Stability of Discrete-Time Conewise Linear Inclusions and Switched Linear Systems

Jinglai Shen and Jianghai Hu

***Abstract*— This paper addresses the stability of discrete-time conewise linear inclusions (CLIs) and its connection with that of switched linear systems (SLSs). The CLIs form a class of switched linear systems subject to state dependent switchings. Strong and weak stability concepts of the CLIs are considered and the equivalence of asymptotic and exponential stability is established. To characterize stability of the CLIs, a Lyapunov framework is developed and a converse Lyapunov theorem is obtained. Furthermore, stability of general SLSs is studied and is shown to be closely related to that of the CLIs through a family of properly defined generating functions.**

I. Introduction

Stability analysis of hybrid and switched dynamical systems has received tremendous interest in the systems and control community, driven by important applications in large scale and complex systems with hierarchical and multi-modal structure [8]. Switched dynamical systems can be roughly divided into two groups: those subject to arbitrary, state independent switchings, and those subject to state dependent switchings. There is a large body of the literature on switched systems of the first kind; and various stability criteria have been proposed, e.g., the Lie-algebraic approach [7] and the Lyapunov framework [11]. In the latter case, different forms of Lyapunov functions have been considered, such as common Lyapunov functions [9] and composite quadratic Lyapunov functions [5]. The converse Lyapunov theorem and stabilization issue have also been addressed for this class of switched systems [2], [12]. In comparison, stability of switched systems subject to state dependent switchings receives relatively less attention, despite its importance in robotics, dynamic optimization and other fields. This is largely due to the fact that state-dependent switching usually complicates fundamental dynamic and control analysis. Particularly it poses great difficulty in obtaining less conservative and easily verified stability conditions.

The present paper addresses the stability of a class of discrete-time, switched linear systems subject to state-dependent switchings, i.e., conewise linear inclusions (CLIs). Such a system partitions the state space into finitely many cones; and the system dynamics is linear on each cone. Switching occurs as a state trajectory exits from one cone and enters another. The state dynamics may have multiple values on the boundary of two cones, thus making the system a class of linear inclusions. In spite of its simple structure, the stability knowledge of the CLIs is far from complete, except a few results for some special conewise linear systems, e.g., [1] studies the planar case only and [10] focuses on a CLI with a Lipschitz continuous right-hand side. In this paper, we consider strong and weak stability concepts of the general CLIs and establish the equivalence of (strong) asymptotic and exponential stability. We then characterize the stability of the CLIs from the Lyapunov perspective and develop a converse Lyapunov result. Furthermore we establish a connection between the stability of CLIs and that of SLSs. Specifically, a new stability criterion for the stability of SLSs is proposed based on generating functions. Stability of the SLSs is shown to be equivalent to that of two families of CLIs obtained from the SLSs using the generating functions.

The rest of the paper is organized as follows. In Section II, we introduce the CLI and its strong and weak stability concepts. It is shown that (strong) asymptotic stability is equivalent to exponential stability. Subtle technical conditions that yield such equivalence are discussed. Section III focuses on Lyapunov characterization of strong and weak exponential stability; both Lyapunov and converse Lyapunov theorems are established. The latter theorem ensures the existence of a piecewise quadratic Lyapunov function for strong exponential stability. Section IV discusses stability of SLSs. In contrast to Lyapunov approach, a novel generating function based characterization is proposed for the stability analysis of SLSs. It is further shown in Section V that the exponential stability of such SLSs can be characterized by the weak stability of certain CLIs defined by suitable energy functions. This result provides a new perspective to characterizing the stability of SLSs.

II. Conewise Linear Inclusion and Stability

Let $\Xi \equiv \{\mathcal{X}_i\}_{i=1}^{\ell}$ be a finite family of nonempty closed cones whose union is $\mathbb{R}^n$, namely, $\cup_{i=1}^{\ell}\mathcal{X}_i = \mathbb{R}^n$. Each $\mathcal{X}_i$ is not necessarily polyhedral or even convex, and two cones in Ξ may overlap. For a given $x \in \mathbb{R}^n$, let the index set $\mathcal{I}(x) \equiv \{ i \in \{1, \cdots, \ell\} \,|\, x \in \mathcal{X}_i \}$. We assign to each cone $\mathcal{X}_i$ an $n \times n$ matrix A_i that defines the linear mapping $x \mapsto A_i x$ if $x \in \mathcal{X}_i$. This gives rise to a linear set-valued mapping $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by $f(x) = \{ A_i x \,|\, i \in \mathcal{I}(x) \}$. For the given f, define the following discrete-time conewise linear inclusion (CLI):

$$x(t+1) \,\in\, f(x(t)), \quad t \in \mathbb{Z}_+. \tag{1}$$

For an initial state $x^0 \in \mathbb{R}^n$, let $x(t, x^0)$ denote its trajectory of (1). It is noted that for a given x^0, there are possibly

Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21250. Email:shenj@umbc.edu. This research was partially supported by the NSF grant ECCS-0900960.

School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907. Email:jianghai@ecn.purdue.edu. This research was partially supported by the NSF grant CNS-0643805.

multiple trajectories corresponding to x^0 in general.

Definition 1 (Strong Stability of CLI) At $x_e = 0$, the CLI (1) is called

- *(strongly) stable* in the sense of Lyapunov if, for each $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ such that $\|x^0\| \le \delta_\varepsilon \Rightarrow \|x(t, x^0)\| < \varepsilon, \forall\ t \in \mathbb{Z}_+$ for any trajectory $x(t, x^0)$ starting from x^0;
- *(strongly) asymptotically stable* if it is (strongly) stable and $\delta > 0$ exists such that $\|x^0\| < \delta \Rightarrow \lim_{t\to\infty} x(t, x^0) = 0$ for any trajectory $x(t, x^0)$ starting from x^0;
- *(strongly) exponentially stable* if there exist $\delta > 0$, $\kappa \ge 1$, and $\rho > 0$ such that $\|x^0\| < \delta \Rightarrow \|x(t, x^0)\| \le \kappa\|x^0\|e^{-\rho t}, \forall\ t \in \mathbb{Z}_+$ for any $x(t, x^0)$ starting from x^0.

