Dynamics of Globally and Dissipatively Coupled Resonators

This work explores the dynamics of arrays of globally and dissipatively coupled resonators. These resonator arrays are shown to be capable of exhibiting seemingly new collective behaviors which are highly sensitive to the dispersion of the natural frequencies of the constituent resonators in the array, the intrinsic damping of the resonators in the array, and the magnitude of the global coupling coefficient that captures the strength of the dissipative coupling. These behaviors have been identified within the work as group attenuation, confined attenuation, and group resonance. Group and confined attenuation are associated with an absence of energy and are strongly dependent on the dispersion of the natural frequencies. In cases of moderate dissipative coupling, the effects of group and confined attenuation could be interpreted as frequency-dependent damping. In cases where the global coupling coefficient is large, group resonance is significant. This effect is synonymous with the resonances of the constituent resonators being shared and occurring at frequencies in between the isolated resonators' natural frequencies. Accordingly, one could view group resonance as the antithesis of localization, in that the localization of the modes of a conservatively coupled system with a finite dispersion of the constituent resonators' natural frequencies is most significant when the coupling is weak. The authors believe that collective behaviors, such as those described herein, have direct applicability in new single-input, single-output resonant mass sensors, and, with extension, a variety of other sensing and signal processing systems. [DOI: 10.1115/1.4029226]

1 Introduction

In engineering, scientific, and mathematical contexts, examples of coupled resonator and oscillator arrays are diverse and abundant [1–7]. However, when the technical scope is limited to mechanical systems or structures, research typically focuses on arrays of resonators in which the coupling between subunits is conservative and nearest-neighbor in nature [8–14]. When these subunits are nominally identical and the coupling is weak, conventional modal and/or perturbation analyses can be applied and localization [8,9] or the spatial confinement of energy in distinct or limited regions can be observed. This effect is a strong function of the coupling strength between resonators and the dispersion characteristics of the system’s uncoupled natural frequencies. In contrast, if the coupling is global and dissipative in nature, very different collective behaviors are observed, namely, confined attenuation, group attenuation, and group resonance. Note that the dynamics of globally coupled systems with conservative coupling have been studied before [15] with classic localization effects being observed; thus, it is the dissipative nature of the global coupling considered here that produces these different collective behaviors. As opposed to the classic localization phenomena, group and confined attenuation are associated with the absence of energy within specific frequencies ranges in either a majority or subset, respectively, of the resonators. In addition, group resonance is the antithesis of localization in that instead of each resonance being correlated to a specific resonator, all resonators share common resonances. Interestingly, all three of these collective behaviors become more pronounced as the coupling strength increases.

This work systematically investigates the dynamics of globally and dissipatively coupled systems by first developing a closed-form solution for the steady-state behavior of an arbitrarily sized array under direct excitation. Following this derivation, analyses of the three identified collective behaviors that globally and dissipatively coupled systems can exhibit are presented. Note that both theoretical and numerical analyses are conducted to validate the results. This work closes with some concluding remarks, after which (in the Appendix) a short model derivation of a representative mechanical system that is globally and dissipatively coupled is presented.

2 Motivation for the Study of Globally and Dissipatively Coupled Systems

As referenced in the Introduction, examples of coupled resonator and oscillator arrays are plentiful and there are many systems that are globally coupled via dissipative mechanisms. The original system that inspired the authors to study globally and dissipatively coupled systems is an array of electromagnetically transduced microresonators. A representative array is shown in Fig. 1 [16] and a model that describes its dynamic response is presented in the Appendix. Other systems whose models possess a similar mathematical structure include flexible cylinders that are coupled via an axially flowing fluid [3] and, potentially, arrays of electrostatically transduced microresonators. In the latter case, relatively recent works have experimentally demonstrated that Ohmic dissipation greatly affects the dynamics of these systems [17]. Accordingly, if certain structures, such as the fixed–fixed nanobeam resonators detailed in Ref. [18], were connected in series, such that the source of one device was connected to the drain of another, the collective phenomena referenced in Sec. 1 and subsequently studied herein may be relevant.

