Variable Neural Adaptive Robust Output Feedback Control of Uncertain Systems

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Abstract—The design of an adaptive robust output feedback controller is presented for a class of multi-input multi-output uncertain systems. The proposed output feedback controller uses a variable-structure radial basis function (RBF) network to approximate unknown system dynamics. The output feedback implementation is realized employing a high-gain observer. The structure variation of the RBF network is taken into account in the stability analysis of the closed-loop system using the piecewise quadratic Lyapunov function. The performance of the proposed variable neural adaptive robust output feedback controller is illustrated with simulations.

I. INTRODUCTION

Adaptive control strategies such as adaptive feedback linearization [1], adaptive backstepping [2], nonlinear damping and swapping [3], and switching adaptive control [4] have been proposed to deal with systems with unknown dynamics and/or disturbances. In particular, adaptive controllers have been developed for single-input single-output (SISO) [5]–[7] and multi-input multi-output (MIMO) [8]–[11] feedback linearizable nonlinear systems. However, the proposed controllers often require the full access of the system states, which are usually not available in practice. A possible solution to the problem of inaccessible states is to use observers to estimate the system states and then use the estimated states in the controller implementation. For example, high-gain observers have been employed in [7], [12] to design output feedback based adaptive controllers. The advantage of using high-gain observers is that the control problem can be formulated in a standard singular perturbation form and then the singular perturbation theory can be applied to analyze the closed-loop system stability. Furthermore, when the speed of the high-gain observer is sufficiently high, the performance of the output feedback controller recovers the performance of the state feedback controller [12].

Many types of function approximators have been utilized in the construction of adaptive controllers to approximate unknown system dynamics. For example, fixed-structure radial basis function (RBF) networks have been employed in [5], [7], [11]. The disadvantage of the fixed-structure RBF network is that the set of basis functions needs to be selected off-line. To overcome this problem, multilayer neural network (MLNN) based adaptive robust control strategies were proposed in [6], [8]. Although it is not required for MLNNs to choose basis functions off-line, it is still necessary to pre-determine the number of hidden neurons. On the other hand, the RBF network has a simpler structure than that of the MLNN, faster computation time and superior adaptive performance. Recently, RBF networks with dynamically adjustable structures have been proposed in the design of adaptive controllers for SISO feedback linearizable uncertain systems in [13], [14]. Such networks preserve the advantages of RBF networks and, at the same time, overcome the limitations of fixed-structure RBF networks. However, the effect of the structure variation was not considered in the stability analysis in [13], [14], where the system stability is analyzed only for each fixed structure.

In this paper, we consider the problem of output tracking control for a class of MIMO feedback linearizable uncertain systems. The same problem was also considered in [8]–[11], [15]. In [15], an adaptive robust state feedback control strategy was proposed, where a variable-structure RBF network is used to approximate unknown system dynamics. The employed variable-structure RBF network avoids selecting basis functions off-line by allocating RBFs on-line dynamically. It adds RBFs to improve the approximation accuracy and removes RBFs to prevent the network redundancy. To avoid fast switching between different structures, the dwell time requirement was introduced. However, the implementation of the proposed controller in [15] requires a direct access of the system states, which may not be feasible in practice. To overcome this restriction, we propose a variable neural adaptive robust output feedback controller. We incorporate a high-gain observer into the state feedback control strategy proposed in [15]. The structure variation of the RBF network is also considered in the stability analysis employing a piecewise quadratic Lyapunov function approach that has been widely used in the stability analysis of switched and hybrid systems [16], [17].

II. PROBLEM STATEMENT

A. System Description

We consider a class of uncertain systems consisting of \( p \) coupled subsystems modeled by the following equations,

\[
g_{i}^{(m)}(x) = f_i(x) + \sum_{j=1}^{p} g_{ij}(x)u_j + d_i, \quad i = 1, \ldots, p, \tag{1}
\]

where \( g_{i}, u_{i} \) and \( d_{i} \in \mathbb{R} \) are the output, input and disturbance of the \( i \)-th subsystem, respectively, and \( f_i(x) \) and \( g_{ij}(x) \) are
unknown functions with
\[ x = \begin{bmatrix} y_1 \cdots y_1^{(n_1-1)} \cdots y_p \cdots y_p^{(n_p-1)} \end{bmatrix}^\top \]
being the state vector of the whole system. The disturbance \( d_i \) may depend on time \( t \) and the state vector \( x \). Let \((A_i, b_i)\) be the canonical controllable pair that represents a chain of \( n_i \) integrators, and let \( c_i = [1 \ 0 \cdots 0]_{1 \times n_i}, \ y = [y_1 \cdots y_p]^\top, \ u = [u_1 \cdots u_p]^\top \) and \( d = [d_1 \cdots d_p]^\top \). The system (1) can be represented in a compact form as
\[ \dot{x} = Ax + B(f(x) + G(x)u + d) \]
\[ y = Cx, \]
where \( A = \text{diag}[A_1 \cdots A_p], B = \text{diag}[b_1 \cdots b_p], \ C = \text{diag}[c_1 \cdots c_p], f(x) = [f_1(x) \cdots f_p(x)]^\top \) and \( G(x) = [g_{ij}(x)]_{p \times p} \). The above system is often referred to as a square system because the number of inputs is the same as the numbers of outputs. We assume that \( f(x) \) and \( G(x) \) are Lipschitz continuous vector- and matrix-valued functions of \( x \), respectively, and that \( d \) is Lipschitz continuous in \( x \) and piecewise continuous in \( t \) with \( |d| \leq d_o \). In addition, we consider the case where the input matrix \( G(x) \) is definite with eigenvalues bounded away from zero for all \( x \) of interest. Without loss of generality, we assume that \( G(x) \) is positive definite with \( g_{ij} \leq G(x) \leq \gamma I_p \), where \( g \) and \( \gamma \) are positive constants.