Definition 2 (Weak Stability of CLI) The CLI (1) is called *weakly stable* (respectively, *weakly asymptotically stable* and *weakly exponentially stable*) at $x_e = 0$ if the corresponding condition in Definition 1 for its strong counterpart is satisfied for at least one (instead of any) trajectory $x(t, x^0)$ starting from x^0. In particular, the CLI (1) is weakly asymptotically stable if it is weakly stable and $\delta > 0$ exists such that $\|x^0\| < \delta \Rightarrow \lim_{t\to\infty} x(t, x^0) = 0$ for some trajectory $x(t, x^0)$ from x^0.

Since the trajectories of the CLI are homogeneous in initial states, the local and global stability notions are equivalent. Moreover, it is shown below that (strong) asymptotic stability is equivalent to its exponential counterpart.

Theorem 3 The CLI (1) is asymptotically stable at $x_e = 0$ if and only if it is exponentially stable at $x_e = 0$.

Proof. It suffices to show the "only if" part. To reach this end, we prove the following claim pertaining to uniform asymptotic stability first:

Claim: if the CLI (1) is asymptotically stable at $x_e = 0$, then for any small $\delta > 0$ and a given scalar $0 < c < 1$, there is a scalar $T_{\delta,\, c} \in \mathbb{Z}_+$ (depending on δ and c only) such that $\|x^0\| \le \delta \ \Rightarrow\ \|x(t, x^0)\| \le c\,\delta,\ \forall\ t \ge T_{\delta,\, c}$ for any $x(t, x^0)$ starting from x^0.

For given $\delta > 0$ and $0 < c < 1$, suppose the claim fails. Hence, there exist an initial state sequence $\{x^0_k\} \subseteq \mathcal{B}_\delta$, the corresponding trajectories $\{x(t, x^0_k)\}$, and an increasing time sequence $\{t_k\} \subseteq \mathbb{Z}_+$ with $\lim\limits_{k\to\infty} t_k = \infty$ such that $\|x(t_k, x^0_k)\| > c\,\delta$. Furthermore, it follows from the stability of $x_e = 0$ that two positive scalars r (with $r > \delta$) and μ (with $\mu < \delta$) exist such that (i) $\|x(t, x^0_k)\| \le r, \forall\ t \in \mathbb{Z}_+$ for all k; and (ii) $x^0 \in \mathcal{B}_\mu \Rightarrow \|x(t, x^0)\| \le c\,\delta,\ \forall\ t \in \mathbb{Z}_+$. By (ii) and the semi-group property, we have $\|x(t, x^0_k)\| \ge \mu$ for all $t \in \{0, 1, \cdots, t_k\}$. Since $\mu \le \|x^0_k\| \le \delta$ for all k, there exists a subsequence of $\{x^0_k\}$ convergent to x^0_* with $\mu \le \|x^0_*\| \le \delta$. Without loss of generality, let $\{x^0_k\}$ be that subsequence convergent to x^0_*. In view of (i)-(ii) and the construction of $\{t_k\}$, we see that the sequence $\{x(1, x^0_k)\}_{k\ge t_1}$ satisfies $\mu \le \|x(1, x^0_k)\| \le r$ for all $k \ge t_1$. Thus it has a subsequence converging to x^1_* with $\mu \le \|x^1_*\| \le r$. Due to the closedness of $\mathcal{X}_i$'s, we obtain a neighborhood $\mathcal{U}$ of x^0_* such that $\mathcal{U} \subseteq \cup_{i\in\mathcal{I}(x^0_*)}\mathcal{X}_i$. Note that $x(1, x^0_k) = A_j\, x^0_k$ for some j and $x^0_k \in \mathcal{U}$ for all large k. Furthermore, since the index set $\mathcal{I}(x^0_*)$ is finite, we deduce that there exist a subsequence $\{x(1, x^0_{k'})\}$ of $\{x(1, x^0_k)\}_{k\ge t_1}$ and an index $j_1 \in \mathcal{I}(x^0_*)$ such that $x(1, x^0_{k'}) = A_{j_1} x^0_{k'}$ for all k' with $x(1, x^0_{k'}) \longrightarrow x^1_*$ and $x^0_{k'} \longrightarrow x^0_*$. This shows that $x^1_* = A_{j_1} x^0_*$. Recalling $j_1 \in \mathcal{I}(x^0_*)$, we have $x^1_* \in f(x^0_*)$. Repeating this argument and using induction, we obtain $\{x^t_*\}_{t\in\mathbb{Z}_+}$ such that (i) $\mu \le \|x^t_*\| \le r$ for all $t \in \mathbb{Z}_+$, and that (ii) for each $t \in \mathbb{Z}_+$, $x^{t+1}_* \in f(x^t_*)$. This shows that the trajectory $x(t, x^0_*) = \{x^t_*\}_{t\in\mathbb{Z}_+}$ is such that $\|x(t, x^0_*)\| \ge \mu, \forall\ t \in \mathbb{Z}_+$. This contradicts the asymptotic stability of the CLI. Hence the claim holds true.

Finally, using the above claim, the homogeneity of the CLI and a similar argument as in [6, Theorem 3.9] (for linear time-varying systems), one can show that the CLI is exponentially stable at $x_e = 0$. $\square$

It is worth mentioning that the closedness of $\mathcal{X}_i$'s plays a key role in establishing the above equivalence. The following example shows that if the closedness is dropped, then the two stability notions may not be equivalent in general.

