Distributed Continuous-Time Protocol for Network Localization Using Angle-of-Arrival Information

Guangwei Zhu, Jianghai Hu

***Abstract*— In this paper, we propose a distributed continuous-time linear dynamics for solving the localization problem based on signal angle-of-arrival information. We first formulate the AOA localization problem within the framework of formation graph theory, which is an extension of the classical graph theory by incorporating the positional information of the vertices. Solving the AOA localization problem is equivalent to finding the solution to a system of linear equations. To avoid matrix inversion, we propose a continuous-time dynamics whose global asymptotic equilibrium is the desired localization. This dynamics turns out to have a very similar expression to that of the continuous-time consensus dynamics. The convergence and delay performance of the protocol is also studied. We finally argue that through optimizing the condition number of the stiffness matrix, the convergence and delay performance of the protocol can be simultaneously improved.**

***Index Terms*— localization, angle-of-arrival (AOA), estimation, stiffness matrix, optimization, distributed algorithm**

I. Introduction

Network localization is one of the primary functions that are commonly desired in multi-agent systems, as the positional information may crucially help deciding an agent's behavior or identify the meaning of the data collected by the agents, e.g., in sensor networks for forest fire monitoring [1]. Directly using positioning devices such as GPS to localize every agent in the work is often infeasible for systems with a large number of agents that require relatively low resolution localization , due to cost and energy issues. A more practical method is to equip a small number of agents with high precision positioning capability, while letting other agents localize themselves through local inter-agent measurements. Finding effective localization schemes that only utilize local measurements has become a popular research topic in the study of multi-agent systems [2].

Based on the type of measurements adopted, those localization schemes can be roughly classified into two categories: distance-based schemes and direction-based schemes. In the distance-based schemes, the neighboring agents measure their relatively distances and then attempt to recover the global positional configuration of the system. However, a common issue with such schemes is that eliminating ambiguity is sometimes very difficult, and the generated solution may tremendously deviate from the true configuration due to reflection ambiguity [3]. This is mainly because the distance-based localization problem is essentially NP-hard [4], [5], and the uniqueness of the solution is guaranteed only when the graph is globally rigid [6]. Although the problem may be significantly simplified [7] with some additional assumptions, the distance-based localization remains an open and challenging problem.

On the other hand, the direction-based localization methodology uses the angle measurement (angle of arrival (AOA) of the signal) instead of relative distance. Direction-based localization schemes have not received adequate attention mainly because of two drawbacks. First, localization schemes based on relative angle measurements is also an NP-hard problem [8]. However, if we allow compasses to be installed to create a global coordinate for the measured angles, the direction-based localization could be as easy as solving a set of linear equations [9], [10]. Second, measuring AOA of RF signals requires more advanced hardware, such as antenna arrays, which are usually expensive and large in size due to wavelength constraint. Nevertheless, in many environmental monitoring applications, the agents/sensors are already equipped with acoustic sensing devices [11], thus acquiring AOA information via acoustics may be a natural inexpensive extension [2]. As the signal measurement technologies and hardware manufacturing advance, these drawbacks could be alleviated to a point where the AOA localization becomes the desired method for localization.

In this paper, we first formulate the AOA localization problem within the framework of formation graph theory, which is an extension of the classical graph theory by incorporating the positional information of the vertices. We show that the AOA localization problem is equivalent to finding the solution to a system of linear equations [9], [10]. To avoid the matrix inversion, we propose a continuous-time protocol whose global asymptotic equilibrium is the desired localization result. The continuous-time AOA localization protocol turns out to be similar to the continuous-time consensus protocol in [12] and also the distributed formation controller proposed in [13]. The convergence and delay performance of the protocol is also studied. We argue that through optimizing the condition number of the state matrix, the performance of the protocol can be improved.

The main contribution of this paper is that we formulate and study the AOA localization problem from a continuous-time dynamics point of view, and provide a new perspective on the performance analysis of such localization process. Many important properties regarding the underlying graph topology are revealed, which can help designing multi-agent systems that are easier and more robust for localization purpose.

The organization of this paper is as follows. Notations

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and background on formation graph theory are introduced in Section II. In Section III, the AOA localization problem is formulated and solved, and a necessary and sufficient condition on the AOA localizability is given. In Section IV we propose a distributed continuous-time protocol to obtain the solution of the AOA localization problem, with its convergence and delay performance analyzed and quantified. Numerical examples are illustrated in Section V. We conclude this paper and give some prospective research directions in Section VI.