**B. Problem Formulation and Notation**

The control objective is to design a tracking, output feedback, controller so that the \( i \)-th closed-loop system output \( y_i \), \( i = 1, \ldots, p \), tracks a reference signal \( y_{d_i} \), when each \( y_{d_i} \) has bounded derivatives up to the \( n_i \)-th order. We propose a variable-structure RBF network based adaptive robust output feedback controller. To proceed, we define the desired system state vector as \( x^* = [y_1 \cdots y_1^{(n_1-1)} \cdots y_p \cdots y_p^{(n_p-1)}]^\top \) and let \( y^*_d = [y^*_d(1) \cdots y^*_d(n_p)]^\top \). Because the reference signal \( y_{d_i} \) has bounded derivatives up to the \( n_i \)-th order, we have \( x_{d} \in \Omega_{x_d} \) and \( y_{d}^{(n_p)} \in \Omega_{y_d} \), where \( \Omega_{x_d} \) and \( \Omega_{y_d} \) are compact subsets of \( \mathbb{R}^n \) and \( \mathbb{R}^p \), respectively, and \( n = \sum_{i=1}^n n_i \). Let \( e_{y_i} = y_{d_i} - y_i \) denote the \( i \)-th output tracking error. We define the output tracking error as \( e_i = [e_{y_1} \cdots e_{y_p}]^\top \) and the state tracking error as \( e = x - x^* \). It follows from (2) that the state tracking error dynamics are modeled by
\[ \dot{e} = Ae + B\left(f(x) + G(x)u - y^*_d + d\right). \]  
Let \( K = \text{diag}[k_1 \cdots k_p] \) be selected such that \( A_{m_1} = A_i - b_i k_i \) is Hurwitz. Thus, \( A_{m_1} = \text{diag}[A_{m_1} \cdots A_{m_p}] \) is Hurwitz. Let \( \Omega_{e_o} \) be a compact set that contains all initial state tracking errors and let \( e_{e_o} = \max_{e \in \Omega_{e_o}} \frac{e^\top P_m e}{\frac{1}{2} e^\top P_m e} \), where \( P_m = \text{diag}[P_{m_1} \cdots P_{m_p}] \) is the solution to the continuous Lyapunov matrix equation \( A_{m_1}^\top P_m + P_m A_{m_1} = -2Q_m \) for \( Q_m = \text{diag}[Q_{m_1} \cdots Q_{m_p}] \) with \( Q_{m_i} = Q_{m_i}^\top > 0 \). Choose \( e_c > e_{e_o} \) and let \( \Omega_{e} = \{ e : \frac{1}{2} e^\top P_m e \leq e_c \} \). Let \( \Omega_{x_d} = \{ x : x = e + x_d, e \in \Omega_e, x_d \in \Omega_{x_d} \} \), denote the compact set over which the unknown system dynamics are approximated by a variable-structure RBF network.

**III. VARIABLE-STRUCTURE RBF NETWORK**

The variable-structure RBF network that we use to approximate unknown functions \( f(x) \) over a compact set \( \Omega_x \) is an improved version of the one proposed by us in [15]. It has \( N \) different admissible structures, where \( N = \) a design parameter. For each admissible structure, the RBF network consists of \( n \) input neurons, \( M_v \) hidden neurons, where \( v \in \{1, \ldots, N\} \), and \( p \) output neurons. The \( k \)-th output of the RBF network with the \( v \)-th admissible structure can be represented as
\[ \hat{f}_{k,v}(x) = \sum_{j=1}^{M_v} \omega_{kj,v} \xi_{j,v}(x), \]
where \( \omega_{kj,v} \) is the weight from the \( j \)-th hidden neuron to the \( k \)-th output neuron and \( \xi_{j,v}(x) \) is the radial basis function for the \( j \)-th hidden neuron. Let \( W_v = [w_{1,v} \cdots w_{p,v}] \) with \( w_{i,v} = [w_{1,v} \cdots w_{iM_v,v}]^\top \) and \( \xi_{j,v}(x) = [\xi_{1,v}(x) \cdots \xi_{j,v}(x)]^\top \). We have \( \hat{f}_{j,v}(x) = W_v \xi_{j,v}(x) \), where \( \hat{f}_{j,v}(x) = [f_{j,v}(x) \cdots f_{p,v}(x)]^\top \). In this paper, we employ the same raised-cosine RBF as used in [15] instead of the commonly used Gaussian RBF.