Example 4 Consider the CLI on $\mathbb{R}^2$ with $\Xi = \{\mathcal{X}_i\}_{i=1}^4$, where $\mathcal{X}_1 = \mathbb{R}^2_{++} \equiv \{x \in \mathbb{R}^2 \,|\, x > 0\}$ (namely $\mathcal{X}_1$ is the interior of the nonnegative orthant of $\mathbb{R}^2$ and thus is open), $\mathcal{X}_2 = \{(0, x_2)^T \in \mathbb{R}^2 \,|\, x_2 > 0\}$, $\mathcal{X}_3 = \{(x_1, x_2)^T \in \mathbb{R}^2 \,|\, x_1 < 0, x_2 > 0\}$, $\mathcal{X}_4 = \{(x_1, x_2)^T \in \mathbb{R}^2 \,|\, x_2 \le 0\}$. Let the transition matrices for the linear dynamics be

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_2 = A_4 = 0.$$

Since any two cones are disjoint, f is singleton on $\mathbb{R}^2$ and thus becomes a function, albeit discontinuous. Hence, the CLI has a unique trajectory for each initial state. It is easy to verify that for $x^0 = (x^0_1, x^0_2)^T \in \mathcal{X}_1$, the trajectory sequence is $(x^0_1, x^0_2)^T \to (-x^0_1, x^0_2)^T \to (x^0_1, x^0_2 - x^0_1)^T \to \cdots \to (x^0_1, x^0_2 - 2x^0_1)^T \to \cdots$ until the second entry becomes negative so that the sequence ends at the origin. Furthermore, $\{\|x(t, x^0)\|_2\}$ is non-increasing with respect to t. This also holds for any trajectory starting from $\mathcal{X}_3$ (as well as that from $\mathcal{X}_2 \cup \mathcal{X}_4$). As a result, the CLI is asymptotically stable. On the other hand, let $x^0 = (\varepsilon, 1)^T \in \mathcal{X}_1$, where $\varepsilon > 0$ is sufficiently small. Thus $\|x^0\|_2 \le 2$ for all small $\varepsilon > 0$. Following the above argument, we see that $x(t, x^0)$ reaches the origin in about $1/(2\varepsilon)$ steps, namely, the number of convergence steps tends to infinity as $\varepsilon \downarrow 0$. Consequently, the CLI is not exponentially stable.

To further illustrate Theorem 3, consider the CLI defined on the closure of each of the above cones with the same linear dynamics, i.e., $\Xi = \{\text{cls}\mathcal{X}_i\}_{\ell=1}^4$, where cls denotes the closure of a set. Similarly, we conclude that the CLI is not exponentially stable. Note that the initial state $x^0 = (0, 1)^T \in \text{cls}\mathcal{X}_1 \cap \text{cls}\mathcal{X}_2 \cap \text{cls}\mathcal{X}_3$. Therefore x^0 has a trajectory $x(t, x^0)$ with $x(t, x^0) = x^0, \forall\ t \in \mathbb{Z}_+$ (although another trajectory from x^0 is such that $x(t,x^0) = 0, \forall\ t \geq 1$). Therefore the CLI is not asymptotically stable. This example also demonstrates a necessary condition for asymptotic stability, i.e., A_i has no eigenvector in $\mathcal{X}_i$ associated with a real eigenvalue $\lambda \geq 1$. $\square$

The following example shows that the conclusion of Theorem 3 is no longer true if the strong notions of stability are replaced with their weak counterparts.

Example 5 Consider the CLI on $\mathbb{R}^2$ with $\Xi = \{\mathcal{X}_i\}_{i=1}^3$, where $\mathcal{X}_1 = \{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq 0\}$, $\mathcal{X}_2 = \{(x_1,x_2)^T \in \mathbb{R}^2 \,|\, x_1 \leq 0, x_2 \geq 0\}$, $\mathcal{X}_3 = \{(x_1,x_2)^T \in \mathbb{R}^2 \,|\, x_2 \leq 0\}$. Let the corresponding matrices be

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_3 = 0. \qquad (2)$$

This CLI is not strongly asymptotically or exponentially stable as it has a trajectory of periodicity two, $(0,1)^T \to (-1,0)^T \to (0,1)^T \to \cdots$, that fails to converge to the origin. On the other hand, it is weakly asymptotically stable. For example, starting from $x^0 = (x_1^0, x_2^0)^T$ in the interior of $\mathcal{X}_1$, the trajectory sequence is $(x_1^0,x_2^0)^T \to (-x_2^0,x_1^0)^T \to (x_1^0, x_2^0 - x_1^0)^T \to \cdots \to (x_1^0, x_2^0 - 2x_1^0)^T \to \cdots$ which eventually reaches $\mathcal{X}_3$; then the sequence can arrive at 0 at the next time step. Similarly we can verify the existence of at least one convergent trajectory starting from any other initial state. Now consider the trajectories starting from $x^0 = (\varepsilon, 1)^T \in \mathcal{X}_1$ for a small $\varepsilon > 0$. By the above argument, for any trajectory $x(t,x^0)$ starting from x^0, the time steps it takes for $\|x(t,x^0)\|$ to decreases to half of its initial value are at least about $1/\varepsilon$ steps, which grow unboundedly as $\varepsilon \to 0$. Thus, the CLI is not weakly exponentially stable. $\square$

III. Lyapunov and Converse Lyapunov Theorems for CLI

A function $V : \mathbb{R}^n \to \mathbb{R}$ is called *(infinitely) piecewise quadratic* if it is positively homogeneous of degree two along each ray: $V(\lambda x) = \lambda^2 V(x)$, for all $\lambda \geq 0$ and $x \in \mathbb{R}^n$. In particular, V is called *finitely piecewise quadratic* if it is piecewise quadratic and for each $x \in \mathbb{R}^n$, $V(x) = x^T P x$ for at most a finite number of positive semidefinite matrices $P \in \mathbb{R}^{n\times n}$. The following result asserts the equivalence of exponential/asymptotic stability of CLI and the existence of a finitely piecewise quadratic Lyapunov function.