3 A Generic and Closed-Form Steady-State Solution

To study the dynamics of globally and dissipatively coupled systems, the following series of differential equations are considered. As noted above, these equations arose from the study of an electromagnetically transduced microresonator array. A derivation

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of these equations for that system is presented in the Appendix. The particular system of interest is

\[
\frac{d^2}{dt^2} z_n + \frac{1}{Q} \frac{dz_n}{dt} + 2 \sum_{m=1}^{N} \Omega_n \Omega_m z_m + l_n z_n = I_a f(\tau), \quad \text{for } n = 1, \ldots, N \tag{1}
\]

where \(Q\) is the array quality factor, \(\alpha\) is the global coupling coefficient, \(f(\tau)\) is some common forcing applied to the array, \(l_n\) and \(z_n\) are related to the undamped natural frequency and displacement, respectively, of the \(n\)th resonator, \(N\) is the number of resonators in the array, and \((\cdot)'\) denotes a derivative with respect to dimensionless time \(\tau\). Note that when the value of \(I_a\) for a given resonator is close to 1, the array quality factor reduces to the classic definition for quality factor. This definition for the array quality factor arises from the assumption that all of the resonators in the array are subject to the same viscous damping.

Since steady-state solutions are of primary interest, the governing equations in Eq. (1) can be Fourier transformed yielding

\[
(-\Omega^2 + \frac{i\Omega}{Q} + l_n^2) Z_n + z_n \sum_{m=1}^{N} \Omega_n \Omega_m Z_m = I_a F(\Omega) \tag{2}
\]

where \(Z_n(\Omega)\) and \(F(\Omega)\) are the Fourier transforms of \(z_n(\tau)\) and \(f(\tau)\) taken with respect to the dimensionless frequency \(\Omega\), respectively, and \(i\) is the imaginary unit. This system of equations can be expressed in matrix form as

\[
[A] Z(\Omega) = F(\Omega) L \tag{3}
\]

where \([A]\) and \(L\) are column vectors of length \(N\) whose \(n\)th elements are \(Z_n(\Omega)\) and \(l_n\), respectively, and the matrix \([A]\) is defined such that

\[
[A] = [B] + (\alpha \Omega I) L^T
\]

\[
[B] = \left( -\Omega^2 + \frac{i\Omega}{Q} \right) L^T + [L^T]
\]

where \(I\) is a column vector of length \(N\) composed of ones, \([I]\) is an \(N \times N\) identity matrix, and \([L^T]\) is an \(N \times N\) diagonal matrix, whose \(n\)th diagonal element is \(l_n^2\).

Using the Sherman–Morrison formula [19], sometimes referred to as the inverse matrix modification formula, \(Z(\Omega)\) in Eq. (3) can be solved for, in closed-form, by noting that \([A]\) is simply the matrix \([B]\) modified by a rank 1 matrix. Explicitly, the inverse of \([A]\) can be written as

\[
[A]^{-1} = [B]^{-1} - \frac{[B]^{-1} (\alpha \Omega I) L^T [B]^{-1}}{1 + \Omega^2 \Omega^2 + \frac{I_a}{Q} l_n^2}
\]

Typically, the inverse matrix modification formula is employed in situations where the inverse of a matrix is needed and the inverse of another similar matrix is already known. In this case, the inverse of \([B]\) is simple to find since it is diagonal. Thus, with a few manipulations, it can be shown that the frequency response of the \(n\)th resonator, defined as \(H_n(\Omega) = Z_n(\Omega)/F(\Omega)\), is

\[
H_n(\Omega) = \frac{1}{\Omega^2 + \frac{i\Omega}{Q} + l_n^2} \left[ \frac{1}{1 + \sum_{m=1}^{N} l_m^2 \Omega^2 + \frac{i\Omega}{Q} l_m^2} \right]
\]

While not readily apparent based on the results of Eq. (6), the dissipation in the system described by Eq. (1) is complete [20]. This can be shown by rewriting Eq. (1) in matrix form and demonstrating that the associated damping matrix is positive definite, which can be proven using Sylvester’s Criterion [21]. Sylvester’s Criterion states that a real, symmetric matrix is positive definite if and only if all of its principal minors are positive. Using the matrix determinant lemma [22], the \(r\)th principal minor is equal to

\[
\frac{1}{\Omega^2} \left( 1 + \sum_{m=1}^{N} l_m^2 \right)
\]

Thus, if \(Q\) and \(z\) are positive, the dissipation is complete. In addition, if all of the natural frequencies of the uncoupled system are positive, this system is asymptotically stable. In the limiting case that \(1/Q = 0\), all of the principal minors of the damping matrix are equal to zero. This results in a system with incomplete damping, thus it is always assumed that \(Q > 0\) here.