In the following, we provide a description of the improved variable-structure RBF network. The major improvement over [15] lies in the RBF adding and removing operations.

**A. Center Grid**

Recall that the unknown function \( f(x) \) is approximated over a compact set \( \Omega_x \subset \mathbb{R}^n \). To locate the centers of the RBFs inside the approximation region \( \Omega_x \), we utilize an \( n \)-dimensional center grid with layer hierarchy, where each grid node corresponds to the center of one RBF. The grid nodes of the center grid are located at \( \{x_1, \ldots, x_N\} \), where \( S_i \) is the set of locations of the grid nodes in the \( i \)-th coordinate and \( \prod_{i=1}^n S_i \) denotes the Cartesian product. The center grid is initialized inside the approximation region \( \Omega_x \) with \( S_i = \{x_{ij} \mid x_{ij} \in \mathbb{R} \}, \ i = 1, \ldots, n, \) where \( x_{ij} \) and \( x_{im} \) denote the lower and upper bounds in the \( i \)-th coordinate. The \( 2^n \) grid nodes of the initial grid are referred to as the boundary grid nodes, and they are non-removable.

**B. Adding RBFs**

If the time elapsed since the last operation of adding or removing is greater than the dwell time \( T_d \), and \( \|e_{y_i}\| > \varepsilon_{\text{max}} \), where \( \varepsilon_{\text{max}} \) is a prespecified design parameter, for a period of time greater than \( T_d \), then the network attempts to add new RBFs represented by some new grid nodes. First, the nearest neighboring grid node in the center grid, denoted \( c_i(\text{nearest}) \), to the current input \( x \) is located among existing grid nodes. Then the “nearest” neighboring grid node in the center grid denoted \( c_i(\text{nearest}) \) is located, where \( c_i(\text{nearest}) \) is determined so that \( x_i \) is between \( c_i(\text{nearest}) \) and \( c_i(\text{nearest}) \). The adding operation is performed for each coordinate independently. In the \( i \)-th coordinate, if the following conditions are satisfied:

1. \[ |x_i - c_i(\text{nearest})| > \frac{1}{2} |c_i(\text{nearest}) - c_i(\text{nearest})|, \]
2. \[ |x_i - c_i(\text{nearest})| > d_i(\text{threshold}), \]
where \( d_i(\text{threshold}) \) is a design parameter that specifies the minimum grid distance in the \( i \)-th coordinate and thus determines the number of admissible structures denoted by \( N_i \), then a new location at exactly the middle of \( c_i(\text{nearest}) \) and \( c_i(\text{neareast}) \) is added into \( S_i \). Otherwise, no new location is added to \( S_i \). The layer of the newly added location is one level higher than the highest layer of the two adjacent existing locations in the same coordinate.

C. Removing RBFs

If the elapsed time since the last operation of adding or removing is greater than the dwell time \( T_d \), and \( \|e_g\| \leq \rho e_{\text{max}} \), where \( \rho \in (0, 1) \), for a period of time greater than \( T_d \), then the network attempts to remove some of the existing RBFs, that is, some of the existing grid nodes, to prevent network redundancy. The RBF removing operation is also implemented for each coordinate independently. In the \( i \)-th coordinate, if the following conditions are satisfied:

1. \( c_i(\text{nearest}) \notin \{x_k, x_{\text{ui}}\} \)
2. the location \( c_i(\text{nearest}) \) is in the higher than or in the same layer as the highest layer of the two neighboring locations in the same coordinate,
3. \( |x_i - c_i(\text{nearest})| < \theta |c_i(\text{nearest}) - c_i(\text{neareast})|, \theta \in (0, 0.5) \)

then the location \( c_i(\text{nearest}) \) is removed from \( S_i \). Otherwise, no location is removed from \( S_i \).

D. Uniform Grid Transformation

The determination of the radius of the RBF is achieved by uniform grid transformation, whose details can be found in [15]. When implementing the output feedback controller, the state vector estimate, \( \hat{x} \), is used rather than the actual state vector, \( x \). The case when \( \hat{x} \notin \Omega_x \) is analyzed in detail in [14].

