Proceedings of the ASME 2008 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2008
August 3-6, 2008, Brooklyn, New York, USA

DETC2008-49594

THE EFFECTS OF NONLINEARITY ON PARAMETRIC AMPLIFIERS

Jeffrey F. Rhoads*
School of Mechanical Engineering
and Birck Nanotechnology Center
Purdue University
West Lafayette, Indiana 47907
Email: jfrhoads@purdue.edu

Steven W. Shaw
Department of Mechanical Engineering
Michigan State University
East Lansing, MI 48824
Email: shawsw@msu.edu

ABSTRACT

Mechanical and electromechanical parametric amplifiers have garnered significant interest, as of late, due to the increased need for low-noise signal amplification in resonant micro/nanosystems. While these devices, which are traditionally designed to operate in a linear range, potentially represent an elegant, on-chip amplification solution, it is not readily apparent that this technical approach will suffice in all micro/nanoresonator implementations, due to the scale-dependent nature of a mechanical or electromechanical amplifier's dynamic range. The present work investigates whether the aforementioned linear dynamic range constraint is truly a practical limitation, by considering the behavior of a representative degenerate parametric amplifier driven within a nonlinear frequency response regime. The work adopts a comparatively simple lumped-mass model for analysis and proceeds with the characterization of pertinent performance metrics, including gain/pump and gain/phase behaviors. Ultimately, the work concludes that parametric amplification can be realized in a nonlinear context, but such implementations generally lead to inferior amplifier performance.

INTRODUCTION

Recently, mechanical and electromechanical parametric amplifiers have garnered significant interest, due to the increased need for low-noise signal amplification in resonant micro/nanotransducers. While these devices utilize the same fundamental mode of operation as their classical, purely-electrical counterparts, mechanical and electromechanical parametric amplifiers typically feature a scale-dependent dynamic range. Because of this, many micro/nanotransducers feature, at best, a narrow window of forcing amplitude in which a parametrically-amplified, linear (Lorentzian) frequency response can exist. Accordingly, if robust, low-noise signal amplification is to be realized in practical application, it may need to be done in a nonlinear context.

The present work investigates the feasibility of implementing parametric amplification in a nonlinear, micro/nanoscale, mechanical or electromechanical resonator by characterizing the effects of a cubic nonlinearity on a classical degenerate parametric amplifier. The work adopts a comparatively-simple, lumped-mass model for analysis and utilizes standard perturbation methods to characterize and evaluate pertinent performance metrics, including gain/pump and gain/phase behaviors. The paper ultimately concludes with a brief overview and a discussion of ongoing research, including planned experimental investigations at both the micro- and nanoscales.

1 SYSTEM MODELING AND ANALYSIS

Though micro/nanomechanical parametric amplifiers vary greatly in form, most are typically modeled as simple linear resonators driven by combined, parametric and direct, excitations (see, for example, Refs. [1–4]). While such a model suffices for resonators with an appreciable dynamic range, in practice

*Address all correspondence to this author.

Copyright © 2008 by ASME

many micro/nanoresonators (including nanotube and nanowire devices with large length-to-cross-section aspect ratios) operate in the presence of an elevated noise floor and lowered 'nonlinear ceiling' [5], and thus, when parametrically amplified in ambient conditions, exhibit a nonlinear frequency response structure. While the effects of nonlinearity vary significantly on a system by system basis, in a very generic sense, they can be characterized by incorporating a simple cubic nonlinearity in a representative degenerate amplifier's equation of motion and examining the resulting impact on system performance. With this in mind, this preliminary effort utilizes a non-dimensional governing equation of the form

$$z'' + 2\varepsilon\zeta z' + z + \varepsilon\lambda\cos(2\Omega\tau)z + \varepsilon\alpha z^3 = \varepsilon\eta\cos(\Omega\tau + \phi) \quad (1)$$

for analysis, where z represents the mechanical or electromechanical amplifier's displacement, ζ captures the effects of linear dissipation, λ represents the effective parametric pumping amplitude, Ω specifies the system's forcing frequency, τ represents a non-dimensional time variable, α dictates the system's effective cubic nonlinearity, η specifies the direct excitation amplitude, and ϕ represents a relative phase term utilized for amplifier tuning, which is necessary here due to the present study's emphasis on phase-dependent, degenerate amplification. Note that each of the dissipation, nonlinear, and excitation terms included here have been assumed to be of $O(\varepsilon)$ for ease of analysis. Future work will expand upon this baseline by considering more accurate nonlinear models, which incorporate parametric nonlinearities, dissipative nonlinearities, and $O(1)$ excitations, amongst others effects. Also note that though equations of motion similar to that presented in Eqn. (1) have been explored in other research contexts, including sheet metal rolling and rotating discs, this work is believed to be the first to consider the system's dynamics within the context of parametric amplification [6–11].