4 Collective Behaviors in Globally and Dissipatively Coupled Resonators

In Secs. 4.1–4.3, the system-level dynamics of globally and dissipatively coupled resonators are considered as characterized by different distributions of \(l_n\). Using the results of Sec. 3, the unique collective behaviors that are observable in these systems are identified and characterized.

4.1 Nearly Identical Resonators and Group Attenuation.

Possibly the simplest case to consider is the case wherein all of the \(l_n\) are identical. Here

\[
H(\Omega) = \frac{1}{\Omega^2 + \frac{i(1 + NQ z)\Omega}{Q} + l_n^2}
\]

In this limiting case, the coupling coefficient only serves to decrease the effective quality factor of the array, in a way which scales with the size of the array \(N\). While this result may seem obvious, it may lead one to incorrectly conclude that if the distribution of \(l_n\) is sufficiently narrow, a response similar to the one in Eq. (8) may be observed. To explore how the parameters \(Q, z, N,\) and the dispersion of the \(l_n\) influence the response, Figs. 2 and 3 show the response of the array, or a subset of the response, for different values of these parameters. The rows of Figs. 2 and 3
correspond to $N = 5, 25,$ and $1000$, respectively, and the columns of these plots correspond to $QN = 1/50, 1/10, 1,$ and $5$, respectively. In these two figures, for a given row, the $l_n$ were selected from a normally distributed random variable with a mean of $1$ and a standard deviation $r$ of $0.001$. After a set of $l_n$ for a given row was generated, the $l_n$ were sorted from lowest to highest. Since it is impractical to show all of the responses in the cases where $N$ is large, a subset of the responses is shown. These responses were selected such that they are equally spaced (i.e., for $N = 1000$, the responses for $n = 1, 200, 400, 600, 800,$ and $1000$ are shown). Finally, there are differences in damping between Figs. 2 and 3, namely, $Q = 100$ in Fig. 2 and $Q = 1000$ in Fig. 3.

Figures 2 and 3 reveal a few distinct trends. The first columns of both figures show that if $aQN$ is much less than $1$, the response is very similar to the uncoupled response. As this critical product, $aQN$, increases, Figs. 2 and 3 differ. In the second column of Fig. 2, effects related to group attenuation, or the near uniform attenuation of the amplitude responses across the vast majority of the array, are observed. Similar trends are observed in Fig. 3, but in the third column. Since the given $l_n$ is equal to the undamped natural frequency of the corresponding resonator, one measure for characterizing how similar one resonator is to another is the standard deviation of the $l_n$. It can be shown that the variance of $l_n$ is equal to $2\sigma^2(2 + \sigma^2)$, where $\sigma$ is the standard deviation of the $l_n$. For small $\sigma$, however, the standard deviation of $l_n$ is approximately $2\sigma$. When damping is small and the undamped natural frequency is close to $1$, the passband of an uncoupled resonator is equal to $1/Q$. Thus, a set of resonators with a mean natural frequency of $1$ can be said to be similar if $\sigma Q$ is close to or less than $1$. It can be concluded that provided $\sigma Q$ is less than $1$, effects related to group attenuation become relevant as $aQN$ approaches $1$. The caveat with this finding is that effects related to group attenuation may be observed for lower $aQN$, as was shown in the second column of Fig. 2, when $\sigma Q$ is less than $1$, but as $\sigma Q$ decreases, the response predicted by Eq. (8) should be similar to the one observed.

While effects related to group attenuation, for resonators with a $\sigma Q$ product less than $1$, become more pronounced as $aQN$ approaches $1$, a different effect known as confined attenuation, or the selective attenuation of the resonator responses, becomes significant as $aQN$ increases beyond $1$. The fourth columns of Figs. 2 and 3 show that for resonator arrays that can be characterized as similar, for an $aQN$ product greater than $1$, resonators with $l_n$ values near the fringes of the selected set are attenuated less than resonators with $l_n$ near the mean of the $l_n$. Confined attenuation is
characterized by a selective attenuation of some of the peak amplitudes of the resonators, where the responses of resonators that are classified to not be similar are attenuated relatively less. In addition, it can also have distinct frequency regions where kinks are observed in the amplitude of the frequency response. These kinks are related to antiresonance effects and will be discussed in greater detail in Sec. 4.2. Note, however, that in general antiresonance effects are only relevant in frequency regions where the response of the uncoupled resonator is already small, as compared to its value at resonance.