IV. HIGH-GAIN TRACKING ERROR OBSERVER

We assume that the system states are not available for the feedback implementation. Thus, we apply the high-gain observer, as in [7], [12],

\[
\dot{\hat{e}}_i = A_i \hat{e}_i + l_i (c_{y,i} - c_i \hat{e}_i), \quad i = 1, \ldots, p
\]

(5)
to estimate the tracking error \( e_i \) of the \( i \)-th subsystem. The observer gain \( l_i \) is chosen as \( l_i = [\alpha_{i1}/\epsilon \cdots \alpha_{in}/\epsilon^{n_i}]^\top \), where \( \epsilon \in (0, 1) \) is a design parameter and \( \alpha_{ij} \), \( j = 1, \ldots, n_i \), are selected such that the roots of the characteristic polynomial equation, \( s^{n_i} + \alpha_{i1}s^{n_i-1} + \cdots + \alpha_{i(n_i-1)}s + \alpha_{in_i} = 0 \), have negative real parts. Let \( \hat{e} = [\hat{e}_1 \cdots \hat{e}_p]^\top \) and \( L = \text{diag}(l_1 \cdots l_p) \). Then the high-gain observer (5) can be represented as

\[
\dot{\hat{e}} = A \hat{e} + L (e_y - C \hat{e}).
\]

(6)

To facilitate stability analysis of the closed-loop system, we represent the estimation error dynamics in the singularly perturbed form. Let \( \zeta = [\zeta_1 \cdots \zeta_p]^\top \), \( \zeta_i = [\zeta_{i1} \cdots \zeta_{in_i}]^\top \) with

\[
\zeta_{ij} = \frac{e_{ij}^{(j-1)} - e_{ij}^{(j-1)}}{\epsilon^{n_i - j}}, \quad j = 1, \ldots, n_i.
\]

(7)

It follows from (7) that \( \dot{\hat{e}} = e - D \zeta \), where \( D = \text{diag}(D_1 \cdots D_p) \) and \( D_i = \text{diag}(e^{n_i-1} \cdots 1) \). Note that \( \|D\| = 1 \). Combining (3), (6) and (7) yields

\[
\dot{c} \zeta = A_c \zeta + cB (f(x) + G(x)u - y_d^{(n)} + d),
\]

(8)

where \( A_c = \text{diag}(A_{c1} \cdots A_{cp}) \) and \( A_{ci} = cD_i^{-1}(A_i - l_i c_j)D_i \) is a Hurwitz matrix. Applying the method of [18], we can prove the following proposition.

**Proposition 1:** Suppose that the control input \( u \) is globally bounded. There exists a constant \( c_i^* \in (0, 1) \) such that if \( c \in (0, c_i^* \epsilon) \), then \( |(\zeta(t))| \leq \beta \epsilon \), with some \( \beta > 0 \) for \( t \in [t_0 + T_1(\epsilon), t_0 + T_3) \), where \( T_3(\epsilon) \) is a finite time and \( t_0 + T_3 \) is the moment when the tracking error \( e(t) \) leaves the compact set \( \Omega_e \) for the first time. Moreover, we have \( \lim_{t \to 0^+} T_1(\epsilon) = 0 \) and \( c_{e*} = \frac{1}{2} e(t_0 + T_1(\epsilon)) \top P_m e(t_0 + T_1(\epsilon)) < c_e \).

We define the estimate of the system state vector as \( \hat{x} = x_d + \hat{e} \). In the following Section, we present the design of the proposed variable neural adaptive robust output feedback controller.

V. OUTPUT FEEDBACK CONTROLLER

A. Controller Structure

We consider adaptive robust output feedback controller,

\[
\dot{u} = \hat{u}_{a,v} + \hat{u}_{s,v},
\]

\[
= G_0^{-1} \left( -\hat{f}_v(\hat{x}) + y_d^{(n)} - K \hat{e} \right) + \hat{u}_{s,v},
\]

(9)

where \( G_0 \) is a chosen positive definite matrix, \( \hat{f}_v(\hat{x}) = W_v \hat{x}, \xi_v(\hat{x}) \), and \( \hat{u}_{s,v} \) is the robustifying component to be designed later. It is obvious that the controller architecture varies as the structure of the RBF network changes. Hence, the controller has \( N \) different architectures, which are referred to as modes.

To proceed, recall that \( W_v = [\omega_{1,v} \cdots \omega_{p,v}] \). We constrain the weight vectors \( \omega_{i,v} \) to reside in the compact sets \( \Omega_{i,v} = \{ \omega_{i,v} : \omega_{i,v} \leq \omega, 1 \leq j \leq M_v \} \), where \( \omega \) is the maximum weight, \( \omega \), and \( \omega \), \( i = 1, \ldots, p \), are design parameters. Let \( W_v^* = [\omega_{1,v}^* \cdots \omega_{p,v}^*] \) denote the “optimal” constant weight matrix corresponding to the \( v \)-th network structure, that is,

\[
W_v^* = \arg \max_{\omega_{i,v} \in \Omega_{i,v}} \max_{\omega \in \Omega_v} \| \hat{f}_v(\hat{x}) - W_v^\top \xi_v(\hat{x}) \|.
\]

