of the nodes that have nonzero outdegrees.

Proposition 5: Let $\mathcal{F} = \{P_1, \ldots, P_m\} \subset \mathbb{S}_{++}^n$ be given, and suppose $(\mathcal{J}, \{\mathcal{H}_j\}_{j \in \mathcal{J}}) = \mathbf{Algo}(\mathcal{F})$. Then, $c(\mathcal{H}_j, P_j) < 0$, $\forall j \in \mathcal{J}$ and (12) hold.

Remark 6: The condition of Proposition 5 is an LMI and can be checked using the SDP solvers. Therefore, for a given SLS (1), if the condition of Proposition 5 holds, and the associated graph has no sink, then $V_i(x) = x^T P_i x$, $i \in \mathcal{V}$ is a graph Lyapunov function, and by Proposition 4, then SLS (1) is switching stabilizable.

Remark 7: Note that [7, Theorem 1] provides a set-theoretic necessary and sufficient condition to check the stabilizability of the SLS (1). It was proved that the SLS (1) is stabilizable if and only if the the proposed algorithm [7, Algorithm 1] ends within a finite iterations. In this sense, [7, Algorithm 1] is a nonconservative test of the stabilizability. A natural question is as follows: is the proposed algorithm (Algorithm 2 and the test to check that the associated graph has no sink) nonconservative? The answer is no because, as also pointed out in [8], the proposed algorithm relies on sufficient LMI tests, and hence, even if the SLS (1) is switching stabilizable, the algorithm can fail to identify its stabilizability.

Algorithm 2 $[\mathbf{Algo}(\cdot)]$

Input: $\mathcal{F} \subset \mathbb{S}_{++}^n$
Output: $\mathcal{J} \subset \mathbb{N}_+$, $\{\mathcal{H}_j\}_{j \in \mathcal{J}} \subset \mathbb{S}_{++}^n$

Specify proper values for $\varepsilon \in \mathbb{R}_+$ and $h_{\max} \in \mathbb{N}_+$
$\mathcal{H}_j = \emptyset$, $\forall j \in \mathcal{V}$, $\mathcal{J} = \emptyset$
for $j \in \mathcal{V}$ **do**
  $\mathcal{H}_j \leftarrow \mathbf{Tree}_j(\mathcal{F}, \varepsilon, h_{\max})$
  if $\mathcal{H}_j \neq \emptyset$ **then**
    $\mathcal{J} \leftarrow \mathcal{J} \cup \{j\}$
  end if
end for
return $(\mathcal{J}, \{\mathcal{H}_j\}_{j \in \mathcal{J}})$

E. Periodically and aperiodically piecewise quadratic Lyapunov functions [7], [8]

The concept of the periodic and aperiodic Lyapunov functions can be extended to more general classes of Lyapunov functions. For the stabilization of the SLS (1), a piecewise quadratic function has been suggested in [4]. It can be written as a pointwise minimum of a finite number of quadratic functions as follows:

$$V_{\mathcal{F}}(z) := \min_{P \in \mathcal{F}} z^T P z, \tag{13}$$

where $\mathcal{F} := \{P_1, P_2, \ldots, P_m\}$ is a set of positive definite matrices. From [4, Theorem 3], the SLS (1) is stabilizable if the minimum of the one-sample difference of $V_{\mathcal{F}}(z)$ is negative definite, i.e., $\min_{i \in \mathcal{M}} V_{\mathcal{F}}(A_i z) - V_{\mathcal{F}}(z) < 0$, $\forall z \in \mathbb{R}^n \backslash \{0_n\}$, which can be represented by

$$V_{\rho_{\mathcal{M}}(\mathcal{F})}(z) - V_{\mathcal{F}}(z) < 0, \quad \forall z \in \mathbb{R}^n \backslash \{0_n\}. \tag{14}$$

A sufficient condition to check (14) is given below.

Lemma 1: ([4, Corollary 1]) If $c(\rho_{\mathcal{M}}(\mathcal{F}), P) < 0$, $\forall P \in \mathcal{F}$, then (14) holds.

This class of piecewise quadratic Lyapunov functions can be generalized to the *periodically piecewise quadratic Lyapunov function* that satisfies $\min_{(i_1, \ldots, i_h) \in \mathcal{M}^h} V_{\mathcal{F}}(A_{i_h} \cdots A_{i_1} z) - V_{\mathcal{F}}(z) < 0$ for all $z \in \mathbb{R}^n \backslash \{0_n\}$, or equivalently,

$$V_{\mathcal{P}_h}(z) - V_{\mathcal{F}}(z) < 0, \quad \forall z \in \mathbb{R}^n \backslash \{0_n\} \tag{15}$$

with $\mathcal{P}_0 = \mathcal{F}$. Similarly to Lemma 1, we have the following result.

Proposition 6: If $c(\mathcal{P}_h, P) < 0$, $\forall P \in \mathcal{F}$ with $\mathcal{P}_0 = \mathcal{F}$, then (15) holds.