Given that Eqn. (1) fails to offer a *tractable* closed-form solution, it proves convenient to characterize the representative amplifier's behavior through the use of perturbation methods. For ease of analysis, the method of averaging is utilized here. To facilitate this approach, a constrained, Cartesian coordinate transformation of the form

$$\begin{aligned} z(\tau) &= X(\tau)\cos(\Omega\tau) + Y(\tau)\sin(\Omega\tau), \\ z'(\tau) &= -X(\tau)\Omega\sin(\Omega\tau) + Y(\tau)\Omega\cos(\Omega\tau), \end{aligned} \quad (2)$$

is introduced into Eqn. (1). Likewise, since near-resonant behavior is of principal interest, a detuning parameter, $\sigma = (\Omega - 1)/\varepsilon$, is also incorporated. Separating the equation that results from substitution, as well as the implied constraint equation, in terms of X and Y and averaging over the period $2\pi/\Omega$ results in the system's averaged equations, which are given to $O(\varepsilon)$ by

$$\begin{aligned} X' &= -\frac{1}{8}\varepsilon\left(2\lambda Y + 8\sigma Y + 8\zeta X - 3\alpha X^2 Y - 3\alpha Y^3 - 4\eta\sin\phi\right), \\ Y' &= -\frac{1}{8}\varepsilon\left(2\lambda X - 8\sigma X + 8\zeta Y + 3\alpha X Y^2 + 3\alpha X^3 - 4\eta\cos\phi\right). \end{aligned} \quad (3)$$

With these equations in hand, the system's steady-state behavior can be recovered by setting $(X', Y') = (0, 0)$ and solving for the steady-state values of X and Y. The nonlinear amplifier's performance can then be examined by converting the result into polar coordinates and evaluating the resulting system gain for various normalized pump amplitude (λ), relative phase (ϕ), direct excitation amplitude (η), and nonlinear stiffness (α) values. Note that for present purposes, the aforementioned amplifier gain has been defined according to

$$G = \frac{a_1}{a_1|_{\lambda=0}}, \quad (4)$$

where a_1 represents the steady-state amplitude of the amplifier's upper response branch in polar coordinates. A closed-form expression for G is omitted here due to its nontrivial dependence on the value of α, amongst other parameters.

2 AMPLIFIER METRICS AND THE EFFECTS OF NONLINEARITY

Due to the nonlinear nature of the parametric amplifiers under consideration here, it is prudent to briefly consider the frequency response structures associated with these systems before characterizing pertinent amplifier metrics. To assist with this, three representative frequency response plots have been included in Fig. 1. Note that though only responses with hardening nonlinearities ($\alpha > 0$) are presented here, softening-like ($\alpha < 0$) behaviors can be easily characterized through symmetry arguments.

The frequency response structure depicted in the upper pane of Fig. 1 is representative of a nonlinear parametric amplifier driven near resonance, but below its principal parametric instability threshold (i.e. $\lambda < 4\zeta$ at $\sigma = 0$). As evident, this response exhibits a Duffing-like structure, wherein the amplifier exhibits three distinct response branches. This structure is attributable to the fact that there is a single 'active' resonance associated with this operating condition – that associated with the system's direct excitation. The middle and lower panes of Fig. 1 depict representative frequency responses for an amplifier driven near resonance, but slightly above (middle) and well above (bottom) the parametric instability, respectively. As evident, in each of these scenarios the system features five distinct response branches.