Note that this \( W_v^* \) is used only for analytical purpose. Let \( \Phi_v = W_v - W_v^* = [\phi_{1,v} \cdots \phi_{p,v}] \), with \( \phi_{i,v} = \omega_{i,v} - \omega_{i,v}^* \), and let \( c = \sum_{i=1}^{p} c_i \), where

\[
c_i = \max_{v} \left( \max_{\omega_{i,v} \in \Omega_{i,v}} \frac{1}{2} \omega_{i,v} \phi_{i,v}^\top \phi_{i,v} \right),
\]

(10)

where \( \kappa > 0 \) is a design parameter, and \( \max_{c_i} \) denotes the maximization taken over all the admissible structures of the RBF networks. It is obvious that \( c_i \) decreases as \( \kappa \) increases. Let \( \sigma = B^\top P_m \hat{e} \). The following projection based weight matrix adaptation law is employed,

\[
\dot{W}_v = \text{Proj} \left( W_v, \kappa \xi_v(\hat{x}) \sigma^\top \right),
\]

(11)
where \( \text{Proj}(W_v, \Theta_v) \) denotes \( \text{Proj}(\omega_{ij,v}, \theta_{ij,v}) \) for \( i = 1, \ldots, p \) and \( j = 1, \ldots, M_v \) and \( \text{Proj}(\omega_{ij,v}, \theta_{ij,v}) \) is the discontinuous projection operator used in [14]. It is easy to verify that

\[
\frac{1}{\kappa} \text{trace} \left( \Phi_v^T \left( \dot{W}_v - \kappa \sigma_v (\hat{x}) \sigma_v^T \right) \right) \leq 0. \tag{12}
\]

Choose two positive constants \( d_f \) and \( d_g \) such that

\[
d_f \geq d_f' = \max \left\{ \max_{x \in \Omega_v} \| f(x) - W_v^T \xi_v(x) \| \right\}
\]
and \( d_g \geq d_g' = \max_{x \in \Omega_v} \| G(x) - G_0 \| \). Then we propose to use in (9) the robustifying component \( \hat{u}_{s,v} \) of the form,

\[
\hat{u}_{s,v} = \begin{cases} 
- \frac{k_{s,v}}{\bar{u}} \sigma & \text{if } \| \sigma \| \geq \nu \\
- \frac{k_{s,v}}{\bar{u}} \sigma & \text{if } \| \sigma \| < \nu,
\end{cases}
\]

where \( \hat{k}_{s,v} = d_f + d_g \| \hat{u}_{a,v} \| + d_o \) and \( \nu > 0 \) is a design parameter.

Recall that there exist peaking phenomena [19] associated with the high-gain observer described by (6). Thus, we cannot directly apply the output feedback controller given by (9). To eliminate the peaking phenomena, we introduce saturation to the control input (9). Let \( \Omega = \{ e : \frac{1}{2} e^T P_m e \leq c_v \} \), where \( c_v > c_r \). Let \( \hat{u} \) denote the state feedback controller obtained by replacing \( \hat{x} \) with \( x \) in (9), and let \( \hat{u}_i \geq \max \{ \max_{x \in \Omega_v} \| u_{ae} \| (\hat{x}, x_i, y_i, \omega_i) \} \), where \( x_i \) is the \( i \)-th element of \( \hat{u} \) and the inner maximization is taken over \( e \in \hat{\Omega}, \hat{\sigma} \in \hat{\Omega}_s, y_i \in \hat{\Omega}_{di} \) and \( \omega_i \in \hat{\Omega}_{wi} \). Then the proposed adaptive robust output feedback controller takes the form

\[
u^* = [U_1 \text{sat} \left( \frac{\hat{u}_{11}}{U_1} \right) \ldots U_p \text{sat} \left( \frac{\hat{u}_{p1}}{U_p} \right)]^T, \tag{13}
\]

where sat is the saturation operator.

Remark 1: The existence of a unique solution to (2) with the proposed variable neural adaptive robust output feedback controller can be established using the arguments of [15].

B. Stability Analysis

Note that the control input \( u^* \) is globally bounded by construction. It follows from Proposition 1 that if \( e \in (0, e_2^* \), then \( e(t) \in \Omega_e \) and \( \| \xi(t) \| \leq \beta_e \) for \( t \in [t_0 + T_1(e), t_0 + T_3) \). Thus, we have \( \| e(t) - \hat{e}(t) \| \leq \| D \| \| \xi(t) \| \leq \beta_e \) for \( t \in [t_0 + T_1(e), t_0 + T_3) \). There exists a constant \( c_2 \) such that \( \| e(t) - \hat{e}(t) \| \leq \beta c_2 \), which implies \( \hat{e}(t) \in \Omega_e \). Let \( e_2^* = \min \{ e_1, c_2^* \} \). If \( e \in (0, e_2^* \), then the adaptive robust output feedback controller (13) is in the saturated mode, that is, \( u^* = \hat{u} = \hat{u}_{a,v} + \hat{u}_{s,v} \) for \( t \in [t_0 + T_1(e), t_0 + T_3) \). Substituting \( u^* = \hat{u}_{a,v} + \hat{u}_{s,v} \) into the state tracking error dynamics (3) gives