II. Preliminaries

A. Basic Notation

For symmetric matrices A, B, we write $A \succeq 0$ if A is positive semidefinite. For $\mathbf{v} = [a \ b]^\top \in \mathbb{R}^2$, we use the notation $\angle\mathbf{v}$ to denote the principal value of argument (within the range $[0, 2\pi)$) of the complex number $a + bi$ in the complex plane. We use the notation $\mathbf{p}_i^\perp$ for any $\mathbf{p}_i \in \mathbb{R}^2$ to denote the vector $\mathbf{p}_i$ rotated by 90 degrees counterclockwise. If $\mathbf{p}$ is a stacked vector with $\mathbf{p}_i$ as its components, then $\mathbf{p}^\perp$ has components $\mathbf{p}_i^\perp$. We also define the symbolic operator Q as $Q : \mathbf{p} \mapsto \mathbf{p}^\perp$ and $Q^{-1} : \mathbf{p} \mapsto -\mathbf{p}^\perp$, regardless of the dimension of $\mathbf{p}$.

B. Formation Graph Theory

In what follows we will give a quick introduction to the theoretic framework we use to study the general applications concerning the formations of multi-agent systems, of which network localization is one instance.

Definition 1 (Formation Graph): A formation graph is a triple $(\mathcal{V}, \mathbf{p}, K)$ consisting of the following:

- $\mathcal{V} = \{1, 2, \ldots, n\}$ is the index set of n vertices (agents, sensor vertices, etc.) on the plane;
- $\mathbf{p} = \begin{bmatrix}\mathbf{p}_1^\top & \mathbf{p}_2^\top & \cdots & \mathbf{p}_n^\top\end{bmatrix}^\top \in \mathbb{R}^{2n}$ is the (position) *configuration* of the n vertices, with the assumption $\mathbf{p}_i = \mathbf{p}_j \Leftrightarrow i = j$, where $\mathbf{p}_i \in \mathbb{R}^2$ denotes the position of vertex i;
- $K = [k_{ij}]_{i,j\in\mathcal{I}} \in \mathbb{R}^{n\times n}$ is the *connectivity matrix*, where for each pair of vertices $i, j \in \mathcal{I}$, k_{ij} is the *connectivity coefficient* between them and satisfies $k_{ii} = 0$, $k_{ij} \geq 0$, and $k_{ij} = k_{ji}$. Denote by $\mathcal{K}$ the set of all such K.

Definition 2 (Anchored Formation Graph): A formation graph $(\mathcal{V}, \mathbf{p}, K)$ associated with a nonempty set $\mathcal{A} \subset \mathcal{V}$ is called an *anchored formation graph*, denoted by a quadruple $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$, where each vertex in $\mathcal{A}$ is called an *anchor* and each vertex in $\mathcal{F} \triangleq \mathcal{V} \setminus \mathcal{A}$ a *free vertex*.

In the applications of network localization and formation control, anchors usually refer to the special nodes in the network which know their absolute locations, e.g., by using positioning devices.

Eren *et. al* showed that for the distance-based localization, the agents' positions can be uniquely recovered from the distance measurements only when the underlying formation graph is *globally rigid* [6]. But to determine whether a formation graph is globally rigid is generally not easy, especially in a distributed fashion. Alternatively, one may analyze the rigidity of formation graph $(\mathcal{V}, \mathbf{p}, K)$ from an infinitesimal perspective as follows. Let the distance between each pair of connected ($k_{ij} > 0$) vertices be constant, i.e.,

$$k_{ij}\|\mathbf{p}_j - \mathbf{p}_i\|^2 \equiv k_{ij} d_{ij}^2 \quad (i \neq j)$$

Assume there is virtual movement of the vertices. Taking the derivative with respect to time, we have

$$k_{ij}(\mathbf{p}_j - \mathbf{p}_i)^\top (\dot{\mathbf{p}}_j - \dot{\mathbf{p}}_i) = 0 \quad (i \neq j)$$

These equations can be put into a matrix form as below,

$$\Lambda_K R \dot{\mathbf{p}} = 0, \tag{1}$$

where R is the *normalized complete rigidity matrix* whose rows are $\mathbf{r}^{(ij)\top}$ for $i < j$,

$$\mathbf{r}^{(ij)\top} \triangleq \begin{bmatrix}\mathbf{0} & \cdots & \mathbf{0} & \underbrace{\mathbf{e}_{ij}^\top}_{i\text{-th block}} & \cdots & \underbrace{\mathbf{e}_{ji}^\top}_{j\text{-th block}} & \mathbf{0} & \cdots & \mathbf{0}\end{bmatrix},$$

$$\mathbf{e}_{ij} \triangleq \frac{\mathbf{p}_j - \mathbf{p}_i}{\|\mathbf{p}_j - \mathbf{p}_i\|},$$

and Λ_K is the diagonal matrix whose diagonal entries are the k_{ij}. Here we introduction the definition of *stiffness matrix* based on (1).