Copyright © 2008 by ASME

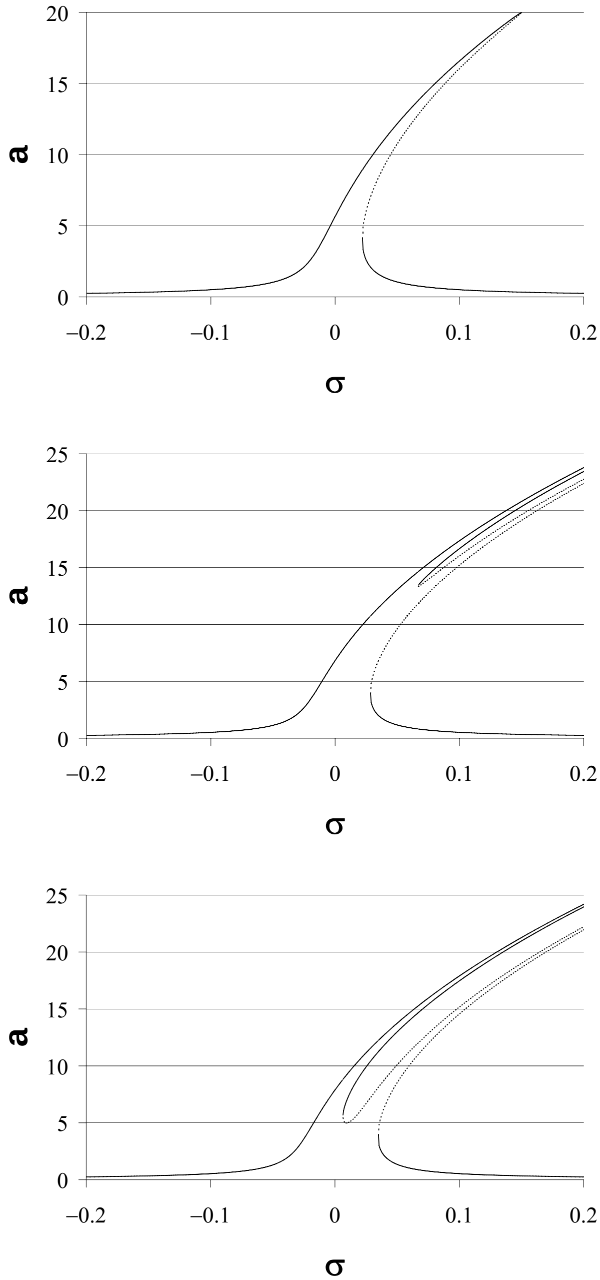

Figure 1. FREQUENCY RESPONSE, a VERSUS σ, FOR A REPRESENTATIVE AMPLIFIER ($\phi = -\pi/4$, $\eta = 0.1$, $\alpha = 0.001$, AND $\zeta = 0.01$) DRIVEN (TOP) SLIGHTLY BELOW ITS PARAMETRIC INSTABILITY THRESHOLD ($\lambda = 0.03$), (MIDDLE) SLIGHTLY ABOVE ITS PARAMETRIC INSTABILITY THRESHOLD ($\lambda = 0.055$), AND (BOTTOM) WELL ABOVE ITS PARAMETRIC INSTABILITY THRESHOLD ($\lambda = 0.08$). NOTE THAT SOLID LINES ARE USED TO INDICATE STABLE STEADY-STATE SOLUTIONS AND DASHED LINES UNSTABLE STEADY-STATE SOLUTIONS.

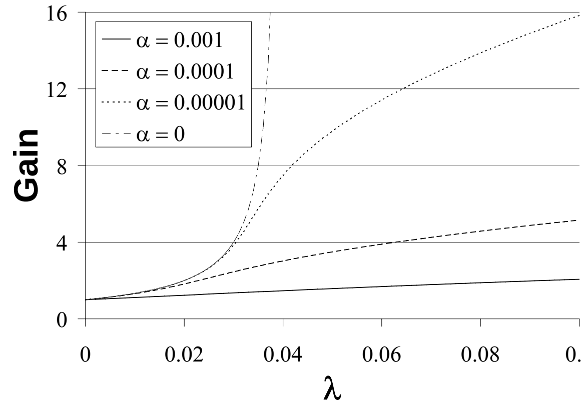

Figure 2. AMPLIFIER GAIN $G(\sigma = 0)$ VERSUS PARAMETRIC PUMP AMPLITUDE λ FOR A REPRESENTATIVE AMPLIFIER WITH $\phi = -\pi/4$, $\eta = 0.1$, AND $\zeta = 0.01$.

The two additional response branches present here, comparable in magnitude yet distinct in phase from the other stable/unstable, upper response branch pair, arise from the coexistence of two resonances within the amplifier: one induced by the system's direct excitation and another caused by parametric effects. Though these additional branches could prove problematic in some applications, it is important to note that the stability and amplitude of the amplifier's upper branch remain qualitatively unchanged, regardless of whether the system is driven above or below the parametric instability threshold. Because near-resonant operation on the aforementioned upper branch is believed to be the preferred operating state for nonlinear parametric amplifiers, the present work, in contrast to those focusing solely on linear amplification, characterizes the representative amplifier's performance metrics with only passing concern for the location of the system's operating point with respect to the parametric instability threshold.