\[
\dot{e} = A e + B \left( f(x) + G(x) \hat{u}_{a,v} + \hat{u}_{s,v} - y_i \right) + d
\]
\[
= A_m e + B K e - \hat{e} + B G(x) \hat{u}_{a,v} + Bd
\]
\[
+ B \left( f(x) - \hat{f}_v(\hat{x}) \right) + B \left( G(x) - G_0 \right) \hat{u}_{a,v}, \tag{14}
\]

Now we consider the piecewise quadratic Lyapunov function candidate

\[
V_v = \frac{1}{2} e^T P_m e + \frac{1}{2 \kappa} \text{trace} \left( \Phi_v^T \Phi_v \right) \tag{15}
\]
for \( t \in [t_0 + T_1(e), t_0 + T_3) \) and \( e \in (0, e_2^* \), whenever the proposed adaptive robust controller (9) is in the \( \nu \)-th mode. It follows from (10) that

\[
\frac{1}{2 \kappa} \text{trace} \left( \Phi_v^T \Phi_v \right) = \sum_{i=1}^{p} \frac{1}{2 \kappa} \phi_i^T \phi_i \leq \sum_{i=1}^{p} c_i = c. \tag{16}
\]

This Lyapunov function has jump discontinuities when the proposed adaptive robust output feedback controller switches between different modes. The time derivative of \( V_v \) evaluated along the solutions of (14) is

\[
\dot{V}_v = -\frac{1}{\kappa} e^T P_m e + \frac{1}{\kappa} \text{trace} \left( \Phi_v^T \Phi_v \right) + \frac{1}{2 \kappa} \text{trace} \left( \Phi_v^T \left( \dot{W}_v - \kappa \sigma_v (\hat{x}) \sigma_v^T \right) \right)
\]
\[
+ \frac{1}{\kappa} \text{trace} \left( \Phi_v^T \left( W_v - \kappa \sigma_v (\hat{x}) \sigma_v^T \right) \right) + \dot{\sigma} d
\]
\[
+ \sigma^T \left( f(x) - W_v^T \xi_v(\hat{x}) \right) + (\sigma - \hat{\sigma})^T d
\]
\[
+ \sigma^T (G(x) - G_0) \hat{u}_{a,v} + \sigma^T (G(x) - G_0) \hat{u}_{s,v}
\]
\[
+ (\sigma - \hat{\sigma})^T \left( f(x) - \hat{f}_v(\hat{x}) \right)
\]
\[
+ (\sigma - \hat{\sigma})^T (G(x) - G_0) \hat{u}_{a,v}. \tag{17}
\]

For \( t \in [t_0 + T_1(e), t_0 + T_3) \), if \( e \in (0, e_2^* \), then \( \| \xi(t) \| \leq \beta_e \), \( \| e(t) - \hat{e}(t) \| \leq \beta_e \), and \( e(t) \in \Omega_e \). \( \hat{e}(t) \in \Omega_e \). \( x_i(t) \in \Omega_{x_d} \) and \( y_i(t) \in \Omega_{y_d} \). Hence, \( \sigma(t), \sigma(t), x_i(t), \hat{x}(t), \hat{u}_{a,v}(t) \) \( \) and \( \hat{u}_{s,v}(t) \) are all bounded for \( t \in [t_0 + T_1(e), t_0 + T_3) \). Thus, we have

\[
\| \sigma K (e - \hat{e}) \| \leq r_1 \epsilon \tag{18}
\]
and

\[
\| \sigma - \hat{\sigma} \| \leq r_2 \epsilon \tag{19}
\]
for some \( r_1, r_2 > 0 \). On the other hand, it follows from the Lipschitz continuity of the raised-cosine RBF that

\[
\| W_v^T \xi_v(x) - W_v^T \xi_v(\hat{x}) \| \leq L_v \| x - \hat{x} \|
\]
for some \( L_v > 0 \). Let \( L = \max_v L_v \). Taking into account that \( \| x - \hat{x} \| = \| e - \hat{e} \| \) gives

\[
\| \sigma^T \left( f(x) - W_v^T \xi_v(\hat{x}) \right) \|
\leq \| f(x) - W_v^T \xi_v(\hat{x}) \|
\| \hat{\sigma} \|
\leq d_1 \| \hat{\sigma} \| + E \| e - \hat{e} \| \| \hat{\sigma} \|
\leq d_2 \| \hat{\sigma} \| + r_3 \epsilon \tag{20}
\]
for some \( r_3 > 0 \). It follows from (12) and (17)–(20) that

\[
\dot{V}_v \leq -\epsilon^T Q_m e + \kappa \sigma \| \hat{\sigma} \| + \sigma^T (G(x) \hat{u}_{s,v} + r_1 \epsilon, \tag{21}
\]
where $r = r_1 + r_2 + r_3$ and $\hat{k}_{s,v} = d_f^* + d_g^* \| \hat{u}_{a,v} \| + d_o \leq \hat{k}_{s,v}$.