Definition 3 (Stiffness Matrix [14]): The stiffness matrix S is defined as

$$S \triangleq R^\top \Lambda_K R, \tag{2}$$

Remark 1: The stiffness matrix S is square and has the block structure $S = [S_{ij}]$, each $S_{ij} \in \mathbb{R}^{2\times 2}$ for $i, j \in \mathcal{V}$, where

$$\begin{cases} S_{ii} = \sum_{j\in\mathcal{V}\setminus\{i\}} k_{ij} P_{ij}, \\ S_{ij} = -k_{ij} P_{ij}, & \text{if } i \neq j \end{cases} \tag{3}$$

and P_{ij} is the project matrix defined by $P_{ij} \triangleq \mathbf{e}_{ij}\mathbf{e}_{ij}^\top$. Therefore, all the diagonal blocks are positive semidefinite, while each offdiagonal blocks are negative semidefinite, and each block row and block column adds up to zero. This structure resembles that of the well-known graph Laplacian [15], with the difference that the stiffness matrix S has twice the dimension as the graph Laplacian.

Using the stiffness matrix, the solutions to (1) are equivalently given by the normal equation $S\dot{\mathbf{p}} = 0$ of (1). Note that the solution space of (1) must contain a three-dimension subspace corresponding to the simultaneous translations and rotation (i.e., rigid body motions) of the vertices. Therefore, it can be seen that $\operatorname{rank}(S) \leq 2n - 3$.

Definition 4 (Infinitesimal Rigidity [14], [16]): A formation graph is *infinitesimally rigid* if $\operatorname{rank}(S) = 2n - 3$.

An intuitive interpretation of infinitesimal rigidity is that no infinitesimal perturbation which maintains the distances between connected vertices will lead to shape deformation.

For the anchored formation graphs, we would like to derive a property that correponds to the infinitesimal rigidity. Recall that the vertex set $\mathcal{V}$ can be divided into free vertices $\mathcal{F}$ and anchors $\mathcal{A}$. We may partition the normalized complete rigidity matrix R accordingly as $R = \begin{bmatrix}R_f & R_a\end{bmatrix}$, where R_f

(R_a) contains the i-th block column of R if $i \in \mathcal{F}$ ($i \in \mathcal{A}$). Consequently, the stiffness matrix S can be partitioned as

$$S = \begin{bmatrix} R_f^\top \\ R_a^\top \end{bmatrix} \Lambda_K \begin{bmatrix} R_f & R_a \end{bmatrix} = \begin{bmatrix} S_{ff} & S_{fa} \\ S_{af} & S_{aa} \end{bmatrix}. \quad (4)$$

We also use the notation $\mathbf{p}_f$ and $\mathbf{p}_a$ to denote the stacked vector of the components in $\mathbf{p}$ that correspond to the vertices in $\mathcal{F}$ and $\mathcal{A}$, respectively.

Definition 5 (Fixability): An anchored formation graph $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ is *fixable* if S_{ff} is nonsingular.

Theorem 1: If $(\mathcal{V}, \mathbf{p}, K)$ is infinitesimally rigid, then any anchored formation graph $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ with $|\mathcal{A}| \geq 2$ is fixable.

Proof: Suppose $\mathbf{u} = \begin{bmatrix} \mathbf{u}_f^\top & \mathbf{u}_a^\top \end{bmatrix}^\top$ with $\mathbf{u}_a = \mathbf{0}$ satisfies $S\mathbf{u} = \mathbf{0}$. We have $S\mathbf{u} = \mathbf{0} \Rightarrow S_{ff}\mathbf{u}_f = \mathbf{0}$. Since $(\mathcal{V}, \mathbf{p}, K)$ is infinitesimally rigid, the nullity of S is 3. Specifically,

$$\begin{bmatrix} \mathbf{u}_f \\ \mathbf{0} \end{bmatrix} \in \operatorname{null}(S) = \operatorname{span}\left\{ \begin{bmatrix} \mathbf{x}_f \\ \mathbf{x}_a \end{bmatrix}, \begin{bmatrix} \mathbf{y}_f \\ \mathbf{y}_a \end{bmatrix}, \begin{bmatrix} \mathbf{p}_f^\perp \\ \mathbf{p}_a^\perp \end{bmatrix} \right\},$$

where $\mathbf{x}, \mathbf{y}, \mathbf{p}^\perp$ denote the simultaneous translations along x, y directions and the rotation around the origin. We claim that $\left\{\mathbf{x}_a, \mathbf{y}_a, \mathbf{p}_a^\perp\right\}$ is a linearly independent set; otherwise, $\mathbf{p}_a^\perp = \alpha\mathbf{x}_a + \beta\mathbf{y}_a$ for some $\alpha, \beta \in \mathbb{R}$, which implies that all anchors share the same position — a contradiction to Definition 1. Therefore, $\mathbf{u}_f = \mathbf{0}$ is the only solution to the equation $S_{ff}\mathbf{u}_f = \mathbf{0}$, and S_{ff} is nonsingular. ∎

The following corollary is useful in determining fixability from infinitesimal rigidity.