Proposition 6 The CLI (1) is exponentially stable at $x_e = 0$ if and only if there exists a finitely piecewise quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

(a) there exist $c_1 > 0$ and $c_2 > 0$ such that $c_1\|z\|^2 \leq V(z) \leq c_2\|z\|^2$ for all $z \in \mathbb{R}^n$;
(b) there exists $c_3 > 0$ such that $V(z') - V(z) \leq -c_3\|z\|^2,\ \forall\, z' \in f(z)$ for all $z \in \mathbb{R}^n$.

Proof. The sufficiency follows from the standard argument (even without the finitely piecewise quadratic property for V) and thus is omitted. In the sequel, we consider its converse and let $\|\cdot\|$ be the 2-norm. Since the CLI is (globally) exponentially stable at $x_e = 0$, there exist $\kappa \geq 1$ and $\rho > 0$ such that $\|x(t,x^0)\| \leq \kappa\, e^{-\rho t}\|x^0\|, \forall\ t \in \mathbb{Z}_+$ for all $x^0 \in \mathbb{R}^n$. Hence, there exists $T_* \in \mathbb{Z}_+$ such that $\kappa^2\, e^{-2\rho(T_*+1)} \leq \frac{1}{2}$. In the following, let $\mathcal{W}(z,T)$ denote the family of all the trajectories starting from $z \in \mathbb{R}^n$ on the interval $[0,T]$ with $T \in \mathbb{Z}_+$. Thus for each $T < \infty$, $\mathcal{W}(z,T)$ contains finitely many trajectories. We shall show that

$$V(z) \equiv \max_{x(t,z)\in\mathcal{W}(z,T_*)} \sum_{t=0}^{T_*} \|x(t,z)\|^2 \qquad (3)$$

is a desired finitely piecewise quadratic Lyapunov function as follows. It is clear that for each $z \in \mathbb{R}^n$, $V(z) \geq \|z\|^2$. Hence $c_1 = 1$. Moreover, $V(z) \leq \sum_{t=0}^{T_*} \kappa^2\, e^{-2\rho t}\, \|z\|^2 \leq c_2\|z\|^2$ where $c_2 \equiv \kappa^2/(1-e^{-2\rho})$. Therefore (a) holds true. To prove (b), we note that for any $z' \in f(z)$ and any trajectory $\widehat{x}(t,z')$ starting from z', the concatenation of z followed by $\widehat{x}(t,z')|_{t=0,\cdots,T_*}$ is a trajectory in $\mathcal{W}(z,T_*+1)$. Therefore

$$\begin{aligned} & \|z\|^2 + \sum_{t=0}^{T_*} \|\widehat{x}(t,z')\|^2 \\ \leq\ & \max_{x(t,z)\in\mathcal{W}(z,T_*+1)} \sum_{t=0}^{T_*+1} \|x(t,z)\|^2 = \sum_{t=0}^{T_*+1} \|\widetilde{x}(t,z)\|^2, \end{aligned}$$

where $\widetilde{x}(t,z)$ is a trajectory in $\mathcal{W}(z,T_*+1)$ that achieves the above maximum. Since $\widetilde{x}(t,z)|_{t=0,\cdots,T_*}$ is a trajectory in $\mathcal{W}(z,T_*)$ and $\|\widetilde{x}(T_*+1,z)\|^2 \leq \frac{1}{2}\|z\|^2$ by the choice of T_*, we have

$$\begin{aligned} \sum_{t=0}^{T_*+1} \|\widetilde{x}(t,z)\|^2 &= \sum_{t=0}^{T_*} \|\widetilde{x}(t,z)\|^2 + \|\widetilde{x}(T_*+1,z)\|^2 \\ &\leq V(z) + \frac{1}{2}\|z\|^2. \end{aligned}$$

Combining the above two inequalities, we obtain

$$\sum_{t=0}^{T_*} \|\widehat{x}(t,z')\|^2 \leq V(z) - \frac{1}{2}\|z\|^2,$$

for any $\widehat{x}(t,z') \in \mathcal{W}(z',T_*)$. Therefore, $V(z') \leq V(z) - \frac{1}{2}\|z\|^2$, and $V(z)$ indeed satisfies (b). Finally, we show that V is finitely piecewise quadratic. Indeed, define the positive definite $P_j \equiv I + A_{t_1}^T A_{t_1} + \left(A_{t_2}A_{t_1}\right)^T\left(A_{t_2}A_{t_1}\right) + \cdots + (\prod_{i=T_*}^1 A_{t_i})^T(\prod_{i=T_*}^1 A_{t_i})$, where each $t_i \in \{1,\cdots,\ell\}$. Since there are at most ℓ^{T_*} such P_j's and $V(z) \in \{z^T P_j z\}$ for each z, V is finitely piecewise quadratic. $\square$

The converse Lyapunov result established in Proposition 6 can be extended to weak exponential stability.

Proposition 7 The CLI (1) is weakly exponentially stable at $x_e = 0$ if and only if there exists a (possibly infinitely) piecewise quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

(a) there exist $c_1 > 0$ and $c_2 > 0$ such that $c_1\|z\|^2 \leq V(z) \leq c_2\|z\|^2$ for all $z \in \mathbb{R}^n$;

(b) there exists $c_3 > 0$ such that for each $z \in \mathbb{R}^n$, there exists $z' \in f(z)$ such that $V(z') - V(z) \leq -c_3\|z\|^2$.