Figure 2 details the gain/relative pump amplitude behavior of a nonlinear amplifier, driven at resonance, for various values of nonlinear stiffness (α). As evident, even comparatively small cubic nonlinearities significantly limit the amplifier's performance. Because of this, gains comparable in magnitude to those reported in prior literature (see, for example, Refs. [1–4]), are not expected to be obtainable in a nonlinear context. On the positive side, because the classical limitation on pump amplitude ($\lambda = 4\zeta$), predicted by linear theory, can be largely disregarded in a nonlinear amplifier, limited, yet meaningful, gains can still be realized with strong parametric pumping. Accordingly, parametric amplification can be plausibly realized in a micro/nanoresonator with a limited, or even nonexistent, linear dynamic range.

Because the system of interest is designed to operate in a de-

Copyright © 2008 by ASME

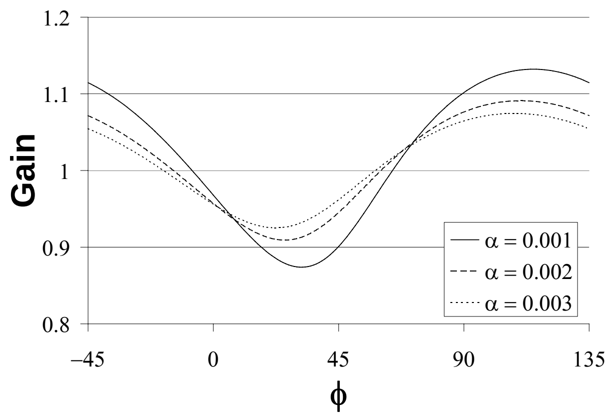

Figure 3. AMPLIFIER GAIN $G(\sigma = 0)$ VERSUS RELATIVE EXCITATION PHASE ϕ FOR A REPRESENTATIVE AMPLIFIER WITH $\eta = 0.1$, $\lambda = 0.01$, AND $\zeta = 0.01$.

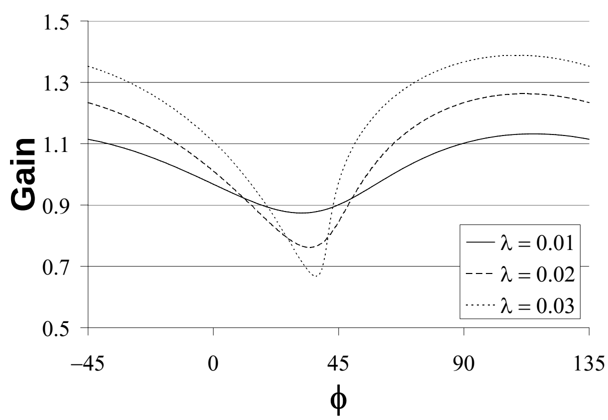

Figure 4. AMPLIFIER GAIN $G(\sigma = 0)$ VERSUS RELATIVE EXCITATION PHASE ϕ FOR A REPRESENTATIVE AMPLIFIER WITH $\eta = 0.1$, $\alpha = 0.001$, AND $\zeta = 0.01$.

generate mode, phase-dependence is predicted for all amplifier implementations, including those operating within a linear frequency response regime. As evident from Figs. 3 and 4, however, the typical phase-periodic gain variance seen in linear amplifiers is significantly distorted in the presence of nonlinearity. Specifically, increasing the magnitude of the system's cubic stiffness induces an asymmetry in the gain/phase relationship. Though this asymmetry ultimately has minimal effect on amplifier performance, it does alter the phase value at which maximum amplification takes place and thus modifies the resonator's optimal operating condition.

Though classical linear, degenerate amplifiers are usually driven at resonance ($\sigma = 0$), it should be noted that small variations in gain can be acquired in a nonlinear parametric amplifier through slight detuning. To demonstrate this, Fig. 5 depicts the gain/relative pump amplitude behavior of the representative amplifier driven at various values of σ. As evident, in this particular scenario, a minor improvement in system gain can be realized at small pump amplitudes with slight overtuning. Likewise, at comparatively larger pump amplitudes, a minor improvement in gain can be realized with slight undertuning. Though such improvements may be helpful in implementation, it is important to note that the gain/detuning trend depicted here is highly sensitive to variations in a number of system parameters. As such, the trend should be re-examined prior to implementation on a device-by-device basis.