More generally, the *aperiodically piecewise quadratic Lyapunov function* can be defined as $\min_{\substack{(i_1, \ldots, i_j) \in \mathcal{M}^j \\ j \in \{1, \ldots, h\}}} V_{\mathcal{F}}(A_{i_j} \cdots A_{i_1} z) - V_{\mathcal{F}}(z) < 0$ for all $z \in \mathbb{R}^n \backslash \{0_n\}$, and equivalently,

$$V_{\mathcal{H}_h}(z) - V_{\mathcal{F}}(z) < 0, \quad \forall z \in \mathbb{R}^n \backslash \{0_n\}, \tag{16}$$

where $\mathcal{H}_h := \bigcup_{i=1}^{h} \mathcal{P}_i$ is defined in (7), and $\mathcal{P}_k = \rho_{\mathcal{M}}(\mathcal{P}_{k-1})$, $\mathcal{P}_0 = \mathcal{F}$. This class of Lyapunov functions was considered in [8, Proposition 8]. The following result can be readily derived.

Proposition 7: If $c(\mathcal{H}_h, P) < 0, \forall P \in \mathcal{F}$ with $\mathcal{P}_k = \rho_{\mathcal{M}}(\mathcal{P}_{k-1})$, $\mathcal{P}_0 = \mathcal{F}$, then (16) holds.

The aperiodically piecewise quadratic Lyapunov function can be regarded as one of the most general classes of Lyapunov functions presented in this paper, but still can be interpreted as a graph Lyapunov function. The condition in Proposition 7 is computationally demanding to check but possibly less conservative than others.

Interpretation: By comparing Propositions 7 and 5, it can be observed that the condition of Proposition 5 with $\mathcal{H}_j = \mathcal{H}_h, \forall j \in \mathcal{J}$ reduces to the condition of Proposition 7. Generally, the aperiodically piecewise quadratic Lyapunov function can be viewed as a graph Lyapunov function whose digraph is forced to be a complete digraph in which every pair of nodes is connected by a bidirectional edge. In this respect, the graph Lyapunov function approach is more general then the aperiodically piecewise quadratic Lyapunov approach. For instance, consider the SLS (1) with two subsystems and its aperiodically piecewise quadratic Lyapunov function with two quadratic functions V_1 and V_2 and $h = 2$. The digraph of the aperiodically piecewise quadratic Lyapunov function is shown in Fig. 5, where $\mathcal{A}_{1\to1} = \mathcal{A}_{1\to2} = \mathcal{A}_{2\to1} = \mathcal{A}_{2\to2} = \{A_1, A_2, A_1A_1, A_1A_2, A_2A_1, A_2A_2\}$, while an example of the digraph of a graph Lyapunov function is illustrated in Fig. 5, where $\mathcal{A}_{1\to1}$, $\mathcal{A}_{1\to2}$, and $\mathcal{A}_{2\to1}$ are not forced to have the same structure, and each node can be assigned to its own set of words, i.e., $\mathcal{A}_{1\to1} = \{A_1, A_2\}$, $\mathcal{A}_{1\to2} = \{A_2A_1, A_2A_2\}$, and $\mathcal{A}_{2\to1} = \{A_2, A_1A_1\}$. In this regard, a benefit of considering the graph Lyapunov inequality is that it allows more flexibility in the analysis and the control design of the SLS (1).

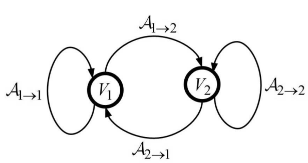

Fig. 5. Complete digraph of a aperiodically piecewise quadratic Lyapunov function with two quadratic functions V_1 and V_2 and $h = 2$, where $\mathcal{A}_{1\to1} = \mathcal{A}_{1\to2} = \mathcal{A}_{2\to1} = \mathcal{A}_{2\to2} = \{A_1, A_2, A_1A_1, A_1A_2, A_2A_1, A_2A_2\}$.

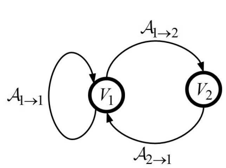

Fig. 6. Digraph of a quadratically graph Lyapunov function with two quadratic functions V_1 and V_2, where $\mathcal{A}_{1\to1} = \{A_1, A_2\}$, $\mathcal{A}_{1\to2} = \{A_2A_1, A_2A_2\}$, and $\mathcal{A}_{2\to1} = \{A_2, A_1A_1\}$.

On the other hand, the graph Lyapunov inequality with the maximum length of the words equal to $h_{\max}$ can be regarded as a aperiodically piecewise quadratic Lyapunov inequality with a reduced structure and with the maximum period $h_{\max}$.

IV. Conclusion

In this paper, a class of graph Lyapunov functions has been introduced for the stabilization of the SLSs. It unifies several types of Lyapunov functions including non-monotonic Lyapunov functions, the periodic and aperiodic Lyapunov functions.

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