Nonlinear Dynamics and Its Applications in Micro- and Nanoresonators

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This review provides a summary of work on the resonant nonlinear dynamics of micro- and nanoelectromechanical systems. This research area, which has been active for approximately a decade, involves the study of nonlinear behaviors arising in small scale, vibratory, mechanical devices that are typically integrated with electronics for use in signal processing, actuation, and sensing applications. The inherent nature of these devices, which includes low damping, desired resonant operation, and the presence of nonlinear potential fields, sets an ideal stage for the appearance of nonlinear behavior. While nonlinearities are typically avoided in device design, they have the potential to allow designers to beneficially leverage nonlinear behavior in certain applications. This paper provides an overview of the fundamental research on nonlinear behaviors arising in micro-/nanoresonators, including direct and parametric resonances in individual resonators and coupled resonator arrays, and also describes the active exploitation of nonlinear dynamics in the development of resonant mass sensors, inertial sensors, and electromechanical signal processing systems. This paper closes with some brief remarks about important ongoing developments in the field.
[DOI: 10.1115/1.4001333]

1 Introduction

This paper describes past developments, ongoing work, and a vision of future topics for a research area with a relatively short history—the application and exploitation of nonlinear dynamic behavior in micro- and nanoelectromechanical systems (M/NEMS). M/NEMS resonators are commonly used for applications such as sensing, signal processing, switching, and timing. Many of these implementations are driven by the recognition that some purely electrical components can be advantageously replaced by electromechanical analogs. The benefits of such an approach include smaller size, lower damping, and improved performance metrics. These devices offer the desirable feature of being easily integrated with solid-state circuits, thus enabling the development of integrated circuit (IC) chips with integrated mechanical and electrical functionality. The transition to electromechanical components requires that analysts and device designers deal with dynamic behavior resulting from classical forces acting on the mechanical components, as well as multiphysics interactions, many of which do not have counterparts in purely electrical or macroscale mechanical systems. Accordingly, the vast experience developed for the analysis and design of electrical circuits and for mechanical structures are not, in themselves, sufficient for understanding the behavior of most resonant micro-/nanosystems.

M/NEMS resonators are usually quite simple in terms of mechanical design and typically consist of common elements such as beams, plates, or lumped masses. They range in length scales from 10^1-10^4 μm, often with feature sizes on the order of a few nanometers, and have been designed with natural frequencies in the range 10^4-10^9 Hz and dissipation corresponding to Q values in the range 10^0-10^5. They are fabricated using a wide variety of additive and subtractive methods adopted from classical IC fabrication techniques. The first notable progress in the area of microresonators was the work of Nathanson and co-workers on resonant gate transistors [1,2]. The driving force that made the fabrication of such devices possible came from the electronics industry, specifically the production of miniaturized ICs. The field has progressed rapidly since that time, with most efforts based on linear resonance behavior in M/NEMS, as summarized in previous reviews [3–5].

Despite the early research emphasis on micro-/nanosystems based on linear resonators, it is quite easy to realize nonlinear behavior in M/NEMS. In fact, M/NEMS offer tremendous flexibility in terms of designing devices that actively exploit the rich behavior of nonlinear systems, as continually demonstrated in the literature over the past decade. Given that research efforts related to nonlinear micro- and nanoresonators span only this brief time window, a quite thorough overview of previous and ongoing research is possible. The authors have attempted to provide such herein, and in the conference paper on which this article is based, which includes a more comprehensive review of the literature [6]. These papers cover most areas of nonlinear M/NEMS resonators with the exception of atomic force microscopy (AFM) and other forms of probe-based microscopy, which utilize micro-/nanoresonators, reviews of which are available in Refs. [7,8]. Also worth noting is the recent review by Lifshitz and Cross [9], which covers some of the same ground as this paper and Ref. [6], and also includes a detailed description of relevant analytical techniques and results. The present paper compliments [9] by providing an engineering perspective of nonlinear micro- and nanoresonators, with a broader view and further discussion of applications.

The present work is organized as follows: a discussion of modeling fundamentals and an illustrative physical model, systems with direct excitation, systems with parametric excitation, systems with combined direct and parametric excitation, and coupled resonator arrays. Reviews on the topics of pull-in, self-excited systems, vibro-impact systems, linear resonator arrays, and single-element systems with a few degrees of freedom can be found in the companion conference paper [6]. This paper closes with some thoughts about the potential for further progress in this field and a brief discussion of emerging research topics that will become increasingly important as the area advances.

2 Modeling and Sources of Resonances

This section starts with a general discussion about the considerations made when modeling and analyzing M/NEMS resonators and proceeds with the derivation of a simple model for a generic electromechanical device, which demonstrates how resonances arise naturally in these systems. The section then closes with a description of electrostatic actuation using comb drives. Note that the discussion presented here focuses on simple lumped-mass

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Contributed by the Dynamic Systems Division of ASME for publication in the Journal of Dynamic Systems, Measurement, and Control. Manuscript received January 19, 2009; final manuscript received February 10, 2010; published online April 16, 2010. Assoc. Editor: Ahmet S. Yigit.

models for individual resonators, from which one can build models of systems with additional degrees-of-freedom, such as coupled resonator arrays. These low degree-of-freedom models are often the outcome of a more detailed modeling approach, which begins with partial differential equations or finite element models and subsequently employs model reduction techniques. Though this approach dominates within the nonlinear dynamics community, many researchers have adopted an alternative modeling approach, wherein lumped-mass models are developed through the use of a basic, phenomenologically derived equations of motion, buttressed by experimental system identification techniques, which can be used to predict frequencies, quality factors, and nonlinear characteristics. The latter approach is particularly appealing since there are many common effects that are notoriously difficult to model from first principles, including fabrication-induced prestresses, three-dimensional electrostatic and/or electromagnetic forces, and (precise) boundary conditions. Note that reviews of system modeling in the presence of mechanical and electrostatic interactions, incorporating quite extensive reference lists, can be found in Refs. [10,11]. The books of Senturia [12], Pelesko and Bernstein [13], Lobontiu [14], and Cleland [15] also provide systematic views of M/NEMS modeling.

Many of the forces that act on micro- and nanoresonators are the same as those encountered at the macroscale, such as those arising from elastic, magnetic, electromagnetic, and aerodynamic sources. In general, these devices are large enough that one can use continuum approaches as the starting point for modeling the aforementioned effects. Accordingly, standard finite element packages, such as ANSYS™, are useful in MEMS modeling. Likewise, a number of computational modeling tools have been developed specifically for M/NEMS applications, including SUGAR [16] and COVENTORWARE™. Some of these modeling tools are able to capture many important nonlinear effects, for example, nonlinear elastic and electrostatic effects attributable to finite displacements. Other effects, especially nonconservative ones such as squeeze film damping and other forms of dissipation, as well as adhesion, are more difficult to include in computational models, and often require special treatment (see, for example, Refs. [17–29]).

Due to the length scales involved, forces not important at the macroscale can often play a significant role in M/NEMS as well, including van der Waals/Casimir-Polder, adhesion, and electrostatic forces. The van der Waals and Casimir–Poulder forces arise from atomic-level interactions and are very weak except at close distances (approximately 1–10 nm), wherein they become increasingly attractive as interaction lengths decrease. At very close distances, however, these forces can become repulsive, due to the fact that electron orbitals cannot overlap. This effect is very strong at extremely close interaction distances and, in fact, becomes essentially unbounded near contact (to avoid penetration). The combined attraction and repulsion forces prove particularly important in probe-based microscopy applications, where they are conveniently modeled by Lennard-Jones potentials, which can be used to capture the qualitative nature of the interactions. It is worth noting that very basic notions from harmonically forced nonlinear oscillations were instrumental in providing some of the first experimental evidence of the aforementioned forces, including the Casimir–Polder force (see, for example, Refs. [30,31]).

M/NEMS actuation can be achieved using many of the forces noted above and can also be realized through base excitation and induced stresses, both of which are commonly generated by piezoelectric elements. It is the combination of all of the aforementioned effects, and the fact that the mechanical device is often integrated directly with electronics, that provides both challenges and opportunities for the design of dynamic M/NEMS.