Corollary 1: Given a formation graph $(\mathcal{V}, \mathbf{p}, K)$ and an anchor set $\mathcal{A} \subset \mathcal{V}$ with $|\mathcal{A}| \geq 2$, if $(\mathcal{V}, \mathbf{p}, \hat{K})$ is infinitesimally rigid, then the anchored formation graph $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ is fixable. Here $\hat{K}$ is the augmented matrix obtained from K by setting strictly positive connectivity coefficients to every pair of anchors.

Proof: Observe from (3) that S_{ff} is independent from the connectivity coefficients between anchors. Hence, we may arbitrarily increase k_{ij} for $i, j \in \mathcal{A}$, without affecting the fixability of the anchored formation graph. ∎

III. Angle-of-Arrival (AOA) Localization Problem

Suppose n free vertices and m anchors form an anchored formation graph $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$. Assume each free vertices can only measure the relative orientations of its neighbors with respect to itself using a built-in compass, while the anchors have precise knowledge of their own locations $\mathbf{p}_a$. The objective is to recover the free vertices' positions from these information.

This problem can be formulated as follows.

Definition 6 (AOA Localization Problem): The AOA Localization Problem is to find the solution $\mathbf{p}_f$ to the problem description $(\mathcal{V}, \mathcal{A}, \mathcal{E}, \Theta, \mathbf{p}_a)$, where

- $\mathcal{V}$ is the set of vertices;
- $\mathcal{A} \subset \mathcal{V}$ is the set of anchors (and $\mathcal{F} \triangleq \mathcal{V} \setminus \mathcal{A}$ is the set of free vertices);
- $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges and we assume that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$;
- $\Theta \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$ is the AOA information matrix satisfying $\theta_{ij} \in [0, 2\pi), |\theta_{ij} - \theta_{ji}| = \pi$ for every $(i, j) \in \mathcal{E}$;
- $\mathbf{p}_a$ is the precise locations of the anchors,

such that for every $(i, j) \in \mathcal{E} \cap (\mathcal{F} \times \mathcal{V})$,

$$\frac{\mathbf{p}_j - \mathbf{p}_i}{\|\mathbf{p}_j - \mathbf{p}_i\|} = \begin{bmatrix} \cos\theta_{ij} \\ \sin\theta_{ij} \end{bmatrix}.$$

It is possible that an AOA Localization Problem described by $(\mathcal{V}, \mathcal{A}, \mathcal{E}, \Theta, \mathbf{p}_a)$ has no feasible solution due to inconsistency in measured data. To avoid this, we formulate the AOA Localization Problem using data derived from a feasible anchored formation graph $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ by letting $\mathcal{E}$ be the set of all vertices pairs (i, j) with $k_{ij} > 0$, and $\Theta = [\theta_{ij}]$ where $\theta_{ij} = \angle(\mathbf{p}_j - \mathbf{p}_i)$. Evidently, solutions to this problem exist (e.g., the original $\mathbf{p}_f$). We call an anchored formation graph $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ *AOA localizable* if the corresponding AOA Localization Problem has a *unique* solution.

Analytic solutions to the AOA localization problem is possible and very simple, as was previously shown in [9], [10]. First observe that $\mathbf{p}^\perp$ points along the direction of an infinitesimal rotational motion of the formation about the origin O, which implies that $\mathbf{p}^\perp \in \operatorname{null}(S)$, or

$$\begin{bmatrix} S_{ff} & S_{fa} \\ S_{af} & S_{aa} \end{bmatrix} \begin{bmatrix} \mathbf{p}_f^\perp \\ \mathbf{p}_a^\perp \end{bmatrix} = \mathbf{0}.$$

Particularly, the following equation holds true,

$$S_{ff}\mathbf{p}_f^\perp + S_{fa}\mathbf{p}_a^\perp = \mathbf{0}. \quad (5)$$

If the anchored formation graph $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ is fixable, which implies that S_{ff} is nonsingular, we have

$$\mathbf{p}_f^\perp = -S_{ff}^{-1} S_{fa}\mathbf{p}_a^\perp. \quad (6)$$

It can been seen from the following theorem that fixability of the anchored formation graph is necessary and sufficient for its AOA localizability.