Proof. For sufficiency, suppose there exists a Lyapunov function V satisfying (a) and (b). For any initial state x^0, it follows from (b) that there exists $z' \in V(x^0)$ such that $V(z') \leq V(x^0) - c_3\|x^0\|^2 \leq \eta V(x^0)$, where $\eta \equiv 1 - c_3/c_2$ via (a). To avoid triviality, we may assume $\eta \in (0, 1)$. Letting $x(1, x^0) \equiv z'$ and repeating the above argument, we obtain an exponentially decaying trajectory $x(t, x^0)$ (with the decay rate determined by η). Hence the CLI is weakly exponentially stable at $x_e = 0$.

For necessity, assume the CLI (1) is weakly exponentially stable at $x_e = 0$. Then starting from any $z \in \mathbb{R}^n$, there exists at least one trajectory $x(t, z)$ satisfying $\|x(t, z)\| \leq \kappa\, e^{-\rho t}\|z\|$, $\forall\, t \in \mathbb{Z}_+$. Thus, the function $V(z)$ defined by $V(z) \equiv \inf_{x(t,z)} \sum_{t=0}^{\infty} \|x(t, z)\|^2$, where the infimum is taken over all the trajectories from z, is finite for each z. Furthermore, it satisfies $\|z\|^2 \leq V(z) \leq c\,\|z\|^2$ for some constant $c \geq 1$, where the existence of c is due to the weak exponential stability. Moreover, being the value function of an infinite-horizon optimal control problem, $V(z)$ satisfies the Bellman equation:

$$V(z) = \min_{z' \in f(z)} \left\{ \|z\|^2 + V(z') \right\} = \|z\|^2 + \min_{z' \in f(z)} V(z').$$

The minimum in the above equation is achieved by some $z'_* \in f(z)$. Then we have $V(z) = \|z\|^2 + V(z'_*)$, i.e., $V(z'_*) - V(z) = -\|z\|^2$. Thus, $V(z)$ satisfies both (a) and (b). □

IV. Stability of Switched Linear Systems

Switched linear systems (SLS) are a natural extension of linear systems. In this section, some basic notions of the stability of switched linear systems are reviewed.

For a given discrete-time switched linear system, its state $x(t) \in \mathbb{R}^n$ evolves by switching among a finite family of linear dynamics indexed by $\mathcal{M} \equiv \{1, \ldots, m\}$:

$$x(t + 1) = A_{\sigma(t)} x(t), \quad t \in \mathbb{Z}_+. \tag{4}$$

Here, $\sigma(t) \in \mathcal{M}$ for $t \in \mathbb{Z}_+$, or simply σ, is called the switching sequence; and $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$, are the subsystem state dynamics matrices. Starting from the initial state x^0, the trajectory of the SLS depends on the switching sequence σ, and will be denoted by $x(t, x^0, \sigma)$. Note that the SLS (4) can be viewed as a special instance of CLIs by setting the family of closed cones to be $\mathcal{X}_i = \mathbb{R}^n$, $i \in \mathcal{M}$. In this case, $f(x) = \{A_i x \,|\, i \in \mathcal{M}\}$ for all $x \in \mathbb{R}^n$.

A. Stability of SLS

Definition 8 (Stability of SLS) The SLS (4) is called

- *exponentially stable under arbitrary switching* if there exist $\kappa \geq 1$ and $\rho > 0$ such that starting from any initial state x^0 and under any switching sequence σ, the state trajectory $x(t, x^0, \sigma)$ satisfies $\|x(t, x^0, \sigma)\| \leq \kappa\|x^0\|e^{-\rho t}, \forall\, t \in \mathbb{Z}_+$.
- *exponentially stable under proper switching* if there exist $\kappa \geq 1$ and $\rho > 0$ such that starting from any initial state x^0, there exists at least one switching sequence σ for which the state trajectory $x(t, x^0, \sigma)$ satisfies $\|x(t, x^0, \sigma)\| \leq \kappa\|x^0\|e^{-\rho t}, \forall\, t \in \mathbb{Z}_+$.

Similarly, we can define the notions of stability (in the sense of Lyapunov) and asymptotic stability for SLS under both arbitrary and proper switching. Clearly the local and global versions of these stability notions are equivalent. Moreover, by Theorem 3, asymptotic stability and exponential stability of SLS under arbitrary switching are equivalent.

B. Generating Functions

We next define some functions that will be useful later on. For each $z \in \mathbb{R}^n$, define the *strong generating function* $G(\cdot, z) : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ of the SLS as [4]

$$G(\lambda, z) \equiv \sup_{\sigma} \sum_{t=0}^{\infty} \lambda^t \|x(t, z, \sigma)\|^2, \quad \forall\ \lambda \in \mathbb{R}_+, \tag{5}$$

where the supremum is taken over all switching sequences σ of the SLS. Obviously, $G(\lambda, z)$ is monotonically increasing in λ, with $G(0, z) = \|z\|^2$ when $\lambda = 0$. As λ increases, however, it is possible that $G(\lambda, z) = +\infty$.

It is often convenient to study $G(\lambda, z)$ as a function of z for fixed λ. Thus, for each $\lambda \in \mathbb{R}_+$, define the function

$$G_\lambda(z) \equiv G(\lambda, z), \quad \forall\ z \in \mathbb{R}^n. \tag{6}$$

From the definition (5), $G_\lambda(z)$ is homogeneous of degree two in z, with $G_0(z) = \|z\|^2$.

The *radius of convergence* λ^* for the SLS (4) is defined as $\lambda^* \equiv \sup\{\lambda \,|\, \exists$ a finite constant c such that $G_\lambda(z) \leq c\|z\|^2,\ \forall\ z \in \mathbb{R}^n\}$.