4.2 N–1 Nearly Identical Resonators and Confined Attenuation. Consider the case where \( l_1 = l \) and \( l_n = 1 \) for all \( n \neq 1 \). With this distribution of \( l_n \), it can be shown that \( H_1(\Omega) \) has an antiresonance or a zero amplitude response at a specific frequency, provided \( 0 < l < 1 \), and

\[
\alpha = \frac{l}{(1-l)(N-1)Q} \tag{9}
\]

at a frequency of \( \Omega = 1 \). In addition, \( H_n(\Omega) \) for \( n \neq 1 \) has an antiresonance if \( l > 1 \)

\[
\alpha = \frac{1}{l(l-1)Q} \tag{10}
\]

and \( \Omega = \hat{\Omega} \). At first glance, these results may appear to have limited applicability, but given that \( \partial H_1/\partial \alpha, \partial H_1/\partial \Omega, \partial H_1/\partial l, \partial H_n/\partial \alpha, \partial H_n/\partial \Omega, \partial H_n/\partial l \) are nonzero, provided \( Q \neq 0 \), and the values for the coupling coefficients, \( l \) and \( \Omega \), are within the neighborhood of the critical values, the response is very close to an antiresonance. It also implies that if the \( l_n \) for \( n \neq 1 \) are close to one, this result is still approximately true and can still be used to consider the case that all of the \( l_n \) are distinct, but \( N – 1 \) of them are similar. Using this result about the continuity of the response near antiresonant behavior and comparing Eq. (9) to Eq. (10) reveal that as \( \alpha \) is gradually increased from a trivial value, antiresonance behavior is first observed in the resonator with the \( l_n \) that is characterized as being significantly different. It is also worth noting that the requisite coupling coefficient needed to observe antiresonance effects decreases as \( l \) diverges from 1. While this effect may be considered obvious, the response of the significantly different resonator at \( \Omega = 1 \) dramatically decreases as \( l \) diverges. This antiresonance effect was observed in Sec. 4.1 with resonators that have an \( l \) near the fringe of the set. This effect can be explained by considering one of the fringe resonators as the significantly different resonator and noting that the magnitude of the ratio \( l/(1-l) \) can be significantly less than 1 when \( l \) is not close to 1.

Comparing Eq. (9) to the coupling coefficient used to characterize group attenuation when one resonator is significantly different, \( Q(N-1)^{-1} \), reveals that Eq. (9) is only smaller when \( l \) is less

Fig. 3 The responses of various arrays when \( \sigma = 0.001 \) and \( Q = 1000 \). The rows and columns of this figure are laid out in the same manner as in Fig. 2. Since the resonators in this figure are less similar than the resonators in Fig. 2, a larger \( \alpha Q N \) is needed to observe group attenuation. In addition, Figs. 2 and 3 both show that as \( \alpha Q N \) exceeds 1, confined attenuation is observed, where the resonators near the middle of the set are attenuated more than the resonators near the fringes.
and equal to \( [Q(N-1)]^{-1} \). In the uncoupled limit, the significantly different resonator does not have a very large response in the passband of the resonators that are similar. Thus, as the responses show, attenuation of the responses is confined to the resonators in the array that are similar.

4.3 Distinct Resonators and Group Resonance. In Secs. 4.1 and 4.2, the conditions under which group and confined attenuation are relevant depend on whether or not a given resonator has a response within the passband of another resonator or set of resonators. If the coupling coefficient is small, effects related to coupling can be ignored and the passbands are approximately based just on the distribution of the \( l_n \) and the array quality factor. When the coupling coefficient is not small, however, the needed spacing may be much larger. In fact, for very large coupling coefficients, regardless of the spacing, a phenomenon known as group resonance is observed. In this case, all of the resonators in the array share resonances.