Theorem 2: An anchored formation graph $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ is AOA localizable if and only if it is fixable.

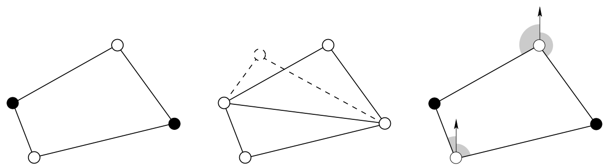
Proof: If $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ is fixable, then by definition S_{ff} is nonsingular, and (6) has a unique solution, which determines the locations of the free vertices.

To prove the converse, suppose $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ is not fixable. By definition, there exists a nonzero vector $\mathbf{u}_f$ such that $S_{ff}\mathbf{u}_f = \mathbf{0}$, equivalently, $R_f\mathbf{u}_f = \mathbf{0}$. Let $\mathbf{u} = \begin{bmatrix} \mathbf{u}_f^\top & \mathbf{u}_a^\top \end{bmatrix}^\top$ where $\mathbf{u}_a = \mathbf{0}$. The following equation then holds for all $i, j \in \mathcal{V}$ such that $k_{ij} > 0$,

$$\mathbf{e}_{ij}^\top (\mathbf{u}_i - \mathbf{u}_j) = 0 \Leftrightarrow \mathbf{u}_i^\perp - \mathbf{u}_j^\perp = \ell_{ij} \cdot \mathbf{e}_{ij} \text{ for some } \ell_{ij} \in \mathbb{R}.$$

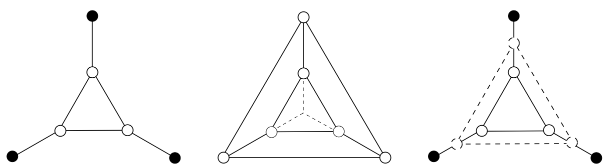
Suppose $\mathbf{p} = \begin{bmatrix} \mathbf{p}_f^\top & \mathbf{p}_a^\top \end{bmatrix}^\top$ are the true locations of the free vertices and the anchors. We show that $\tilde{\mathbf{p}} = \mathbf{p} + \delta \cdot \mathbf{u}^\perp$ induces an ineliminable ambiguity. In fact, note that

$$\begin{aligned} \tilde{\mathbf{p}}_j - \tilde{\mathbf{p}}_i &\triangleq (\mathbf{p}_j - \mathbf{p}_i) + \delta (\mathbf{u}_j - \mathbf{u}_i)^\perp \\ &= (\|\mathbf{p}_j - \mathbf{p}_i\| - \ell_{ij}\delta)\, \mathbf{e}_{ij}. \end{aligned}$$



(a) Fixable (b) Not globally rigid (c) AOA localizable

Fig. 1. AOA localizable formation that is not distance-based localizable



(a) Not fixable (b) Globally rigid (c) Not AOA localizable

Fig. 2. Distance-based localizable formation that is not AOA localizable

For small enough $\delta > 0$, the coefficients on the right hand side remain positive, hence $\tilde{\mathbf{e}}_{ij} = (\tilde{\mathbf{p}}_j - \tilde{\mathbf{p}}_i) / \|\tilde{\mathbf{p}}_j - \tilde{\mathbf{p}}_i\| = \mathbf{e}_{ij}$. This shows that the alternative configuration $\tilde{\mathbf{p}}$ will yield the same AOA information as that by the true configuration $\mathbf{p}$ under the same network topology. Therefore, such $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ is not AOA localizable. ■

Example 1: Two examples are given in Fig. 1 and 2. By Corollary 1, we can determine the fixability by connecting the anchors and checking the infinitesimal rigidity of the unanchored formation graphs. The example in Fig. 1 demonstrates the fact that although a formation with two anchors are in general not localizable using distance measurements due to reflection ambiguity, it can still be AOA localizable. Fig. 2 illustrates the situation where the formation can be uniquely localized base on distance measurements, but it cannot be AOA localized due to the scale ambiguity of the triangular component in the middle.

Remark 2: This result is significant in the sense that fixability, a property derived by the analysis of *distance-based* localization, is neither necessary nor sufficient condition for the *distance-based* localization, but is both for the *direction-based* localization.