Proposition 9 The SLS (4) is exponentially stable under arbitrary switching if its radius of convergence $\lambda^* > 1$;

Proof. Assume $\lambda^* > 1$. Then $G_1(z) \leq c\|z\|^2$ for some c, i.e., for any trajectory $x(t, z, \sigma)$ of the SLS, $\sum_{t=0}^{\infty} \|x(t, z, \sigma)\|^2 \leq c\|z\|^2$. This implies that (i) $\|x(t, z, \sigma)\|$ is bounded by $\sqrt{c}\|z\|$ for all t, hence the SLS is stable; and (ii) $x(t, z, \sigma) \to 0$ as $t \to \infty$. Therefore, the SLS is asymptotically stable. By Theorem 3, the SLS as an instance of CLIs is also exponentially stable. □

The converse statement of the above proposition is also true. Its proof will appear in an upcoming paper.

V. CLI Obtained from SLS

A function $E : \mathbb{R}^n \to \mathbb{R}_+$ is an *energy function* if it is

(i) homogeneous of degree two: $E(\lambda z) = \lambda^2 E(z)$, $\forall\, \lambda \in \mathbb{R}$, $\forall\, z \in \mathbb{R}^n$;
(ii) bounded on the unit sphere: $c_1\|z\|^2 \leq E(z) \leq c_2\|z\|^2$ for some constants $0 < c_1 \leq c_2 < \infty$.

Two CLIs can be defined from the SLS (4) based on an energy function E.

Definition 10 Given the SLS (4) and an energy function E, let $\Xi = \{\mathcal{X}_i\}_{i\in\mathcal{M}}$ be a set of closed cones of $\mathbb{R}^n$ defined as

$$\mathcal{X}_i \equiv \{x \in \mathbb{R}^n \mid E(A_i x) = \min_{j\in\mathcal{M}} E(A_j x)\}, \quad i \in \mathcal{M}.$$

Then a CLI can be defined such that it has the dynamics matrix A_i on the cone $\mathcal{X}_i$. Such a CLI is called the *descending realization* of the SLS (4) with respect to the energy function E. If in the above definition of $\mathcal{X}_i$, minimum is replaced by maximum, while the dynamics matrices A_i remain unchanged, then the resulting CLI is called the *ascending realization* of the SLS (4) with respect to E.

It is easy to see from the above definition that the trajectories $x(t, x^0)$ of the descending (resp. ascending) realization CLI are exactly the trajectories $x(t, x^0, \sigma)$ of the SLS (4) under the particular switching policy σ that tries to decrease (resp. increase) the value of the energy function E as much as possible at each step of the trajectories. If for a given x, $\max_{i\in\mathcal{M}} E(A_i x)$ is achieved by multiple $i \in \mathcal{M}$, then the ascending realization CLI has multiple trajectories starting from x: $f(x) = \{A_i x \mid E(A_i x) = \max_{j\in\mathcal{M}} E(A_j x)\}$. Similarly for the descending realization.

The following two theorems show that the study of exponential stability for SLSs can be reduced to the study of weak exponential stability for the ascending/descending CLIs with respect to properly chosen energy functions.

Theorem 11 A necessary and sufficient condition for the SLS (4) to be exponentially stable under arbitrary switching is that its ascending realization CLI with respect to any energy function E is weakly exponentially stable with the same parameters $\kappa, \rho > 0$.

Proof. The necessary part is trivial once we notice that any trajectory $x(t, z)$ of the ascending realization CLI is also a trajectory of the SLS under some switching sequence σ.

To show sufficiency, suppose the ascending realization CLI of the SLS (4) with respect to any energy function E is weakly exponentially stable. Consider the function $G_\lambda(z)$ defined in (6) and its radius of strong convergence λ^*.

Let $\lambda \in [0, \lambda^*)$ be arbitrary. By the definition of λ^*, $G_\lambda(z)$ is an energy function. Denote by $\mathcal{C}_\lambda$ the the ascending realization CLI of (4) with respect to $G_\lambda(z)$, which by our assumption is weakly exponentially stable. Thus, starting from any initial z, there exists a trajectory $x(t, z)$ of $\mathcal{C}_\lambda$ that satisfies $\|x(t, z)\| \le \kappa e^{-\rho t}\|z\|$, $\forall\, t \in \mathbb{Z}_+$. We observe that this trajectory $x(t, z)$ of $\mathcal{C}_\lambda$ is a trajectory $x(t, z, \sigma^*)$ of the SLS (4) under an optimal switching sequence σ^* that achieves the supremum in (5). Indeed, $G_\lambda(z)$ by its definition is the value function for maximizing the functional $\sum_{t=0}^\infty \lambda^t \|x(t, z, \sigma)\|^2$, and satisfies the Bellman equation:

$$G_\lambda(z) = \|z\|^2 + \lambda \cdot \max_{i\in\mathcal{M}} G_\lambda(A_i z), \quad \forall\, z \in \mathbb{R}^n.$$

At each time, say $t = 0$, the next state $x(1, z, \sigma^*) = A_i z$ of $x(t, z, \sigma^*)$ is chosen to be some $i \in \mathcal{M}$ that achieves the maximum in the above equation. Thus, by the dynamical programming principle, $x(t, z) = x(t, z, \sigma^*)$ is a maximizer of the functional $\sum_{t=0}^\infty \lambda^t \|x(t, z, \sigma)\|^2$. As a result, for any trajectory $x(t, z, \sigma)$ of the SLS (4),

$$\sum_{t=0}^\infty \lambda^t \|x(t, z, \sigma)\|^2 \le G_\lambda(z) = \sum_{t=0}^\infty \lambda^t \|x(t, z)\|^2$$
$$\le \sum_{t=0}^\infty \lambda^t \kappa^2 e^{-2\rho t}\|z\|^2 = \frac{\kappa^2}{1 - \lambda e^{-2\rho}}\|z\|^2, \tag{7}$$

for $\lambda \in [0, \lambda^*)$, and $\lambda < e^{2\rho}$ (for the last equality to hold).