2.1 An Illustrative Electromechanical Device. In this section, we consider a simple electromechanical device that contains the ingredients needed for describing the nonlinear resonances of interest. This system consists solely of a parallel-plate capacitor, with gap g and a voltage $V(t)$ across the electrodes, which has one

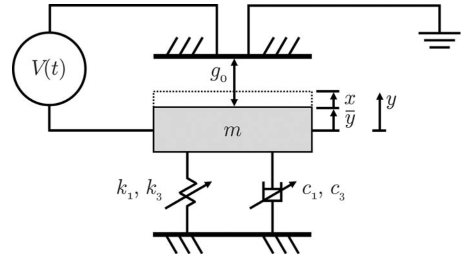

Fig. 1 A schematic of the representative, electrostatically-actuated micro/nanoresonator discussed in Sec. 2

fixed electrode and a second movable electrode, of mass m, which is suspended by a spring, see Fig. 1. The electrostatic attractive force between the electrodes is of the form

$$F_e = \frac{\varepsilon A V^2}{2g^2} \tag{1}$$

where ε is the permittivity of the space and A is the effective area of the electrodes (this simple expression ignores fringe field effects). It is important to note that F_e will, for any nonzero voltage, possess a dc term. Furthermore, if the voltage is composed of dc and ac terms

$$V(t) = V_{\mathrm{dc}} + V_{\mathrm{ac}} \cos(\omega t) \tag{2}$$

then F_e is composed of a dc component, dependent on both V_{ac} and V_{dc}, as well as harmonic components with frequencies ω and 2ω. This form of excitation has some interesting consequences on the amplitude-frequency response of the system, obtained as one sweeps through the excitation parameters V_{ac} and ω, and provides a pathway for the tuning of various effects, as realized through independent manipulation of V_{dc} and V_{ac}.

A model for the dynamics of this single-DOF system can be expressed in terms of y, the inertially measured displacement of the movable element. Note that $y=0$ is the equilibrium of the system for zero voltage, that is, for $V=0$ ($F_e=0$), if one ignores gravity (which is negligible in M/NEMS). Letting g_o denote the gap corresponding to $y=0$, the equation of motion for the suspended mass m can be expressed as

$$m\ddot{y} = F_m(y) + F_e(y,t) + F_d(y,\dot{y}) + F_a(t) \tag{3}$$

where $F_m(y)=-F_m(-y)$ is the (assumed symmetric) conservative mechanical restoring force provided by the spring suspension, F_d accounts for dissipative effects (with $F_d(y,0)=0$), and F_a accounts for time-dependent, externally applied excitations. In the case of no external excitation ($F_a=0$) and no ac voltage ($V_{\mathrm{ac}}=0$), the equilibrium position is given by solving

$$F_m(\bar{y}) + \frac{\varepsilon A V_{\mathrm{dc}}^2}{2(g_o - \bar{y})^2} = 0 \tag{4}$$

that is, by balancing the mechanical and electrostatic forces that result from the dc voltage. At this stage one can already uncover some interesting nonlinear behavior related to this equilibrium condition, namely, the pull-in instability, which is a well-understood buckling phenomenon caused by electrostatic forces. In terms of dynamics, one can use this system to develop a model that exposes terms in the equations of motion that are known to lead to nonlinear resonant behavior. This is done by defining the local coordinate $x=y-\bar{y}$, specifying forms for the elastic stiffness and damping terms, and expanding the equation of motion out to cubic order in x and its derivatives, resulting in

$$
\begin{aligned}
m\ddot{x} + (c_1 + c_3 x^2)\dot{x} &+ \left\{ k_1 + 3k_3\bar{y}^2 - \frac{2\gamma[V_{\text{dc}} + V_{\text{ac}}\cos(\omega t)]^2}{(g_o - \bar{y})^3} \right\} x \\
&+ \left\{ 3k_3\bar{y} - \frac{3\gamma[V_{\text{dc}} + V_{\text{ac}}\cos(\omega t)]^2}{(g_o - \bar{y})^4} \right\} x^2 \\
&+ \left\{ k_3 - \frac{4\gamma[V_{\text{dc}} + V_{\text{ac}}\cos(\omega t)]^2}{(g_o - \bar{y})^5} \right\} x^3 = F_a(t) \\
&+ \frac{\gamma[2V_{\text{dc}}V_{\text{ac}}\cos(\omega t) + V_{\text{ac}}^2\cos^2(\omega t)]}{(g_o - \bar{y})^2}
\end{aligned} \tag{5}
$$

where γ is a coefficient that accounts for the permittivity of the surrounding medium and the area of the electrodes, k_1 and k_3 are the linear and cubic nonlinear mechanical stiffness coefficients, and c_1 and c_3 are the linear and nonlinear damping terms used to capture important dissipative effects [9]. Note that the form of the voltage excitation renders a number of interesting effects. First, it generates both direct and parametric excitations and, in particular, leads to a parametric excitation that acts on the entire restoring force, including its linear and nonlinear components. In fact, when one drives the system near resonance, that is, when ω is close to the natural frequency, there also exist parametric driving terms at 2ω, which are capable of inducing parametric instabilities. Therefore, one cannot apply a resonant excitation to such a system without at least the possibility of these additional resonant interactions. In addition, the ac voltage leads to shifts in the time-invariant linear and nonlinear stiffness coefficients, since $\cos^2(\omega t) = \frac{1}{2}[1 + \cos(2\omega t)]$ results in a dc term. This implies that the linear natural frequency and the strength of the time-independent nonlinearities (which cause shifts in the frequencies of oscillation) both depend on the amplitude of the excitation. This does not occur in most macroscale mechanical systems, where system parameters such as the natural frequency and nonlinear stiffness are independent of the excitation parameters. Finally, note that simple parametric resonances can be induced by driving the system with a frequency ω that is close to twice the system natural frequency, in which case the higher harmonics are nonresonant.

In some cases it is desirable to have a simpler form for the excitation. Accordingly, one can utilize a voltage of the form $V(t) = V_o\sqrt{1 + \alpha\cos(\omega t)}$ with $|\alpha| \leq 1$, which renders a single frequency ac component of amplitude αV_o^2 and frequency ω, in addition to a dc component V_o^2 [32]. In this case, the ac voltage (captured by the parameter α) has only an ω harmonic, but the other effects described above are present. The key to this excitation is that one can independently drive the system into primary or parametric resonance, without interference. If one drives the system with ω near the natural frequency, direct excitation is achieved (in this case the parametric terms are nonresonant). Similarly, by driving ω near twice the natural frequency parametric resonance is obtained (here the direct excitation term is nonresonant).

It should be noted that the particular details of Eq. (5) are not as important as its general form, which is encountered in a number of situations involving electrostatic, and in some cases piezoelectric or electromagnetic, actuation. For example, it is found in the modeling of electrostatically actuated torsional devices, which are mounted by mechanical supports with torsional flexibility and actuated by placing electrodes away from the axis of rotation, thereby generating driving torques. Likewise, these effects arise in multi-degree-of-freedom systems, in which the voltage differences between adjacent resonators produce coupling effects.

With this in mind, the essential features of the systems of interest are those of lightly damped nonlinear resonators (or a set of coupled resonators) that are subjected to periodic excitation, either direct, parametric, or combined in nature. Arguably the most unique feature of these resonators is the manner in which the excitation and parameter coefficients are intertwined, and the fact that one can use a variety of actuators to tune devices and achieve complex, yet tailored, dynamic behaviors.