IV. Distributed Continuous-Time Protocol for AOA Localization

A. Protocol Design

Suppose at each vertex $i \in \mathcal{V}$, a vector-valued state variable $\mathbf{x}_i \in \mathbb{R}^2$ is maintained. If vertex i is an anchor, we fix the value of $\mathbf{x}_i$ to be $\mathbf{p}_A^{\perp}$, which is available through a positioning device. If vertex i is free, then we update the state variable $\mathbf{x}_i$ according to the following protocol,

$$\dot{\mathbf{x}}_i = -\sum_{j \in \mathcal{V}} k_{ij} P_{ij} (\mathbf{x}_j(t) - \mathbf{x}_i(t)), \quad \forall\, i \in \mathcal{F}, \tag{7}$$

where P_{ij} is the projection matrix obtained from angle measurement. The protocol defined in (7) is in fact distributed, because for each free vertex i, the information of $\mathbf{x}_j$ is needed only when $k_{ij} > 0$, i.e., j is connected to i.

We can combine the equations (7) for all $i \in \mathcal{F}$ into the large matrix form shown below,

$$\dot{\mathbf{x}} = -S_{ff}\mathbf{x}(t) - S_{fa}\mathbf{p}_a^{\perp}, \tag{8}$$

where $\mathbf{x} = [\mathbf{x}_i]_{i \in \mathcal{F}}$. Recalling (5), the equilibrium point of the above dynamics is given by $\mathbf{p}_f^{\perp}$, which is the desired result of AOA localization. It can also been seen that the point $\mathbf{x} = \mathbf{p}_f^{\perp}$ is globally asymptotically stable if and only if the real symmetric matrix S_{ff} is positive definite. That is, if the underlying anchored formation is fixable, the continuous-time protocol (8) can always converge to the desired solution exponentially fast, starting from any initial guess $\mathbf{x}(0)$.

We may also notice that the format of (8) is very similar to the continuous-time average consensus dynamics, with the difference being that (8) has vector-valued state variables and matrix-valued weights instead of scalars.

B. Convergence and Delay Tolerance

Let $\boldsymbol{\varepsilon}(t) = \mathbf{x}(t) - \mathbf{p}_f^{\perp}$. We are concerned with the error dynamics

$$\dot{\boldsymbol{\varepsilon}}(t) = \dot{\mathbf{x}}(t).$$

Using (6) and (8), we have

$$\begin{aligned}\dot{\boldsymbol{\varepsilon}}(t) &= -S_{ff}\mathbf{x}(t) - S_{fa}\mathbf{p}_a^{\perp} = -S_{ff}\left(\mathbf{x}(t) + S_{ff}^{-1}S_{fa}\mathbf{p}_a^{\perp}\right) \\ &= -S_{ff}\left(\mathbf{x}(t) - \mathbf{p}_f^{\perp}\right) = -S_{ff}\boldsymbol{\varepsilon}(t).\end{aligned}$$

Therefore, the error vanishes exponentially fast with the convergence rate $\lambda_1(S_{ff})$, i.e., $\|\boldsymbol{\varepsilon}(t)\| \le \|\boldsymbol{\varepsilon}(0)\| \, e^{-\lambda_1(S_{ff})t}$, where $\lambda_1(S_{ff})$ is the smallest eigenvalue of the matrix S_{ff}.

Next, the scenario where there is a constant communication delay τ across all links is considered. The localization dynamics with delay can be expressed as

$$\dot{\mathbf{x}}_i = -\sum_{j \in \mathcal{V}} k_{ij} P_{ij} (\mathbf{x}_j(t-\tau) - \mathbf{x}_i(t-\tau)), \quad \forall\, i \in \mathcal{F}. \tag{9}$$

The stability criterion of such delayed system is given by a marginal value of τ as is described in the following proposition.

Proposition 1: Given an anchored formation $(\mathcal{V}, \mathbf{p}, K, \mathcal{A})$ that is fixable, the dynamics (9) is stable if the communication delay $\tau < \tau^*$, where $\tau^* = \frac{\pi}{2\lambda_{\max}(S_{ff})}$.

The proof of Proposition 1 follows a very similar line as that of the consensus algorithm in [12] (though with twice the dimension), hence is omitted.

The following result gives the estimation bounds for the delay tolerance, which can be obtained by the local computation at each agent/sensor, together with a consensus process over the network. Therefore, the delay tolerance for the network can be estimated in distributed fashion. This can be very useful in determining distributedly whether the AOA localization process will converge in the presence of communication delay.