We claim that $\lambda^* \ge e^{2\rho}$. For otherwise suppose $\lambda^* < e^{2\rho}$. Fix an arbitrary trajectory $x(t, z, \sigma)$ of the SLS. Define $F(\lambda) \equiv \sum_{t=0}^\infty \lambda^t \|x(t, z, \sigma)\|^2$. Note that $F(\lambda)$ can be thought of as a power series in λ with nonnegative coefficients $\|x(t, z, \sigma)\|^2$. By (7), $F(\lambda)$ is convergent for $\lambda \in [0, \lambda^*)$. Thus its radius of convergence, $R_F \equiv \sup\{\lambda \mid F(\lambda) < \infty\}$, must be at least λ^*.

We next show that R_F is strictly larger than λ^*. First note that $F(\lambda)$ is an analytic function, hence infinite time differentiable, within its radius of convergence $(-\lambda^*, \lambda^*)$ (see [3]). For any $\lambda \in [0, \lambda^*)$, we compute

$$\begin{aligned}
F'(\lambda) &= \sum_{t=0}^\infty (t+1)\lambda^t \|x(t+1, z, \sigma)\|^2 \\
&= \sum_{t=0}^\infty \sum_{s=0}^t \lambda^t \|x(t+1, z, \sigma)\|^2 \\
&= \sum_{s=0}^\infty \lambda^s \sum_{t=0}^\infty \lambda^t \|x(t+s+1, z, \sigma)\|^2 \\
&\le \sum_{s=0}^\infty \lambda^s \frac{\kappa^2}{1 - \lambda e^{-2\rho}} \|x(s+1, z, \sigma)\|^2 \\
&\le \frac{\kappa^4}{(1 - \lambda e^{-2\rho})^2} \|x(1, z, \sigma)\|^2 \\
&\le \frac{\kappa^4 \max_{i\in\mathcal{M}} \|A_i\|^2}{(1 - \lambda e^{-2\rho})^2} \|z\|^2,
\end{aligned}$$

where in deriving the first inequality, we have used (7) for the trajectory $x(t+s+1, z, \sigma)$, $t \in \mathbb{Z}_+$, that starts from $x(s+1, z, \sigma)$. The ensuing inequality is obtained by applying (7) once again to the trajectory $x(s+1, z, \sigma)$, $s \in \mathbb{Z}_+$, that starts from $x(1, z, \sigma)$. The last inequality follows as $\|x(1, z, \sigma)\| = \|A_i z\| \le \|A_i\|\,\|z\|$ for some $i \in \mathcal{M}$. In an entirely similar way, we can prove that

$$F^{(k)}(\lambda) \le \frac{k!\kappa^{2(k+1)}(\max_{i\in\mathcal{M}} \|A_i\|^2)^k}{(1 - \lambda e^{-2\rho})^{k+1}} \|z\|^2, \quad \forall \lambda \in [0, \lambda^*),$$

for $k = 0, 1, 2, \ldots$. Pick any $\lambda_0 \in [0, \lambda^*)$. Consider the Taylor series expansion of $F(\lambda)$ at λ_0:

$$F_0(\lambda) \equiv \sum_{k=0}^\infty \frac{1}{k!} F^{(k)}(\lambda_0)(\lambda - \lambda_0)^k. \tag{8}$$

Since $F(\lambda)$ is analytic at λ_0, $F(\lambda) = F_0(\lambda)$ in a neighborhood of λ_0. We claim that the power series $F_0(\lambda)$ has a convergence radius of at least a constant δ defined by

$$\delta = \frac{1 - \lambda^* e^{-2\rho}}{\kappa^2 \max_{i\in\mathcal{M}} \|A_i\|^2}, \tag{9}$$

for any $\lambda_0 \in [0, \lambda^*)$. Indeed, for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$,

$$|F_0(\lambda)| \leq \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(\lambda_0)|\lambda - \lambda_0|^k \leq \sum_{k=0}^{\infty} \frac{\kappa^{2(k+1)}(\max_{i\in\mathcal{M}} \|A_i\|^2)^k}{(1-\lambda_0 e^{-2\rho})^{k+1}} \|z\|^2 \delta^k < \infty.$$

Thus, $F_0(\lambda)$ defines an analytic function on $(\lambda_0 - \delta, \lambda_0 + \delta)$. Letting $\lambda_0 \uparrow \lambda^*$, the union of $F(\lambda)$ and $F_0(\lambda)$ yields an analytic continuation of $F(\lambda)$ from $[0, \lambda^*)$ to $[0, \lambda^* + \delta)$. The feasibility of finding such an analytic continuation of $F(\lambda)$ beyond λ^* implies that λ^* cannot be the radius of convergence R_F, for $F(\lambda)$ being a power series of nonnegative coefficients must have R_F as a singular point, beyond which no analytic continuation is possible [3, Theorem 5.7.1]. More precisely, the above argument implies that $R_F \geq \lambda^* + \delta$.