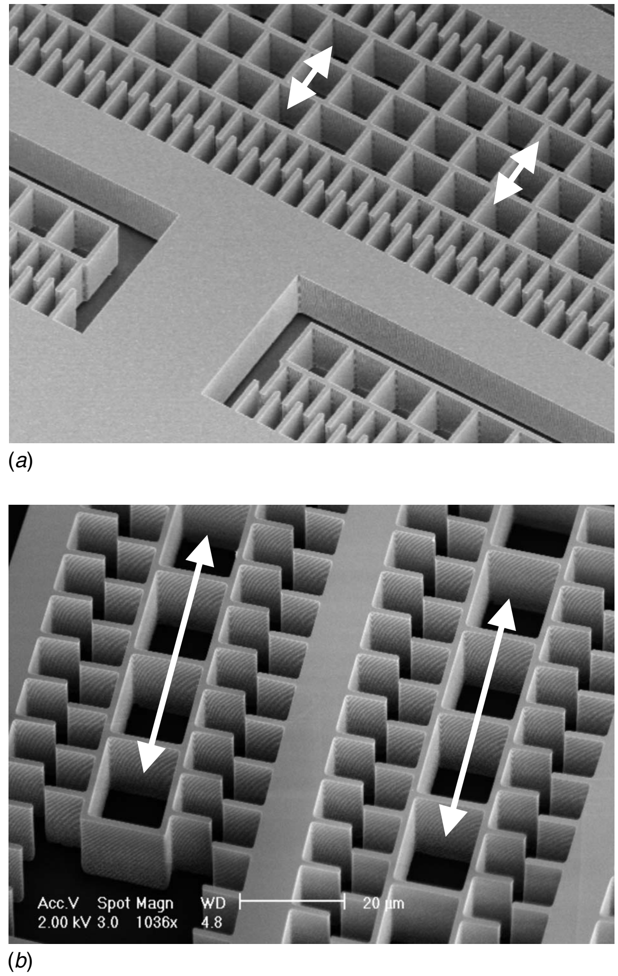

Fig. 2 (***a***) **An interdigitated electrostatic comb drive.** (***b***) **A non-interdigitated electrostatic comb drive. Note that the included arrows indicate the direction of dominant motion for the moveable part of the device. The other banks of comb fingers are fixed (picture courtesy of B. DeMartini).**

2.2 Electrostatic Actuation by Comb Drives. Micro- and nanoresonators based on beam geometries often employ relatively simple electrode configurations that lead to the nonlinear resonant effects described above. In microscale designs involving lumped-mass resonators, alternative electrode embodiments, namely, electrostatic comb drives, are commonly employed. These components use arrays of fingers, as shown in Fig. 2, to allow for an effectively large area over which voltage differences can act. In the more typical case of interdigitated comb fingers, shown in Fig. 2(*a*), two sets of combs move relative to one another in such a manner that the change in area generates a change in capacitance, which results in an electrostatic force of the form

$$F_{ic} = \frac{bV^2}{g} \tag{6}$$

where b is a coefficient that depends on the permittivity of the space, the geometric particulars of the combs, and the number of comb fingers, g is the gap between adjacent fingers, and V is the voltage applied across the fingers. This arrangement is used to induce direct excitation forces, which act along the longitudinal axes of the comb fingers. This arrangement can also be used to capacitively sense the motion of a given device.

Another, less traditional, arrangement is that of noninterdigitated comb drives, shown in Fig. 2(*b*). Here the combs move perpendicular to the finger axes, and the forces are generated by fringe field effects. The most interesting feature of this arrangement is that when the fingers are aligned directly across from one another, or are evenly staggered, zero net force occurs in this symmetric state. If such a configuration corresponds to the mechanical static equilibrium of the device, forces arise only when the symmetry of the finger alignment is broken, that is, by movement of one set of electrodes. A general form for the force in this situation, in terms of the displacement x from the symmetric alignment position, is

$$F_{nic} = f(x)V^2 \tag{7}$$

where V is the voltage across the combs and $f(x)$ describes the displacement-dependent nature of the force, which itself depends on a number of geometric factors [33]. Note that in the symmetric cases $f(0)=0$, and that if there are a large number of fingers (such that end effects are essentially negligible), $f(x)$ is essentially periodic, that is, $f(x+s)=f(x)$, where s is the spacing between fingers. Locally, near the equilibrium position, the electrostatic force can be linearly or nonlinearly hardening or softening [33]. This arrangement is a convenient way to generate parametric excitation in lumped-mass MEMS. It should also be noted that though one generally restricts the movement of noninterdigitated devices such that $x \ll s$, interesting dynamics, including chaos, can arise for larger motions [34].

In summary, comb drives lead to equations of motion of the general form given above in Eq. (5), albeit with different forms for the coefficients. Perhaps the most important aspect of these components is that one can design them to produce desired linear and nonlinear forces, and, in fact, tune these forces by adjusting the amplitude of dc and ac excitation voltages [33,35], or the comb-finger geometry [36].

3 Systems With Direct Excitation: Primary Resonance

Of the numerous investigations of nonlinearity in micro-/nanoelectromechanical resonators, the earliest, and certainly most prevalent, are those focusing on systems with Duffing, or Duffing-like, characteristics, which arise from nonlinear elastic restoring forces and/or electrostatic, electromagnetic, or piezoelectric effects. The present section summarizes the various research investigations of these systems when they respond to direct (as opposed to parametric) excitation, and a primary resonance is excited.

This section begins with the largely observational studies published in the late 1980s and early 1990s, and then continues with more recent analytical and experimental investigations. This section concludes with a review of directly excited nanoscale resonators, a brief discussion of application-specific literature, and an overview of recent results on dynamic transitions between coexistent bistable states.

3.1 Early Investigations. To the best of the authors' knowledge, the earliest report of nonlinear behavior in a resonant microsystem appeared in the 1987 work of Andres et al. [37], which describes the nonlinear hardening response (that is, the resonance peak bends toward higher frequencies) of a silicon microbeam driven by an adjacent piezoelectric element. Though not explicitly detailed within the work, the observed nonlinearity appears to arise from large-amplitude effects, induced by driving the resonator with a hard excitation in a low-pressure environment ($Q=21{,}000$). Subsequent to the publication of that work, a number of research papers reported observations of nonlinear frequency responses, attributable to Duffing-like effects in alternative contexts. Ikeda, Tilmans, and their respective collaborators, for example, reported hardening responses in resonant strain gauges undergoing large elastic deformations [38–40]. Likewise, Nguyen, Legtenberg, Bourouina, Piekarski, and their respective collaborators noted hardening responses in electrostatically actuated, and subsequently electromagnetically and piezoelectrically actuated, devices [41–45]. These latter works are of particular note, as they drew attention to the fact that nonlinearities in resonant microsystems can arise from multiple sources. Specifically, the authors noted the potential for interplay between nonlinearities arising from mechanical mechanisms, which are often hardening, and transduction effects, which are often softening. This observation provided an important research framework for the efforts that would follow.

Building upon the works detailed above, Ayela and Fournier [46] reported in 1998 what is believed to be the first microresonator frequency response with softening characteristics. Though there appears to be some discrepancy regarding the source of the softening response, the authors stated that it arose from mechanical sources. In 2000, Camon and Larnaudie [47] observed softening response characteristics in an electrostatically actuated micromirror excited by a comparatively large actuation voltage. Though not directly addressed in the paper, the softening nature of the recovered response was almost certainly due to the dominance of softening electrostatic nonlinearities, especially in light of the large angles of deflection that were reported.

In 2002, Nayfeh and co-workers [48–52] presented the first of several theoretical works detailing the nonlinear behavior of electrostatically actuated microbeams. These works, which approached microsystems analysis from a classical nonlinear oscillations perspective, emphasized the development and subsequent reduction of distributed-parameter models, incorporating both mechanical and electrostatic nonlinear effects. The authors subsequently analyzed these models with multiple-scale perturbation methods and numerical techniques in order to identify the various nonlinear behaviors that could be recovered with a capacitively driven microbeam. These nonlinear behaviors included not only the classical, hysteretic responses associated with direct excitations applied in the presence of Duffing-like nonlinearities but also super- and subharmonic resonances, internal resonances, and limit cycles.

3.2 Recent Investigations. From 2002 onward, analytical and experimental investigations of directly excited microresonators flourished. Among the various analytical and experimental efforts focusing on electrostatically actuated systems, for example, were (i) the 2004 effort of Kaajakari et al. [53], which examined the nonlinear response of silicon, bulk acoustic wave (BAW) resonators, demonstrating that these systems, in comparison to their flexural counterparts, had appreciably-higher energy storage capabilities; (ii) the 2005 work of Jeong and Ha [54], which developed a predictive model for the linear displacement limits of comb-driven resonant actuators; (iii) the 2007 and 2008 efforts of Agarwal et al. [55,56], which detailed the modeling, analysis, and optimization of double-ended-tuning-fork resonators simultaneously actuated by two, variable-gap electrostatic forces; and finally, (iv) the 2008 investigation of Shao et al. [57], which thoroughly detailed the nonlinear response of free-free microbeam resonators.