Proposition 2: The maximum eigenvalue of S_{ff} is bounded by $\underline{\lambda} \leq \lambda_{\max}(S_{ff}) \leq \overline{\lambda}$, where

$$\underline{\lambda} = \frac{\sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{V}\setminus\{i\}} k_{ij}}{2|\mathcal{F}|},$$

$$\overline{\lambda} = \max_{i\in\mathcal{F}}\left\{\lambda_{\max}\left(\sum_{j\in\mathcal{V}\setminus\{i\}} k_{ij}P_{ij}\right) + \sum_{j\in\mathcal{F}\setminus\{i\}} k_{ij}\right\}.$$

As a result, the delay margin τ^* can be bounded by

$$\frac{\pi}{2\overline{\lambda}} \leq \tau^* \leq \frac{\pi}{2\underline{\lambda}}.$$

Proof: For the lower bound, recall from (3) that

$$\sum_{k=1}^{2|\mathcal{F}|}\lambda_k(S_{ff}) = \mathrm{tr}(S_{ff}) = \sum_{i\in\mathcal{F}}\mathrm{tr}\left(\sum_{j\in\mathcal{V}\setminus\{i\}} k_{ij}P_{ij}\right),$$

where each P_{ij} is a projection matrix whose trace is 1. Therefore, the maximum eigenvalue must be no less than the average.

For the upper bound, we note that each diagonal 2-by-2 block in S_{ff} is real symmetric and positive semidefinite, hence $\|S_{ii}\|_2 = \max\{\lambda_1(S_{ii}), \lambda_2(S_{ii})\}$. For each off-diagonal 2-by-2 block in S_{ff}, we have $\|S_{ij}\|_2 = k_{ij}\|P_{ij}\|_2 = k_{ij}$. We let $\alpha_i \triangleq \lambda_1(S_{ii}) \leq \lambda_2(S_{ii}) \triangleq \beta_i$. According to the generalized Gershgorin Circle Theorem [17], if λ is an eigenvalue of S_{ff}, then the following inequality must hold for at least one $i \in \mathcal{F}$,

$$\begin{aligned}\sum_{\substack{k=1\\k\neq j}}^{n}\|A_{ij}\|_2 &\geq \left(\left\|(A_{ii}-\lambda I_m)^{-1}\right\|_2\right)^{-1}\\ &= \min\{|\alpha_i - \lambda|, |\beta_i - \lambda|\},\end{aligned}$$

which implies that

$$\lambda \leq \beta_i + \sum_{j\in\mathcal{F}\setminus\{i\}} k_{ij}.$$

Taking the maximum value of the right hand side over all $i \in \mathcal{F}$ yields the desired result. ■

Remark 3: The upper bound $\overline{\lambda}$ for $\lambda_{\max}(S_{ff})$ can be further relaxed according to the relation below,

$$\overline{\lambda} \leq \max_{i\in\mathcal{F}}\left\{2\cdot\sum_{j\in\mathcal{F}\setminus\{i\}} k_{ij} + \sum_{j\in\mathcal{A}} k_{ij}\right\}.$$

The relaxed upper bound no longer depends on the positions of the vertices, which may be more desirable in some applications.

C. A Note on Performance Optimization

Assume that the underlying graph topology $\mathcal{E}$ is given, that is, $k_{ij} = 0$ if $(i,j) \notin \mathcal{E}$. We would like to compute the optimal values for parameters k_{ij} in terms of two objectives: 1) to maximize the convergence rate λ_1 while keeping the delay margin no smaller than a given constant $\tilde{\tau} > 0$, and 2) to maximize the delay margin while keeping the convergence rate no smaller than a given constant $r > 0$. In fact, these two cases can be consolidated into the following condition number minimization problem regardless of the values of $\tilde{\tau}$ or r,

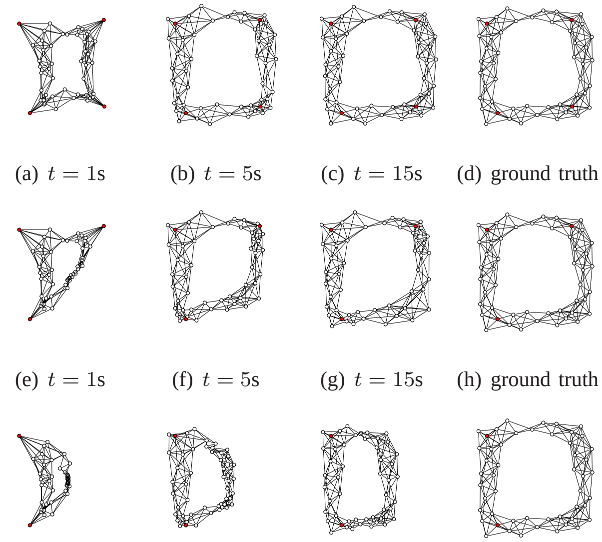
$$\underset{K\in\mathcal{K}(\mathcal{E})}{\text{minimize}} \quad \frac{\lambda_{\max}(S_{ff}(K))}{\lambda_1(S_{ff}(K))}, \tag{10}$$

which can be translated into the following semidefinite program using the relation (4),

$$\begin{aligned}&\underset{K\in\mathcal{K}(\mathcal{E})}{\text{minimize}} && \mu\\ &\text{subject to} && \begin{bmatrix} R_f^\top \Lambda_K R_f - I & O \\ O & \mu I - R_f^\top \Lambda_K R_f \end{bmatrix} \succeq 0\end{aligned} \tag{11}$$

This is because the optimal solution K^* to the problem (11) can be scaled proportionally according to the objective given by r or $\tilde{\tau}$.