If $\| \hat{\sigma} \| > \nu$, then
\[
\hat{k}_{s,v}^* \| \hat{\sigma} \| + \hat{\sigma}^T G(x) \hat{u}_{s,v} \leq \hat{k}_{s,v}^* \| \hat{\sigma} \| - \hat{k}_{s,v} \| \hat{\sigma} \| \leq 0. 
\] (22)

On the other hand, if $\| \hat{\sigma} \| \leq \nu$, then
\[
\hat{k}_{s,v}^* \| \hat{\sigma} \| + \hat{\sigma}^T G(x) \hat{u}_{s,v} \leq \hat{k}_{s,v}^* \| \hat{\sigma} \| - \frac{\hat{k}_{s,v}^* \| \hat{\sigma} \|^2}{\nu} \leq \frac{\hat{k}_{s,v}^*}{4} \nu. 
\] (23)

Combining (22) and (23), we obtain
\[
\hat{k}_{s,v}^* \| \hat{\sigma} \| + \hat{\sigma}^T G(x) \hat{u}_{s,v} \leq \hat{k}_{s,v}^* \| \hat{\sigma} \| + \frac{\hat{k}_{s,v}^* \| \hat{\sigma} \|^2}{4} \leq \frac{\hat{k}_{s,v}^*}{4} \nu. 
\] (24)

where $\hat{k}_{s,v}^* = d_f^* + d_g^* \max_v (\max \| \hat{u}_{a,v} \|) + d_o$ and the inner maximization is taken over $x_d \in \Omega_{x_d}$, $y_d^{(u)} \in \Omega_{y_d}$, and $\hat{e} \in \Omega_e$. It follows from (16), (21) and (24) that
\[
V_\nu \leq -\frac{1}{2} Q_m e + \frac{\hat{k}_{s,v}^*}{4} \nu + \nu \epsilon 
\leq -2\mu_m V_\nu + 2\mu_m c + \frac{\hat{k}_{s,v}^*}{4} \nu + \nu \epsilon 
= -\mu_m V_\nu - \mu_m (V_\nu - 2\epsilon c + \nu \epsilon),
\] (25)

where $\epsilon = c + \frac{\hat{k}_{s,v}^*}{8\mu_m}$. Let $\nu_{t_0}$ and $\nu_{t_f}$ denote the initial and the final time instant, respectively, of a continuous time period when the controller is in the $v$-th mode. It follows from (21) that if $V_\nu(t) \geq 2\epsilon c + \nu \epsilon$ for $t \in [\nu_{t_0}, \nu_{t_f}] \cap [t_0 + T_1(\epsilon), t_0 + T_3]$, then $V_\nu(t) \leq -\mu_m V_\nu(t)$, which implies that
\[
V_\nu(t) \leq \exp(-\mu_m(t-t_0))V_\nu(t_0, \nu_{t_0})
\] (26)

for $t \in [t_0, \nu_{t_0}] \cap [t_0 + T_1(\epsilon), t_0 + T_3]$.

**Theorem 1:** Let $t_1$, $t_2$ and $t_3$ be three consecutive switching time instants in the interval $[t_0 + T_1(\epsilon), t_0 + T_3]$ so that
\[
v = v_1 \text{ for } t \in [t_1, t_2] \text{ and } v = v_2 \text{ for } t \in [t_2, t_3].
\]
Suppose that $V_\nu(t)$ satisfies (26) and $V_\nu(t) \geq 2\epsilon c + \nu \epsilon$ for $t \in [t_1, t_3]$. If the dwell time $T_d$ of the variable-structure RBF network satisfies
\[
T_d \geq \frac{1}{2} \ln \left( \frac{3}{2} \right),
\] (27)

then $V_{\nu_2}(t_2) < V_{\nu_1}(t_1)$ and $V_{\nu_3}(t_3) < V_{\nu_1}(t_2)$.

**Proof:** This can be proved by replacing $2\epsilon$ with $2\epsilon + r \epsilon$ in the proof of Theorem 1 in [15].

**Theorem 2:** For the system (2) driven by the proposed adaptive robust output feedback controller (13) with the adaptation laws (11), suppose that $T_d$ satisfies the condition (27). If $c_\epsilon \geq c_{\epsilon_1} + c$ and $c_{\epsilon} \geq 2c + c$, there exists a constant $\epsilon^* \in (0, 1)$ so that if $\epsilon \in (0, \epsilon^*)$, then $e(t) \in \Omega_e$ and $x(t) \in \Omega_x$ for $t \geq t_0$. Moreover, there exists a finite time $T \geq t_0 + T_1(\epsilon)$ such that
\[
\frac{1}{2} e(t)^T P_m e(t) \leq 2\epsilon c + \nu \epsilon + c
\] (28)

with some $r > 0$ for $t \geq T$. In addition, suppose that there exists a finite time $T_s \geq t_0 + T_1(\epsilon)$ such that $v = v_s$ for $t \geq T_s$. Then a finite time $T \geq T_s$ can be found such that
\[
\frac{1}{2} e(t)^T P_m e(t) \leq 2\epsilon c + \nu \epsilon
\] (29)

for $t \geq T$.