Apart from the aforementioned investigations of electrostatically actuated microresonators, a number of works addressed Duffing-like behaviors that arise in directly excited piezoelectrically actuated devices. Foremost among these are the works of Li, Dick, Mahmoodi, and their respective collaborators [58–61]. The first of these, by Li et al. [59], detailed the modeling and analysis of the clamped-clamped, piezoelectrically actuated microstructure previously described in Ref. [45]. In this work, the authors utilized composite beam theory to develop a distributed-parameter representation of their nonlinear system. This model was subsequently reduced to a lumped-mass analog, and used to characterize the free and forced responses of the device in its postbuckled configuration. The work of Dick et al. [58] extended this effort, demonstrating that parametric identification techniques could be used to extrapolate pertinent model parameters, which, in turn,

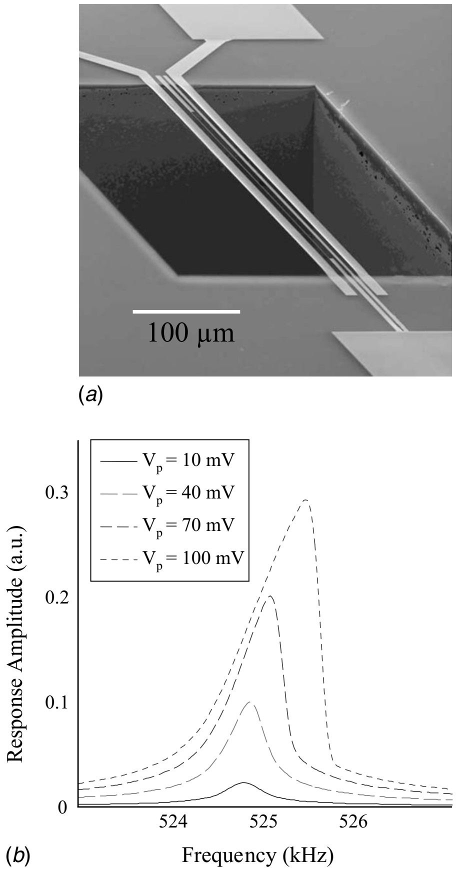

Fig. 3 (*a*) An electrostatically actuated nanoresonator comprised of a clamped-clamped beam. (*b*) Nonlinear frequency response obtained from the device. Note that because these responses were obtained solely through up-sweeps in frequency, hysteresis is not evident (pictures courtesy of E. Buks).

facilitated predictive design. The later works of Mahmoodi et al. [60,61] detailed the nonlinear characteristics of a silicon microcantilever actuated by a ZnO patch. In these works, the authors utilized the constitutive relationships of the piezoelectric material and Euler–Bernoulli beam theory to develop a distributed-parameter representation of their system. This model was subsequently reduced, using Galerkin methods, and experimentally validated through the use of a Veeco DMASP probe. Somewhat surprisingly, the acquired analytical and experimental frequency responses depicted a distinct softening nonlinearity, which softened further with increasing voltage. This was attributed to the dominance of material nonlinearities.

3.3 Nanoscale Systems. As nanofabrication techniques matured, there was a rebirth of largely observational reports of Duffing-like nonlinearities, this time in directly excited nanosystems. In 1999, for example, Evoy et al. [62] reported the existence of hardening frequency response characteristics in nanofabricated paddle resonators. This would be followed by subsequent reports of hardening responses in suspended, electrostatically actuated carbon nanotube resonators [63], magnetomotively excited SiC resonators [64], platinum and silicon nanowires [65,66], and electrostatically actuated nanobeams [67,68]. A typical example of an electrostatically excited clamped-clamped nanobeam and its response are shown in Fig. 3. A key distinction between these reports and earlier reports of Duffing-like behaviors at the microscale was that the nonlinear responses arising in nanoresonators appeared to be endemic to the devices. This was rectified by the work Postma et al. [69] in 2005, which demonstrated that many nanoscale resonators exhibit a severely diminished dynamic range, and, thus, when driven above the thermomechanical noise floor, can transition into a nonlinear response regime rather quickly.

3.4 Applied Investigations. While the majority of studies detailed above approached the investigation of nonlinear behaviors from a largely application-independent perspective, a number of works over the past decade have addressed these nonlinear behaviors within the context of very specific applications. Among these various works are efforts focusing on electromechanical signal processing, signal amplification, resonant mass sensing, and magnetic field detection, as briefly highlighted below.

One of the earliest efforts emphasizing electromechanical signal processing in a directly excited system with Duffing-like nonlinearities is that of Erbe and collaborators from 2000 [70]. In this work, the authors reported the ability to mechanically mix two direct excitation signals with a magnetomotively driven nanoresonator. Specifically, the authors demonstrated that by driving the nanomechanical resonator with two hard excitation signals, one slightly detuned in frequency from the other, they could recover a mechanical output with distinct, tunable side bands. In 2005 and 2006, Alastalo and Kaajakari investigated the impact of electrostatic and mechanical nonlinearities on capacitively coupled filters [71,72]. The first of these efforts utilized a lumped-mass model of the electrostatically actuated system to characterize the filter's third-order intermodulation and intercept points. The latter effort furthered this by incorporating out-of-band distortion mechanisms and developing a closed-form expression for the filter's signal-to-interference ratio. Building upon the success of prior nonlinear electromechanical signal processing efforts, Koskenvuori and Tittonen [73] recently reported the development of a micromechanical down-converter. This work demonstrated, for the first time, that amplitude-modulated signals in the gigahertz frequency range could be converted into megahertz-range signals through the use of a double-ended-tuning-fork resonator. As noted in Ref. [74], these positive results could potentially stimulate the development of MEMS-based radios.

Apart from the investigations of directly excited micro-/nanoresonators detailed above, a series of works by Almog and collaborators has investigated the potential of Duffing-like nonlinearities in signal amplification and noise squeezing applications [75,76]. The first of these works, which appeared in 2006, utilized signal mixing techniques, similar to those described above in relation to the work of Erbe et al. [70], to realize a signal amplifier capable of yielding high intermodulation gains. In particular, the authors demonstrated that by exciting an electrostatically actuated microresonator with two direct excitation signals—a weak carrier signal and an intense pump, driven in the proximity of the system's saddle-node bifurcation—amplifier gains approaching 15 dB could be recovered. The latter work, appearing in 2007, adopted similar methods, in conjunction with a homodyne detection scheme, to realize phase-sensitive signal amplification and noise squeezing in a nanomechanical resonator. Similar principles to those utilized in this latter work were also proposed for use in resonant mass sensing applications by Buks and Yurke in 2006 [77].

Though application-specific investigations of nonlinear, directly excited micro-/nanoresonators with Duffing-like nonlinearities have primarily emphasized signal amplification and signal processing applications, a handful of works, in addition to the work of Buks and Yurke detailed above, have highlighted alternative uses. The 2005 work of Greywall [78], for example, proposed the development of a sensitive magnetometer based on a Duffing-like resonator embedded in an oscillator circuit. This device utilized magnetic-field-induced changes in damping to alter the resonant amplitude of a current-carrying, clamped-clamped beam. This approach resulted in a sensor design theoretically capable of detecting ppm changes in Earth's magnetic field. It should be noted that a recent effort by Choi et al. proposed a similar technical approach

[79]. The recent work of Liu et al. [80] examined the impact of Duffing-like nonlinearities in an alternative context, namely, MEMS microphones. In this effort, the authors detailed the lumped-mass modeling, analysis, and experimental validation of a dual-backplate system. Particular emphasis was placed on system identification, within the paper, as a well-defined system model was deemed essential to future closed-loop, force-feedback studies. Another novel use of primary resonance in nonlinear systems is the work of Chan et al. [30], which used nonlinear resonance measurements of a microscale torsional resonator to verify the existence of the Casimir force, a purely quantum effect.

3.5 Device Tuning. Though the notion of nonlinear tuning in micro-/nanoscale resonators can be traced to the works of Adams et al. [33], among others, a series of works has recently revisited the issue within the context of directly excited resonators. The works of Agarwal et al. [81,82], for example, detailed how, with proper tuning and the effective cancelation of nonlinearity, at least to first order, improved power handling characteristics could be recovered. Specifically, the authors demonstrated how second-order and third-order electrostatic effects could be leveraged against third-order mechanical hardening nonlinearities to "straighten" the nonlinear backbone. Similar results are detailed in the recent work of Shao et al. [83]. Kozinsky et al. [84] subsequently extended the approach to the nanoscale in their investigation of fixed-fixed nanoresonator actuated through magnetomotive and electrostatic effects. In this work, the authors demonstrated the feasibility of bidirectional linear frequency tuning and nonlinear reduction. The latter method proved highly advantageous to dynamic range enhancement, with the authors reporting more than 6 dB of dynamic range improvement in their initial work.