Strictly speaking, there is a paradox in using the positional information for the above optimization, as such information is *a priori* unknown. Nevertheless, if the configuration of the formation can be roughly guessed, such optimization can often identify and strengthen the links that are most likely to cause performance bottlenecks in the formation, consequently improving the performance in the actual execution of the algorithm.



(a) $t = 1$s (b) $t = 5$s (c) $t = 15$s (d) ground truth

(e) $t = 1$s (f) $t = 5$s (g) $t = 15$s (h) ground truth

(i) $t = 1$s (j) $t = 5$s (k) $t = 15$s (l) ground truth

Fig. 3. Simulation results using different sets of anchors

V. Examples

We simulate the distributed continuous-time protocol (8) with a square-shaped formation consisting of 50 agents for three different set of anchors. The connectivity coefficient between each pair of neighboring agents is set to be 2 uniformly. The simulation results are shown in Fig. 3, where the filled dots on the corners denote the anchors and the white dots denote the agents to be localized. In the first case (Fig. 3(a)–(d)), the localization process converges to the true ground configuration fairly quickly. As we remove some of the anchors, as is shown in Fig. 3(e)–(h) and (i)–(l), the AOA

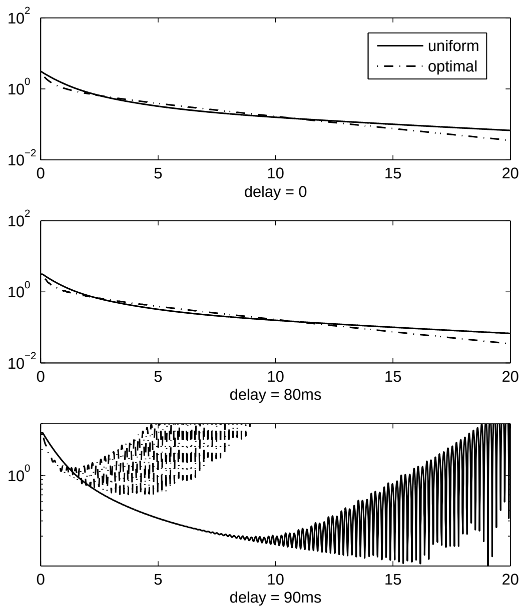


Fig. 4. Comparison on convergence speed and delay performance between uniform connectivity and optimal connectivity

based protocol may become less efficient in localizing the agents that are not geographically surrounded by anchors.

Next, we consider the continuous-time protocol with uniform communication delay with the formation illustrated in Fig. 3(d). Two cases are to be tested using different connectivity matrices. In the first case, all edges have the same connectivity coefficient 2; as a result the partial stiffness matrix S_{ff} has the smallest eigenvalue $\lambda_1 \approx 0.072$. Its largest eigenvalue $\lambda_{\max}$ can be roughly bounded by $14.39 \leq \lambda_{\max} \leq 31.12$ according to Proposition 2, and precise calculation yields $\lambda_{\max} \approx 18.71$. The corresponding delay margin τ^* is hence $\frac{\pi}{2\lambda_{\max}} \approx 84$ milliseconds. We adopt the optimization technique introduced in Section IV-C to obtain K^* which is then used for the second case. Some calculation shows that under the optimal K^* the smallest eigenvalue of S_{ff} increases to $\lambda_1 = 0.14$, which is almost doubled compared to the uniform case, while keeping the same largest eigenvalue as the previous case.

Simulation results of the two test cases are illustrated in Fig. 4. The X axis in the figure represents time (in seconds) and Y axis the total absolute error from the true configuration. From the figure we can infer that the delay margins of both cases lie between 80ms and 90ms. When both dynamics are stable, the one using the optimized connectivity matrix converges to the true configuration faster, as is expected by the objective of the optimization.

VI. Conclusion

In this paper we proposed and analyzed the angle-of-arrival (AOA) localization problem using concepts in formation rigidity theory. We gave the necessary and sufficient condition for AOA localizability as the fixability of an anchored formation. Whenever such condition holds, the network localization problem can be solved through a distributed continuous-time linear dynamics proposed in this paper, with its convergence speed and delay robustness studied. An optimization-based technique of designing the parameters of the dynamics is also introduced to maximize the convergence speed as well as the delay robustness. Finally, the theoretical framework proposed in this paper is tested through several numerical examples.

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