**Proof:** Recall that if $\epsilon \in (0, \epsilon^*)$, then $V_\nu(t)$ satisfies (25) for $t \in [t_0 + T_1(\epsilon), t_0 + T_3]$, where we have $V(t_0 + T_1(\epsilon)) \leq c_{\epsilon_1} + c$. If, at the same time, $T_d$ satisfies the condition (27), it follows from Theorem 1 that
\[
V_\nu(t) \leq \max \{ V(t_0 + T_1(\epsilon)), 2\epsilon c + \nu \epsilon \} 
\leq \max \{ c_{\epsilon_1} + c, 2c + c + \nu \epsilon \},
\] (30)

for $t \in [t_0 + T_1(\epsilon), t_0 + T_3]$.

It follows from Theorem 1 and the above analysis that $V_\nu(t)$ will visit the interval $[0, 2\epsilon c + \nu \epsilon]$ infinitely often for $t \geq t_0 + T_1(\epsilon)$. Let $T \geq t_0 + T_1(\epsilon)$ be the first time such that $V_\nu(T) \leq 2\epsilon c + \nu \epsilon$. The trajectory of $V_\nu(t)$ starting at $t = T$ will stay inside the interval $[0, 2\epsilon c + \nu \epsilon]$ until it jumps outside when the controller switches the mode. However, the jump of $V_\nu(t)$ between different modes satisfies the condition, $|\Delta| \leq c$. Hence, we have $V_\nu(t) \leq 2\epsilon c + \nu \epsilon + c$, which implies that
\[
\frac{1}{2} e(t)^T P_m e(t) \leq 2\epsilon c + \nu \epsilon + c
\] for $t \geq T$. If, in addition, there exists a finite time $T_s \geq t_0 + T_1(\epsilon)$ such that $v = v_s$ for $t \geq T_s$, then it follows from (26) that $V_{\nu_3}^2 \leq -\mu_m V_{\nu_3} - \mu_m (V_{\nu_3} - 2\epsilon c - \nu \epsilon)$ for $t \geq T_s$. Therefore, there exists a finite time $T_s \geq T_s$ such that
\[
\frac{1}{2} e(t)^T P_m e(t) \leq 2\epsilon c + \nu \epsilon
\] (31)

for $t \geq T$. The proof of the theorem is complete. □

**Remark 2:** It can be seen from (28) and (29) that the tracking performance is directly proportional to the magnitude of $c$, $\nu$ and $c$. Recall that the magnitude of $c$ is inversely proportional to $\kappa$. Therefore, large $\kappa$ and small $\nu$ and $c$ yield better tracking performance.

**VI. SIMULATION EXAMPLE**

We illustrate the effectiveness of the proposed variable neural adaptive robust output feedback controller on the model of the planar articulated two-link manipulator used in [15]. Detailed model derivation of this two-link manipulator can be found in [29, p. 394]. Let $q_1$ and $q_2$ denote the angular positions of joint 1 and 2, respectively, and $\tau_1$ and $\tau_2$ denote the applied torques. We assume that there exist input disturbances $\eta_1$ and $\eta_2$ associated with the applied torques $\tau_1$ and $\tau_2$, respectively. Thus, we can represent the dynamics of this two-link manipulator as
\[
\begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
-h\dot{q}_1 -h (\dot{q}_1 + \dot{q}_2) \\
h\dot{q}_1
\end{bmatrix}
= \begin{bmatrix}
\tau_1 + \eta_1 \\
\tau_2 + \eta_2
\end{bmatrix},
\]

where $H_{11} = a_1 + 2a_3 \cos(q_2) + 2a_4 \sin(q_2)$, $H_{12} = H_{21} = a_2 + a_3 \cos(q_2) + a_4 \sin(q_2)$, $H_{22} = a_2$ and $h = a_3 \sin(q_2) -
A variable neural adaptive robust output feedback controller has been proposed for the output tracking control of a class of MIMO uncertain systems. The variable-structure RBF network, which is used to approximate unknown system dynamics, can grow or shrink on-line dynamically according to the tracking performance. The piecewise quadratic Lyapunov function is utilized to analyze the effects of the structure variation of the RBF network. Simulation results confirm the effectiveness of the proposed variable neural adaptive robust output feedback controller.

REFERENCES