3.6 Chaos. Despite the present section's emphasis on conventional, hysteretic behaviors, it is important to note that a number of research efforts have identified or predicted the occurrence of more-complex dynamical behaviors, including chaos, in directly excited micro-/nanoresonators with Duffing-like nonlinearities. In 2002, for example, Scheible et al. [85] noted the existence of chaos in a "clapper" nanoresonator—a free-free beam resonator suspended by an intermediate point—excited through magnetomotive and electrostatic effects. Specifically, the authors demonstrated, through experiment, that in applications like signal mixing, where multiple frequencies are present, there is some degree of likelihood for the emergence of chaos, principally arising through the Ruelle–Takens route [86]. This observation would be further addressed by Gottlieb et al. [87] in 2007. In contrast to the works of Scheible and Gottlieb, Liu et al. [88] in 2004 predicted, using simulation, the emergence of chaos in closed-loop, electrostatically actuated microcantilevers through period-doubling routes. Similar results were reported in the analytical investigations of De and Aluru [89,90]. In 2008, Park et al. [91] noted the potential for chaos control in micro-/nanoscale systems. Specifically, the authors demonstrated, through simulation, that by adopting an appropriate feedback rule, chaotic solution trajectories could be effectively converted into periodic responses. This was shown to not only increase the operating range of the electrostatically actuated microresonator of interest but also increase its effective power output.

3.7 Dynamic Transitions Near Primary Resonance. While the bulk of efforts described above emphasize steady-state behaviors, transient responses are of interest, as well. The utility of transitional behaviors in resonant micro-/nanosystems was first addressed by Aldridge and Cleland in 2005 [92]. In this work, the authors demonstrated that noise-induced transitions between coexistent bistable states in a nanomechanical Duffing resonator could be used to extrapolate the system's resonant frequency and cubic nonlinearity with a degree of accuracy unobtainable in a linear system. This, as the authors noted, facilitates highly sensitive parametric sensing. Along similar lines, Stambaugh and Chan [93,94] characterized noise-activated switching in a torsional micromechanical resonator with softening characteristics. Though the premises of these works were quite similar, differences in activation energy scaling were reported. The latter works' conclusions were recently confirmed in the work of Kozinsky et al. [95], which provided experimentally measured basins of attraction in the bistable region.

While investigations of transient behaviors in bistable micro-/nanomechanical resonators are still in their infancy, a number of practical applications have been reported. In 2004, for example, Badzey et al. [96] detailed the development of a controllable switch based on a Duffing-like response. In this work the authors applied a square-wave modulation signal to a clamped-clamped nanobeam driven near resonance to induce dynamic transitions between the resonator's bistable states. The binary nature of the resulting time response was viewed as an ideal platform for primitive memory elements. Subsequent investigations furthered this work by addressing the temperature dependence of the dynamic switching event, as well as phase-modulation-induced switching [97,98]. In 2006, Chan and Stambaugh [99], applied their previous investigations of noise-induced switching to micromechanical mixing, demonstrating that it can be used to facilitate frequency down-conversion. Almog et al. [100] extended this approach for signal amplification purposes.

4 Parametrically Excited Systems

Systems that experience time-varying parameters that multiply system states are said to be parametrically excited, and this form of forcing arises naturally in many microscale systems, as the illustrative example described above demonstrates. Of interest here are the resonances in M/NEMS that result from periodic parametric excitation. The most basic model that demonstrates this resonance is the linear Mathieu equation,

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + [\omega_0^2 + \beta\cos(\omega t)]x = 0 \qquad (8)$$

which is a simple oscillator with a time-varying stiffness term. The Mathieu equation has been widely studied, and governs the forced motion of a swing, the stability of ships, Faraday surface wave patterns on water, and the behavior of parametric amplifiers based on electronic or superconducting devices. A linear stability analysis predicts that parametric resonances occur at drive frequencies that satisfy $\omega = 2\omega_0/n$, where ω_0 is the system's natural frequency and n is an integer greater than or equal to unity [101]. When instability occurs, the steady-state response is governed by nonlinear effects. It is worth noting that once the instability occurs damping does not have a strong effect on the resulting steady-state amplitude, in contrast with the resonant response of directly forced linear systems.

In macroscopic systems only the first instability region ($n=1$) is typically observed, due to the levels of damping and the exponential narrowing of the regions with increasing n. However, in resonant microsystems, the internal damping is small (when compared with macroscale systems), and the aerodynamic damping can be made very small by operating in vacuum. This leads to a condition wherein parametric resonance effects are significantly more visible and, thus, can be utilized in application. In 1997, Turner et al. [32] experimentally measured five parametric instability regions in a microelectromechanical torsional oscillator. This oscillator, which is shown in Fig. 4, was an electrostatically driven torsional device being developed for data storage applications. An out-of-plane fringing field actuator gave this device its unique properties, wherein the electrostatic torque is nonzero only for nonzero angles of rotation. Following this initial demonstration at the microscale, this phenomenon was seen in other torsional microresonators [102,103], resonant microscanners [104–107], piezoelectrically actuated micro-/nanoresonators [108,109], laser-heated nanoresonators [110], and nanowires [111–113].

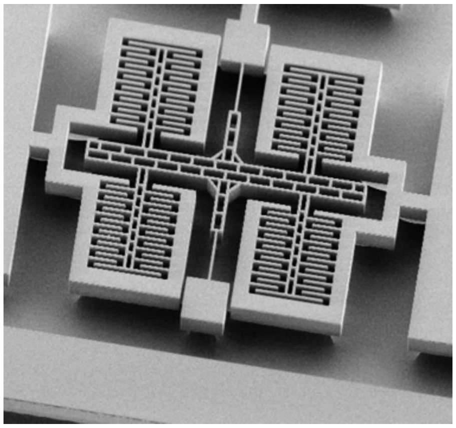

Fig. 4 A parametrically excited torsional microresonator driven through the use of noninterdigitated comb drives

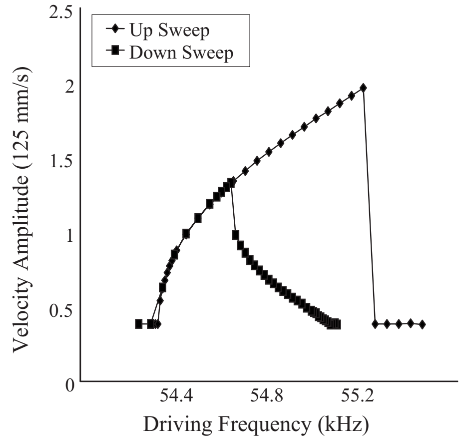

Fig. 5 A representative frequency response obtained from an electrostatically actuated, parametrically excited resonator (a planar variant of the device shown in Fig. 4). Note that the mixed response characteristics displayed here are yet to be observed in any macroscale device (adapted from Ref. [121]).

4.1 Tuning and Applications. In 2002, Zhang et al. [114–116] showed the effects of tuning on parametrically excited microresonators. By designing a device that had fringing field actuators for both tuning and excitation, they were able to independently tune the linear and nonlinear stiffness coefficients of their device, thereby changing the shape of the instability region. Through this tuning, the device performance could be quantitatively and qualitatively adjusted merely by manipulating the amplitudes of the ac and dc excitation signals. This proved to be essential in expanding the application space of parametrically excited micro-/nanoresonators. Napoli et al. [117,118] extended this result to include resonant microbeams and verified the existence of nonlinear parametric instabilities in capacitively actuated cantilevers. As detailed in Sec 5, the combined excitation of electrostatically actuated microbeams has been studied more recently by Zhang and Meng [119,120].

Rhoads et al. [35,121–124] improved on prior modeling efforts, and more comprehensively analyzed parametrically excited microresonators, in order to determine the achievable parameter space, effectively explaining in greater detail previously obtained results [125]. Building upon this result, Rhoads et al. [116,126] created a filter utilizing two MEMS devices, one each tuned to be hardening and softening, such that when their outputs are combined they create a bandpass filter. Although the frequencies were relatively low, as the devices were large, the stage was set for the later creation of logic elements using nonlinear tunable MEMS oscillators [35]. In addition, Rhoads et al. [121] analyzed a model in which the parametric excitation acted on both linear and nonlinear stiffness terms in order to explain certain unique response characteristics, including the unusual mixed hardening/softening behavior shown in Fig. 5.