To show how the needed spacing may change based on the coupling coefficient, Fig. 5 highlights a series of responses for \( Q = 100 \) and \( N = 5 \), where the \( l_n \) were selected such that the nondimensional undamped natural frequencies are separated by 0.1. In Figs. 5(a)–5(f), the coupling coefficient is equal to 0, 1, 5, 10, 20, and 100 times the inverse the array quality factor, respectively. These responses show that when the coupling coefficient is close to or less than the array quality factor, as is the case in Figs. 5(a) and 5(b), the response of a given resonator is only significant near its uncoupled resonant frequency. As the coupling coefficient is further increased in Fig. 5(e), the antiresonances in the responses become more apparent. For a given resonator, the location of these antiresonances is at the natural frequencies of the other resonators. This is in accord with the antiresonance calculation in Sec. 4.2. In conjunction with the antiresonances becoming more prevalent, small lobes in between the uncoupled resonant frequencies appear. For larger coupling coefficients, as is the case in Figs. 5(d) and 5(e), effects related to group resonance are seen as these lobes transition to responses that appear to be resonantlike. In addition, group resonance effects are accompanied by destructive interference near the uncoupled resonant frequencies, as there are no longer resonances close to the uncoupled resonant frequencies. The problem with new resonances being created in between the uncoupled resonant frequencies is that from a system identification perspective, the determination of a given resonator is not as simple as compared to the case when the resonators are uncoupled. An extreme case of this is shown in Fig. 5(f), as all of the resonances in the system occur at frequencies in between the uncoupled resonant frequencies.

Interestingly, there is evidence that suggests that the transition of the side lobes to resonances may be most strongly dependent on \( z \) and weakly dependent on \( Q \) and the relative spacing of the \( l_n \). Shown in Fig. 6 are a series of responses for \( Q = 10,000 \) and \( N = 5 \), where the \( l_n \) were selected such that the nondimensional undamped natural frequencies are separated by 1. In Figs. 6(a)–6(f), the plots show that for \( z \) greater than or equal to 1, the resonances of a given resonator are in between the uncoupled resonant frequencies. Note that for the responses shown in Fig. 6(f), where \( Q \) is very small and \( z \) is very large, these group resonances can be fairly large. Thus, globally and dissipatively coupled systems can exhibit behavior similar to systems with elastic coupling that is nearest-neighbor in nature, with modes transitioning from localized to spatially extended depending on the strength of the coupling between the resonators. A significant difference, however, is that there can be a range of \( z \) for globally and dissipatively coupled systems where the response is significantly attenuated relative to the extreme cases that either \( z \) is very small or very large.

The condition that \( z \) must be close to or greater than 1 in order to observe the transition of the side lobes to resonances can be
supported analytically. In the case that $N = 2$, if it is assumed that $x = 1/e$, where $e \ll 1$, this system has a resonance at

$$
\Omega = \sqrt{l_1 l_2 (l_1 - l_2)^2 - \frac{1}{2Q^2}} + O(e^2) \quad (11)
$$

Note that in this calculation, the $O(e)$ correction is equal to zero; thus, the error in this approximate solution is proportional to $e^2$. As such, the error in this calculation only becomes significant as $x$ approaches and decreases beyond 1. While this result is only for $N = 2$, the responses in Fig. 6 for a larger array corroborate the concept that the resonances in the strongly coupled case are in between the uncoupled resonant frequencies. As is shown in the Appendix, in application, the need for the resonances to be close the uncoupled resonant frequencies (i.e., $x < 1$) places ultimate design limits of a system that can be described as an array of globally and dissipatively coupled resonators.

Fig. 5 Various responses for $Q = 100$ and $N = 5$ when the $l_i$ are selected such that the undamped natural frequencies are separated by 0.1. In Figs. 5(a)–5(f), the coupling coefficient is equal to 0, 1, 5, 10, 20, and 100 times the inverse of the array quality factor, respectively. For small coupling coefficients, such as the cases in Figs. 5(a) and 5(b), the resonators only have a significant response near the uncoupled resonant frequencies. As the coupling coefficient increases, group resonance is observed, wherein individual resonances are lost and all of the resonators share resonances.
5 Conclusions and Future Directions

The dynamics of systems with dissipative and global coupling are discussed herein. As highlighted within, depending on the distribution of the natural frequencies of the resonators in the system and coupling strength, these systems can exhibit three collective behaviors: group attenuation, confined attenuation, and group resonance. Conditions under which these phenomena are relevant are derived via theoretical methods and numerical experiments. To provide an application of this theory, a model for an electromechanical system with this type of coupling is developed (as detailed in the Appendix). In addition, other systems where the dynamic phenomena presented in this work may be relevant are referenced.

Future work will consist of studying other mechanical systems with global coupling and experimentally demonstrating confined attenuation, group attenuation, and group resonance effects. Additional potential endeavors include studying dissipative and global coupling in the presence of not only nonlinear coupling but also nonlinear operation of the constituent resonators. Previous works by these authors have developed models that account for the mechanical nonlinearities of an isolated electromagnetically transduced microresonator and the nonlinearities in the induced electromotive force (EMF) that would couple an array of these devices [23]. Accordingly, a natural extension of the presented results would be the consideration of the aforementioned effects.