Let $\lambda = \lambda^* + \delta/4$ and $\lambda_0 \equiv \lambda - \delta/2 = \lambda^* - \delta/4$. Then since λ is within the radius of convergence of both $F(\lambda)$ and $F_0(\lambda)$, we must have

$$\begin{aligned} F(\lambda^* + \delta/4) &= F_0(\lambda_0 + \delta/2) \\ &\leq \sum_{k=0}^{\infty} \frac{\kappa^{2(k+1)}(\max_{i\in\mathcal{M}} \|A_i\|^2)^k}{(1-\lambda_0 e^{-2\rho})^{k+1}} \|z\|^2 (\delta/2)^k \\ &\leq \frac{2\kappa^2}{1-\lambda^* e^{-2\rho}} \|z\|^2. \end{aligned} \tag{10}$$

Note that the upper bound in (10) is independent of σ. Then

$$G_{\lambda^*+\delta/4}(z) = \sup_{\sigma} F(\lambda^* + \delta/4) \leq \frac{2\kappa^2}{1-\lambda^* e^{-2\rho}} \|z\|^2 < \infty,$$

for all $z \in \mathbb{R}^n$. This leads to a contradiction with the fact that λ^* is the radius of strong convergence for the SLS (4). As a result, $\lambda^* < e^{2\rho}$ cannot be true as we assumed; and we must have $\lambda^* \geq e^{2\rho} > 1$. By Proposition 9, the SLS (4) is exponentially stable under arbitrary switching. □

In reality, however, the above corollary may not be easily implemented as it requires the knowledge of the functions $G_\lambda(z)$, which are typically difficult to obtain.

Theorem 12 A necessary and sufficient condition for the SLS (4) to be exponentially stable under proper switching is that its descending realization CLI with respect to some energy function E is weakly exponentially stable.

Proof. The sufficient part is straightforward. Suppose the descending realization CLI with respect to some E is weakly exponentially stable. Then starting from $z \in \mathbb{R}^n$, any exponentially convergent trajectory $x(t, z)$ of the CLI is also a trajectory $x(t, z, \sigma)$ of the SLS for some suitably σ. Thus, the SLS is exponentially stable under proper switching.

To show necessity, assume that the SLS (4) is exponentially stable under proper switching. Then starting from any $z \in \mathbb{R}^n$, there exists at least one switching sequence σ for which $\|x(t, z, \sigma)\| \leq \kappa e^{-\rho t}\|z\|$ for all t. Thus the function $H(z) = \inf_\sigma \sum_{t=0}^{\infty} \|x(t, z, \sigma)\|^2$ is finite everywhere; and satisfies $\|z\|^2 \leq H(z) \leq \frac{\kappa^2}{1-e^{-2\rho}}\|z\|^2$ for all z. Consider the descending realization CLI of (4) with respect to the energy function $H(z)$. Note that $H(z)$ satisfies the Bellman equation

$$H(z) = \|z\|^2 + \min_{i\in\mathcal{M}} H(A_i z), \quad \forall\ z.$$

Thus, starting from any $z \in \mathbb{R}^n$, we have $H(z') - H(z) = -\|z\|^2$ for any next state $z' \in f(z)$ of the CLI. As a result, $H(z)$ is a piecewise quadratic Lyapunov function of the CLI satisfying the hypotheses of Proposition 6, and the CLI is strongly hence weakly exponentially stable. □

We remark that Theorem 12 remains valid even if the weak exponential stability is replaced by strong exponential stability for the CLI.

Remark 13 The energy functions $E(z)$ in this section are required to be bounded away from both zero and infinity on the unit sphere. To justify this requirement, assume for example $E(z)$ is identically zero (or identically infinity) on $\mathbb{R}^n$. Then the ascending (or descending) CLI will have the same set of trajectories as the SLS (4), thus making the conclusions of Theorem 11 and Theorem 12 trivial.

VI. Conclusion

The stability of conewise linear inclusions is studied. It is found that strong asymptotic stability is equivalent to strong exponential stability for CLIs. Lyapunov and converse Lyapunov theorems are proved for both strong and weak exponential stabilities of CLIs. Finally, it is shown that the exponential stability of switched linear systems is equivalent to that of a family of CLIs obtained from the SLSs via some suitably defined energy functions.

References

[1] A. Arapostathis and M.E. Broucke. Stability and controllability of planar conewise linear systems. *Systems and Control Letters* 56 (2007) 150–158.

[2] W.P. Dayawansa and C.F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Trans. Automatic Control*, Vol. 44(4), pp. 751–760, 1999.

[3] E. Hille. *Analytic Function Theory*. Volume I, AMS Chelsea Publishing (1959).

[4] J. Hu, J. Shen, and W. Zhang. Generating functions of switched linear systems: analysis, computation, and stability applications. Submitted for publication, 2010.

[5] T. Hu, L. Ma and Z. Lin. Stabilization of switched systems via composite quadratic functions. *IEEE Trans. Automat. Contr.*, 54 (2008), pp. 2571-2585.

[6] H. Khalil. *Nonlinear Systems*. 2nd Edition, Prentice Hall (1996).

[7] D. Liberzon. *Switching in Systems and Control*. Birkhauser Boston, MA (2003).

[8] H. Lin and P. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Trans. Automat. Contr.* 54 (2009) pp. 308-322.

[9] P. Mason, U. Boscain, and Y. Chitour. Common polynomial Lyapunov functions for linear switched systems. *SIAM Journal on Control and Optimization*, Vol.45, pp. 226–245, 2006.

[10] J. Shen, L. Han, and J.S. Pang. Switching and stability properties of conewise linear systems. *ESAIM: Control, Optimisation, and Calculus of Variations*, DOI: 10.1051/cocv/2009021, 2009.

[11] R. Shorten, F. Writh, O. Mason, K. Wulf, and C. King. Stability criteria for switching and hybrid systems. *SIAM Review*, Vol. 45(4), pp. 545–592, 2007.

[12] W. Zhang, A. Abate, J. Hu, and M.P. Vitus. Exponential stabilization of discrete-time switched linear systems. *Automatica*, Vol. 45(11), pp. 2526–2536, 2009.