In parametric resonance, the boundary between stability and instability is extremely sharp, and leads to a very dramatic jump in amplitude on the subcritical side of the parametric resonance zone (the frequency step in the original Turner result is 0.001 Hz [32]). In addition, in an unstable region, the amplitude achieved is determined not by the damping, as in a directly forced linear oscillator, but by the nonlinearity in the system. Therefore, parametrically forced systems are capable of significant amplitudes while operating in the unstable region. By tracking this sharp stability boundary in parameter space, a very precise sensing mechanism, based on so-called bifurcation amplification, can be exploited. This has been used with varying degrees of success in mass sensing, noncontact atomic force microscopy, and microgyroscopes, as described below.

The parametrically forced mass sensor was the first microsensor to demonstrate the benefits of such a mechanism. Early attempts at parametric resonance-based mass sensing were completed by Zhang et al. [115,127]. The sensitivity of this phenomenon compares favorably to linear micro-oscillators, showing itself to be approximately one to two orders of magnitude more sensitive in air operation. In this work, Zhang and collaborators detected humidity changes by measuring the mass of condensed water on a planar device. This device utilized a fringing-field, in-plane actuator of the type described in Refs. [33,128]. This was followed up with a nanoscale investigation by Yu et al. [111], who considered mass-loaded nanowires. Furthering the work of Zhang et al. [129], Requa and Turner [130,131] built a self-sensing mass sensor based on parametric resonance (see Fig. 6). This sensor was much smaller and more sensitive, and utilized Lorentz force sensing and actuation. Noise imposes limits on the resolution of these sensors, as described by Cleland [132], and as experimentally investigated by Requa and Turner [133]. Transitions to chaos are also possible in parametrically excited nonlinear systems, and this was investigated in a MEMS resonator by DeMartini et al. [34], both experimentally and by using Melnikov analysis on a system model. This

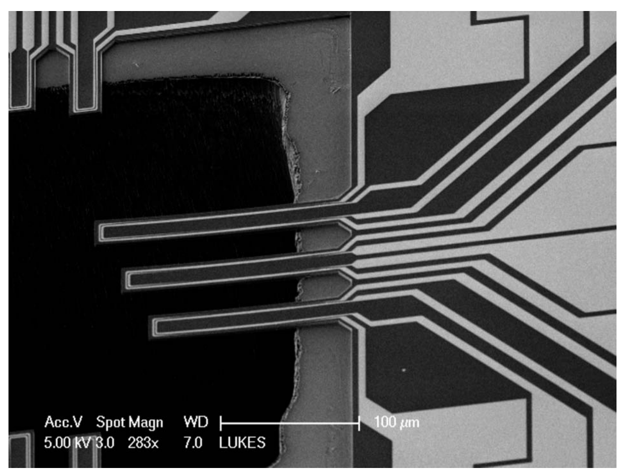

Fig. 6 A set of microbeam resonators actuated by Lorentz forces. The light areas on the beam are deposited wires. A periodic current flowing through the wire at the end of the beam creates an alternating axial load when the beam is placed in a transverse magnetic field, thereby producing parametric excitation (from Ref. [135], picture courtesy of K. (Lukes) Moran).

effort built upon the earlier work of Wang et al. [134].

Parametric resonance has also been applied to the design of microscale gyroscopes. The resonant Coriolis force sensor has been long-studied for rate measurement and inertial tracking (see, for example, Ref. [136]). Contrary to accelerometers, though, the Coriolis rate sensor requires at least two degrees of freedom. In its most straightforward application, the Coriolis force generated by external rotation couples two perpendicular modes, one for drive and the other for sensing. Ideally, in order to achieve full amplification the two modes are tuned to the same resonance frequency. However, in practice this is very difficult to achieve, and significant amplification gains are lost. Oropeza-Ramos et al. [137,138] utilized the broadband response of parametric resonance to achieve robust amplification in a Coriolis force sensor. The sensor operation is similar to the linear types of rate sensor in that it is a single mass, electrostatically driven in one axis, and electrostatically sensed along a perpendicular axis. Here, however, the device is driven into its first parametric instability. This band of excitation is typically broader than a high-Q resonance mode and is limited in amplitude only by nonlinearity. This allows for full sense mode amplification, even in the presence of fabrication-induced irregularities. The device can be driven at the frequency producing the maximum sense direction amplification, *without the need for high precision tuning*. Although this early device was not optimized, the proof of concept is compelling. Subsequent analytical studies by Miller et al. [139,140] using perturbation techniques have shown that these nonlinear devices can be tuned to achieve nearly-linear sense response in the rotation rate.

4.2 Dynamic Transitions Near Parametric Resonance. The discussion above focuses on steady-state behavior resulting from harmonic, deterministic excitation. However, investigations of transient types of excitation, including frequency sweeps and noise, and combinations of these, are also of interest. The work of Requa and Turner [133] considers the effects of frequency sweep rates on the response of a parametrically excited microbeam. Here there is a trade-off, since faster sweeps lead to better sensor performance (in terms of response time), but pay a price in terms of precision, since the ability to precisely locate a subharmonic instability depends on this rate. Noise can also play an important role in these system. Chan et al. [141,142] carried out an analytical and experimental study of noise-induced switching between different steady-states in parametrically excited microsystems. The effects of noise and sweep rates and their interplay will be an increasingly important consideration as devices are reduced in scale.

5 Systems With Combined Excitations

Though the majority of micro- and nanoresonators independently utilize direct or parametric excitation, a number of resonant micro-/nanosystems exploit the excitations simultaneously, using the combined result to render beneficial response characteristics. While there exist a fairly large number of combined excitation studies in the M/NEMS literature, the vast majority of works published to date can be classified into two distinct groups: (i) efforts focusing on parametric amplification and (ii) efforts focusing on nonlinear phenomena induced through combined excitations, resulting from electrostatic or electromagnetic forces. The current section details literature from each of these groups, placing particular emphasis on works that exploit dynamical phenomena that arise *solely* in the presence of a combined excitation. Works that exhibit combined excitation, yet appear to utilize direct or parametric phenomena independently, have been included in alternate sections.

5.1 Parametric Amplification. The most prevalent investigations of combined excitation at the micro-/nanoscale are those emphasizing low-noise parametric amplification—the process of amplifying a harmonic, external drive signal through the use of a parametric pump. While macroscale investigations of this linear phenomenon first appeared in the literature nearly 50 years ago [143–145], microscale implementations have garnered attention only since the early 1990s. In their seminal effort of 1991 [146], Rugar and Grütter demonstrated that the resonant response of a microcantilever, base excited by a piezoelectric bimorph actuator, could be amplified by pumping the system with a parallel-plate electrostatic drive. In this work, the author's specifically utilized the degenerate form of parametric amplification—a phase-dependent variant, wherein the parametric pump is locked at twice the frequency of the resonant, direct excitation signal—to drive the system's resonant amplitude to approximately 20 times its unpumped state. This was achieved by exciting the system with a constant-amplitude harmonic signal and introducing a parametric excitation that was very near, but not above, the Arnold tongue (i.e., the stability boundary) associated with the onset of parametric resonance. By exploiting the phase-dependent nature of the degenerate amplifier's gain, the authors demonstrated that resonant amplitude reduction, subsequently termed parametric attenuation, could be readily obtained, as well.

In subsequent years, a number of works would build upon Rugar and Grütter's efforts by examining the feasibility of both degenerate and nondegenerate (where the parametric pump is locked at a frequency distinct from twice that of the external signal) parametric amplification in torsional microresonators [102,125], optically excited micromechanical oscillators [110], microring gyroscopes [147,148], MEMS diaphragms [149], micromechanical mixers [150], and resonant micro-/nanobeams [151–154]. Of particular note among these latter efforts is the work of Olkhovets et al. [103], which demonstrated experimental gains in excess of 40 dB in a nondegenerate parametric amplifier based on two electrostatically coupled torsional microresonators, and the work of Roukes and collaborators, which demonstrated experimental gains in excess of 65 dB in a degenerate amplifier based on a stiffness-modulated, fixed-fixed nanobeam [152].