Fig. 6 A series of responses for \( Q = 10,000 \) and \( N = 5 \) when the \( I_0 \) are selected such that the undamped natural frequencies are separated by 1. In Figs. 6(a)–6(f), the coupling coefficient is equal to 0, 0.1, 0.5, 1, 10 and 1000, respectively. These responses support the concept that the transition to group resonance is most strongly dependent on the coupling coefficient, and that group resonance is observed when \( \alpha \) is greater than or equal to 1.
system considered here, this section starts with a reduced-order
ics highlighted within this work may emerge. To elucidate the
magnitude similar to the applied potential, the interesting dynam-
tical, resonances in the electrical response near the mechanical
duction wires follow the outer perimeter of the array such that the
red contact pads, is related to the sum of the velocities of the
induced EMF, which can be measured as the potential across
yellow, a Lorentz force suitable for actuation is generated. In turn,
the induced EMF is the superposition of the isolated EMFs. Thus, the total
induced EMF due to the entire array is

\[ S = \sum \text{induced EMF} \]

As referenced earlier, since the array’s vibrations take place in a
magnetic field, an induced EMF is generated. Since the induced
EMF is a product of a time-varying magnetic flux, a model for the
magnetic flux is needed. The magnetic flux \( \Phi(t) \) due to the mag-
netic field \( B \) that is in the neighborhood of the surface \( S \) of a
microcantilever is defined as

\[ \Phi(t) = \int_S B \cdot n dA \]  

where \( n \) is the normal to the surface of the microcantilever. Ignoring
nonlinear effects, or assuming that displacements are small,
this normal vector only has a component that points in the direc-
tion opposite of the end of the microcantilever and has a magni-
tude equal to the angle of deflection. Using the same single-mode
expansion technique that was used to derive Eq. (A1), the mag-
etic flux of an isolated microcantilever is approximated as

\[ \Phi(t) \approx 2Bgz \]  

Using Faraday’s law [25], which states that the induced EMF in a
closed circuit is equal to the opposite of the time derivative of the
magnetic flux enclosed by this circuit, the induced EMF \( V_{\text{EMF}} \) of a
single microcantilever is

\[ V_{\text{EMF}} = -\frac{d\Phi(t)}{dt} = -\frac{d\Phi(t)}{dt} \frac{dt}{d\tau} = -2BgV_0 \sqrt{\frac{EI}{\rho A l_0^2}} \]  

To broaden this result to an array of microcantilevers, the surface \( S \) can be extended to the entire array, yielding that the total
induced EMF is the superposition of the isolated EMFs. Thus, the
total induced EMF due to the entire array is
\[ V_{\text{EMF}} = -2BGy_0 \sqrt{\frac{EI}{\mu A R}} \sum_{n=1}^{N} \gamma_n \]  

(A10)

If it is assumed that the induced EMF due the collective vibrations of the array is significant relative to the supplied voltage, then the supplied current to the array can be written as

\[ i(t) = \frac{(V_{\text{in}} + V_{\text{EMF}})}{R} \]  

(A11)

where \( V_{\text{in}} \) is the excitation voltage and \( R \) is the total resistance of the Au/Cr wire used for actuation, the resistance of the cables that connect the device to the source and the Thévenin equivalent resistance of the source. By incorporating Eq. (A11) into Eqs. (A5) and (A6), Eq. (1) results where

\[ z = \frac{4l_0 B^2 g^2}{\beta^2 R} \sqrt{\frac{\rho AE}{\mu}} \]  

\[ f(\tau) = \frac{2B_g^2 l_0^3}{V_0 EI} \frac{1}{R} V_{\text{in}}(\tau) \]  

(A12)

Note that if the wire trace used for sensing was coiled, the induced EMF would be approximately scaled by the number of turns in the coil. This would, in turn, scale the coupling coefficient \( z \). While coiling the sensing wire trace would help to address insertion loss issues, any effects related to group attenuation, confined attenuation, or group resonance would also be increased. In applications where the resonances must be correlated to a distinct microcantilever in the array or an array designed such that group resonance is not present, Eq. (A12) places ultimate limits on certain parameters. In particular, while one might want a large magnetic field so as to minimize the needed supply current, a very large magnetic field may produce a coupling coefficient that introduces the previously mentioned effects. Equation (A12) also reveals that a large source resistance can be used to decrease the coupling coefficient.

References