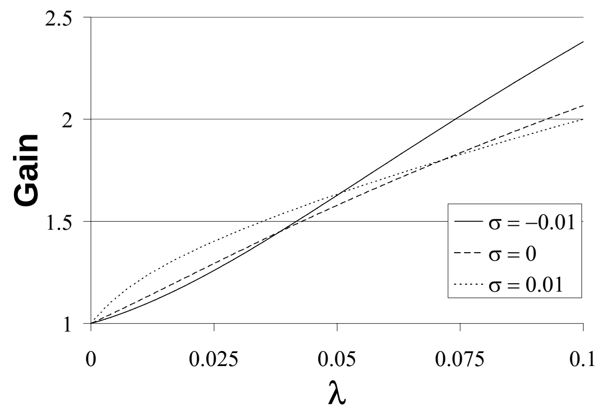

Figure 5. AMPLIFIER GAIN G VERSUS PARAMETRIC PUMP AMPLITUDE λ FOR A REPRESENTATIVE AMPLIFIER WITH $\phi = -\pi/4$, $\eta = 0.1$, $\alpha = 0.001$, AND $\zeta = 0.01$.

3 CONCLUSION

Though the aforementioned results clearly demonstrate that meaningful parametric amplification can be realized in resonant systems driven within a nonlinear frequency response regime, it is important to note that the results concurrently indicate that the performance metrics acquired with such systems will be generally inferior to those of their linear counterparts. While this degradation in performance is unfortunate, parametric amplification does appear to be a feasible option for on-chip, low-noise amplification in dynamic-range limited systems, such as resonant micro/transducers. In light of this promise, ongoing research is aimed at realizing parametric amplification in two distinct classes of micro/nanoresonators, which have been shown to exhibit little to no linear dynamic range; specifically, electromagnetically-actuated microbeams [12] and electrostatically-actuated nanotube resonators. From these ongoing experimental investigations a definitive conclusion on the experimental feasibility of nonlinear parametric amplification is expected.

Copyright © 2008 by ASME

ACKNOWLEDGMENT

This material is based on work supported by the National Science Foundation under grant No. ECS-0428916. Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

REFERENCES

[1] Carr, D. W., Evoy, S., Sekaric, L., Craighead, H. G., and Parpia, J. M., 2000. "Parametric Amplification in a Torsional Microresonator". *Applied Physics Letters,* **77**(10), pp. 1545–1547.

[2] Dana, A., Ho, F., and Yamamoto, Y., 1998. "Mechanical Parametric Amplification in Piezoresistive Gallium Arsenide Microcantilevers". *Applied Physics Letters,* **72**(10), pp. 1152–1154.

[3] Rugar, D., and Grütter, P., 1991. "Mechanical Parametric Amplification and Thermomechanical Noise Squeezing". *Physical Review Letters,* **67**(6), pp. 699–702.

[4] Zalalutdinov, M., Olkhovets, A., Zehnder, A., Ilic, B., Czaplewski, D., Craighead, H. G., and Parpia, J. M., 2001. "Optically Pumped Parametric Amplification for Micromechanical Oscillators". *Applied Physics Letters,* **78**(20), pp. 3142–3144.

[5] Postma, H. W. C., Kozinsky, I., Husain, A. and Roukes, M. L., 2005. "Dynamic Range of Nanotube- and Nanowire-Based Electromechanical Systems". *Applied Physics Letters,* **86**, 223105.

[6] Kim, C. H., Lee, C.-W., and Perkins, N. C., 2005. "Nonlinear Vibration of Sheet Metal Plates Under Interacting Parametric and External Excitation During Manufacturing". *Journal of Vibration and Acoustics,* **127**(1), pp. 36–43.

[7] Yagasaki, K., Sakata, M., and Kimura, K., 1990. "Dynamics of a Weakly Nonlinear System Subjected to Combined Parametric and External Excitation". *Journal of Applied Mechanics,* **57**(1), pp. 209–217.

[8] Szabelski, K., and Warminski, J., 1995. "Self-Excited System Vibrations with Parametric and External Excitations". *Journal of Sound and Vibration,* **187**(4), pp. 595-607.

[9] HaQuang, N., Mook, D. T., and Plaut, R. H., 1987. "A Non-Linear Analysis of the Interactions Between Parametric and External Excitations". *Journal of Sound and Vibration,* **118**(3), pp. 425-439.

[10] HaQuang, N., Mook, D. T., and Plaut, R. H., 1987. "Non-Linear Structural Vibrations Under Combined Parametric and External Excitations". *Journal of Sound and Vibration,* **118**(2), pp. 291-306.

[11] Jana, A., and Raman, A., 2005. "Nonlinear Dynamics of a Flexible Spinning Disc Coupled to a Precompressed Spring". *Nonlinear Dynamics,* **40**(1), pp. 1-20.

[12] Requa, M. V., and Turner, K. L., 2006. "Electromechanically Diven and Sensed Parametric Resonance in Silicon Microcantilevers". *Applied Physics Letters,* **88**, 263508.

Copyright © 2008 by ASME