Though the works noted above considered parametric amplification from a largely linear perspective, recent efforts have considered the effects of nonlinearity on parametric amplifiers [9,155]. These largely analytical efforts were motivated by the discrepancies between analytically predicted and experimentally recovered gains reported in many of the works cited above. In an attempt to address these gain deficits, each of the efforts appended geometric nonlinearities to conventional, linear parametric amplifier models and quantified the resulting impact on amplifier gain. Not surprisingly, the works concluded that nonlinearity not only severely limits the gain of a parametric amplifier but also renders frequency response behaviors that are appreciable more complicated than those predicted by linear theory, due largely to the possible co-existence of resonances.

5.2 Other Combined Excitation Investigations. While the majority of works on combined excitations in MEMS and NEMS emphasize linear, parametric amplification, a handful of works have considered various nonlinear behaviors that arise in the presence of combined excitations resulting from electromagnetic or variable-gap, electrostatic forces that subject the system to both direct and parametric excitations. The work of Zhang and Meng [119,120], for example, considers the nonlinear response of an electrostatically actuated microcantilever, driven by a combined excitation, operating in the presence of quadratic and cubic electrostatic nonlinearities and squeeze-film damping. This effort adopts harmonic balance methods, in addition to numerical techniques, in an attempt to identify regions of periodic, quasiperiodic, and chaotic response. Other notable efforts include those of Abdel-Rahman et al. (see, Ref. [51]), which have been detailed above.

6 Arrays of Coupled Micro-/Nanoresonators

Though SDOF micro-/nanoresonator implementations have historically garnered the most research attention, arrays of coupled resonators have also elicited interest since the early 1990s. While the bulk of investigations in this area have focused on linear device implementations, a number of recent works have demonstrated the potential of coupled systems utilizing nonlinear, resonant subsystems and/or nonlinear coupling. The focus here is on investigations of coupled, *nonlinear* systems, utilizing dynamic phenomena ranging from synchronization to vibration localization, which have built upon the previous investigations of arrays that exhibit linear response characteristics. A more complete review that includes discussion of linear response in arrays can be found in the companion conference paper [6].

While numerous research efforts fall within the broad scope of nonlinear response of M/NEMS arrays, most, if not all, of the current literature on nonlinear micro-/nanoresonator arrays can be categorized into three distinct groups: (i) works focusing on nonlinear extensions of the linear array technologies detailed in Ref. [6]; (ii) works focusing on nonlinear arrays, which exhibit intrinsic localized modes (ILMs), or so-called discrete breathers; and (iii) works focusing on arrays of nonlinear resonators, which synchronize during the course of operation. Each of these categories is considered below.

6.1 Nonlinear Extensions of Linear Investigations of Micro-/Nanoresonator Arrays. To date, the traditional nonlinear dynamics community's contribution to nonlinear micro-/nanoresonator array research has primarily been through the modeling and analysis of nonlinear variants of the linear array technologies detailed in [6]. For example, the works of Hammad et al. [156–159] detail the development of refined nonlinear models for electrostatically actuated, elastically coupled filters similar in design to those originally proposed by Bannon et al. in 1996 [160]. These largely theoretical works utilize multiphysics, continuous-system modeling, model reduction, and perturbation analysis in an attempt to more accurately characterize pertinent filter metrics, pull-in behavior, and the effects that manufacturing imperfections have on a representative system's dynamic response.

Though elastically coupled devices have been the traditional focus of nonlinear micro-/nanoresonantor array research, a number of recent efforts have considered the behavior of comparatively large arrays of microbeams coupled through electrostatic interactions. While these works could be seen as natural extensions of the investigations of Pourkamali, Zhu, Porfiri, and their collaborators [162–165], many, in actuality, predate these linear efforts. Representative among the nonlinear investigations of electrostatically coupled micro-/nanoresonator arrays is the work of Napoli et al. [166], which examines the response of parametrically excited microcantilevers coupled through both elastic and electrostatic interactions, and the collective efforts of Buks, Roukes, Lifshitz, and Cross [9,161,167,168]. The earliest paper in the series of works is an experimental endeavor by Buks and Roukes [161] examining the collective behavior of a 67-element array of electrostatically-actuated, doubly-clamped gold microbeams, driven near the principal parametric resonance; this device is shown in Fig. 7. In this work, the authors utilize the microbeam array as a diffraction grating and integrate it with an optical fiber light source and a photodiode detector to recover real-time representations of the system's modal behavior under varying drive conditions. While some of the relatively complex behaviors reported in this work are yet to be explained, a series of subsequent works by Lifshitz and Cross has addressed the observed collective response using both analytical and numerical techniques [9,167,168]. In Ref. [168], for example, Lifshitz and Cross proposed a nonlinear model for the microbeam array incorporating Duffing-like elastic nonlinearities, in addition to linear electrostatic and nonlinear dissipative coupling. The authors then utilize this model, in conjunction with secular perturbation methods, to characterize the behavior of representative, two- and three-degree-of-freedom systems. Not surprisingly, rich frequency response behaviors are shown to arise in even these comparatively small arrays. The high dimension of Buks and Roukes' 67-element array

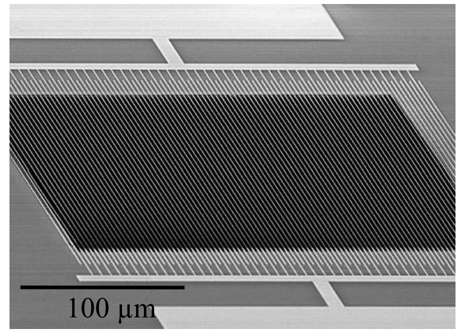

Fig. 7 A 67-element array of electrostatically coupled, fixed-fixed microbeams (adapted from Ref. [161], picture courtesy of E. Buks)

is prohibitive to analytical investigation, but a numerical investigation in Ref. [168] nicely captures the mean response of this system. These authors subsequently continued their work in Bromberg et al. [167] in 2006, by approaching the 67-element array problem from its continuous limit and transforming the spatially discrete system considered in Ref. [168] into a spatially continuous analog. This approach facilitated bifurcation analysis in the proximity of the parametric resonance, and, in turn, opened the problem to predictive system design.

Relevant to the efforts noted above are the recent works of Zhu et al. and Gutschmidt and Gottlieb [169–172]. The first of these efforts, by Zhu et al., extends the work of Lifshitz and Cross to incorporate the nonlinear parametric interactions that arise from higher-order approximations of the electrostatic force. More specifically, the authors employ harmonic balance methods, in conjunction with numerical approaches, to characterize the behavior of a three-resonator system wherein the outer resonators are fixed (in essence, the system represents a variant of the one considered in Ref. [123]). These results are subsequently used to show that the inclusion of nonlinear electrostatic effects renders higher-order subharmonic parametric resonances, which cannot be captured with models incorporating only linear electrostatic coupling. Along similar lines, the works of Gutschmidt and Gottlieb support the efforts of Lifshitz and Cross by adopting a spatiotemporal modeling approach which captures a broader range of excitation inputs [172]. These refined models are subsequently used in conjunction with multiple time scale perturbation methods and numerical continuation techniques to classify the various internal and combinational resonances that arise in two- and three-element arrays of electrostatically coupled microbeams [170,171]. These results capture a wider range of dynamic behavior, although corroboration with experimental results remains incomplete.

6.2 Intrinsic Localized Modes in Micro-/Nanoresonator Arrays. A second category of literature emphasizing nonlinear behaviors which arise in coupled micro/nanoresonator arrays is that focused on ILMs or so-called discrete breathers (DBs)—analogs of classical localized modes, which arise in the presence of strong nonlinearity, rather than structural impurity (mistuning) [173–182]. While the investigation of ILMs dates to the earlier analyses of discrete lattice vibration, ILMs were first reported to occur within microresonator arrays by Sato et al. in 2003 [180]. In this defining effort, the authors concluded that energy could be spatially confined in periodic arrays of identical microresonators, provided a strong mechanical nonlinearity and an appropriate drive mechanism were present. To verify this, the authors fabricated a spatially periodic, 248-element array of alternating length, silicon nitride cantilevers, which were strongly coupled through a common elastic overhang. This coupled system was then driven at the base using a PZT element, and the resulting system response was recorded using a one-dimensional camera. By driving the system at a frequency slightly below the maximum frequency of the array's "optic" band and subsequently chirping the drive to a slightly higher frequency, the authors were able to observe a number of interesting dynamical phenomena [177,178,180]. First, the authors noted, throughout the duration of the chirp, the formation of multiple localized modes well dispersed across the array at seemingly random locations. These ILMs were observed to "hop" around the array, interacting with one another throughout the interval of transient excitation. This validated, in part, previous investigations of ILMs, which emphasized the independence of *intrinsic* energy localization and structural impurity. As the excitation reached a constant frequency state, those ILMs vibrating at the frequency of the excitation signal were observed to persist, while those oscillating at alternative frequencies decayed. The persistent modes remained dominant until the excitation was terminated, whereafter the localized oscillations became "unpinnned," "hopped," to various sites within the array interacting constructively and destructively with their counterparts, and subsequently decayed.

Following the success of their initial work, Sato and co-workers proceeded to investigate various extensions of their ILM research. Notable milestones from these latter works include (i) the realization of ILMs in microbeam arrays with softening nonlinearities (acquired through electrostatic tuning) [179]; (ii) a demonstration of ILMs within the acoustic spectrum—a feat previously deemed to be improbable due to the influence of higher-frequency spectral components on localized responses [181]; and (iii) the manipulation of ILMs through the use of optically induced impurity [179]. Of these works, the last is of particular note, as it demonstrated the ability to manipulate the location of energy confinement within a spatially periodic array through the use of local laser heating. Specifically, the work demonstrated that reducing the linear natural frequency of a single resonator near an existing ILM results in ILM repulsion, if the system is operating in a hardening response regime, and ILM attraction, if the system is operating in a softening response regime. Such spatial control opens doors to a number of micro/nanoscale targeted energy transfer applications.

Apart from the works of Sato et al., detailed above, there have been a number of ILM-related research efforts that have approached the topic from alternative perspectives. Maniadis and Flach [176], for example, utilized nonlinear invariant manifold theories to predict the optimal operating conditions for ILM emergence, and to predict that ILMs can be induced through paths other than frequency modulation. A recent effort by Chen et al. [173] extended this further by demonstrating that ILMs can be induced through chaos. Within the traditional nonlinear dynamics community, recent works by Dick et al. have addressed the existence of ILMs using the theory of nonlinear normal modes [174,175]. This analytical approach, in comparison to those employed in prior works, facilitates the derivation of analytical expressions for ILM amplitude profiles, which should prove invaluable in future design efforts.

6.3 Synchronization in Micro-/Nanoresonator Arrays. The third, well-defined class of literature emphasizing nonlinear behaviors which arise in coupled micro/nanoresonator arrays is that concerned with the synchronization of coupled resonators. While macroscale investigations of this phenomenon date to Huygen's mid-1600s observations of weakly coupled pendulum clocks, investigations of synchronicity in microsystems date only to the early 2000s [183,184]. Earliest among the various works on M/NEMS synchronization is Hoppensteadt and Izhikevich's [183] speculative effort, which proposed the use of globally coupled, limit cycle oscillators as the functional backbone of MEMS-based neurocomputers. This work highlighted the distinct parallels between microelectromechanical resonators driven via positive feedback loops, phase-locked loops (PLLs), and lasers, and utilized theory the authors had previously developed for the latter two systems to convey the potential of MEMS-based autocorrelative associative memories. Cross and collaborators, building upon their earlier efforts related to coupled micro/nanosystems, would further this work, by considering, in appreciable depth, the synchronization of globally coupled, limit-cycle oscillators with distributed frequencies [185,186]. The latter efforts' emphasis on frequency mistunings is of particular note, as it addressed a common concern associated with micro-/nanoresonator technologies—the potentially debilitating effects of process-induced variations.

While the works referenced above adopted a largely device-independent approach to micro-/nanoscale synchronization research, two recent efforts have approached the problem with specific devices in mind. The first of these [187], by Sahai and Zehnder, considers the synchronization of elastically and electrothermally coupled, self-excited dome oscillators similar to those previously detailed in Ref. [188]. In this work, the authors utilized numerical methods to investigate the various operating conditions that result in the synchronization and entrainment of a representative two-resonator system. The second effort, by Shim et al. [189], potentially represents the first experimental demonstration of synchronization in a micro-/nanomechanical array. In this work, the authors utilize a two-element array of elastically coupled, magnetomotively excited resonators to show some degree of frequency locking. While the acquired results are promising, their interpretation has been debated within the nonlinear dynamics community.

7 Conclusion

With this review the authors have attempted to provide a brief account of the history and literature related to the nonlinear dynamics of micro-/nanoresonators, in terms of both fundamental and application-oriented research. The exploitation of nonlinearity in these systems offers significant potential for improving device performance, but thoughtful modeling and analysis must be carried out in order to achieve such designs. This is especially true in applications involving dynamic behavior, where nonlinearity can lead to interesting results—or wreak havoc—if one is not careful. It should be noted that it is still a challenge to convince device engineers to consider designs based on nonlinear behavior, since the goal for many decades has been to steer clear of nonlinearity. However, this trend is changing, and there are several devices based on nonlinear dynamic behavior under development that have promise of reaching fruition.

Ongoing progress in this field is taking place on two broad, not unrelated, fronts. Engineering-oriented research is being carried out with specific target applications in mind, such as mass sensing, inertial sensing, and signal filtering. In these and related areas, one can use existing fabrication techniques and a basic knowledge of nonlinear dynamics to design devices that exhibit desired nonlinear behavior. Though nonlinear design is crucial here, one still must face all the considerations inherent to device design, such as fabrication tolerances, robustness, and reliability. The other front is more physics oriented and is geared toward fundamental experimental research. One example of such work is that emphasizing mesoscale vibration systems, that is, systems that exhibit quantum effects even though they can be modeled as continua. A typical illustration of such an effect would be to measure quantum energy levels in a nanobeam. This, of course, requires that one isolates the beam as much as possible from the environment thermally, mechanically, and electronically (although, of course, one must couple the beam to something in order to make measurements). The driving force behind these developments is the desire for devices that have unprecedented sensitivities and are capable of detecting, for example, the presence of gravity waves. This is a very active area of research in physics, and the reader is referred to the review of Blencowe [190] for further reading on this topic.

As devices become smaller, and yet are required to operate in ambient environments, the effects of noise will become an increasingly important consideration. There are several sources of noise, and these will limit the ultimate resolution of nanoscale sensors [191]. There is also growing interest in the so-called "bifurcation amplifiers" that make use of nonlinear response branch jumping, as described at the end of the section on directly excited systems, for targeted types of detection, and it is interesting to note that these systems actually *require* noise to function effectively. Likewise, the nonstationary effects of controlled parameter variations, for example, frequency sweeps used in sensors, is of interest, since sensor response specifications will depend on the determination of reliable sweep rates. This general topic has received considerable attention in the nonlinear vibrations and physics communities under various subject names, including "passage through resonance" and "nonstationary oscillations," and it has now found a new set of applications. The combination of these effects, namely, nonlinearity, noise, and parameter sweeps, is a class of challenging fundamental problems that will play an important role in the development of M/NEMS resonators. To make progress in this area, and for similar challenges, collaborations between device engineers, physicists, and nonlinear dynamicists will become increasingly important.

Acknowledgment

The authors are grateful to the Organizing Committee of the 2008 ASME DSC Conference for the opportunity to prepare the review paper [6] upon which this article is based. The authors have benefited from stimulating conversations on the subject of nonlinear dynamics in M/NEMS with many colleagues, including Anil Bajaj, Bala Balachandran, Eyal Buks, Bill Conley, Mike Cross, Harry Dankowicz, Barry DeMartini, Mark Dykman, Oded Gottlieb, John Judge, Slava Krylov, Ron Lifshitz, Nick Miller, Jeff Moehlis, Arvind Raman, and Ashwin Vyas. In addition, the authors would like to thank Eyal Buks, Barry DeMartini, and Kari (Lukes) Moran for providing some of the SEM images shown in this work. This effort was supported by the National Science Foundation (NSF) under Grant Nos. NSF-0428916, NSF-0758419, NSF-0826276, and NSF-0846385.

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