1.6 Z-Transform

The *z-transform* is an important tool for filter design and for analyzing the stability of systems.

It is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \tag{1.44}$$

where z is a complex variable, i.e. $z = |z| e^{j\omega}$.

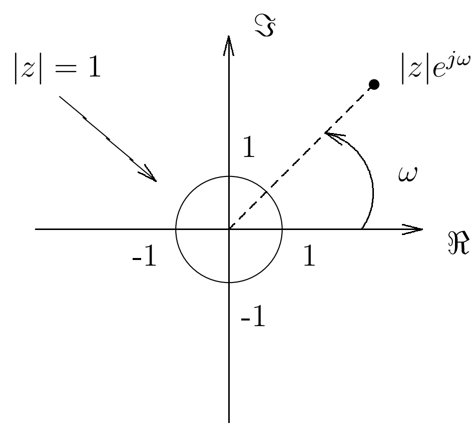


Figure 1.70. Z-transform.

DTFT can be interpreted as the z-transform evaluated on the unit circle,

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}} \tag{1.45}$$

1.6.1 Rational Z-Transform

We will mostly be interested in z-transforms that are rational functions of z, i.e. ratios of two polynomials,

$$H(z) = \frac{P(z)}{Q(z)}, \tag{1.46}$$

where $P(\cdot)$ and $Q(\cdot)$ are polynomials. Rational z-transforms are transfer functions of LTI systems described by linear, constant-coefficient difference equations such as:

$$y(n) = \sum_{i=0}^{N-1} b_i x(n-i) - \sum_{k=1}^{M} a_k y(n-k). \tag{1.47}$$

We now introduce some terminology.

- If there is no second term in the right-hand side of Eq. (1.47), then the system is called *non-recursive*, or *finite-duration impulse response (FIR)* system.

- Otherwise, it is *recursive* (because the current output sample $y(n)$ is expressed in terms of the past output samples). If it cannot be written as non-recursive, then it is an *infinite-duration impulse response* system.

$$\mathrm{ZT}\left\{x(n-i)\right\} = z^{-i}X(z) \tag{1.48}$$

$$\begin{aligned} Y(z) &= \sum_{i=0}^{N-1} b_i z^{-i} X(z) - \sum_{k=1}^{M} a_k z^{-k} Y(z) \\ Y(z)\left(1+\sum_{k=1}^{M} a_k z^{-k}\right) &= X(z)\sum_{i=0}^{N-1} b_i z^{-i} \end{aligned}$$

Therefore, the transfer function can be written as

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{i=0}^{N-1} b_i z^{-i}}{1+\sum_{k=1}^{M} a_k z^{-k}} \\ &= \frac{z^{-N+1}\sum_{i=0}^{N-1} b_i z^{n-1-i}}{z^{-M}\left(z^M+\sum_{k=1}^{M} a_k z^{M-k}\right)} \\ &\overset{\text{assume } b_0\neq 0}{=} b_0 z^{M-N+1}\frac{\prod_{i=1}^{N-1}(z-z_i)}{\prod_{k=1}^{M}(z-p_k)}\,, \end{aligned} \tag{1.49}$$

where the last equality comes from the fact that polynomials of degrees $N-1$ and M have $N-1$ and M roots, respectively. The values of z for which $H(z)=0$ (i.e., the roots of the numerator $z_1, z_2, \cdots, z_{N-1}$) are called the *zeros* of $H(z)$; whereas the values for which $H(z)$ is infinite (i.e., zeros of the denominator $p_1, p_2, \cdots, p_M$) are called the *poles* of $H(z)$. In addition, if $M > (N-1)$, then there are $M-(N-1)$ zeros at $z=0$; if $M < (N-1)$, then there are $(N-1)-M$ poles at $z=0$.

Poles and zeros may also occur at $z=\infty$:

"zero at ∞" means $\lim_{|z|\to\infty} H(z) = 0$,

"poles at ∞" means $\lim_{|z|\to\infty} H(z) = \infty$.

Example 1.30. *Let us consider the z-transform of $a^n u(n)$:*

$$\begin{aligned} ZT\left\{a^n u(n)\right\} &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \\ &= \frac{1}{1-az^{-1}} \quad \textit{if } \left|az^{-1}\right| < 1, \\ &= \frac{z}{z-a} \end{aligned}$$

The region of convergence (ROC) *is the set of all values of z for which the z-transform converges. In this example, it is $|z| > |a|$.*

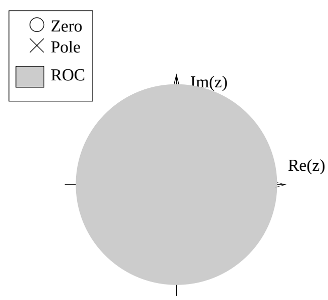


Figure 1.71. The region of convergence of the series in Example 1.30.

Let us now consider the filter whose transfer function is the z-transform of Example 1.30.

Example 1.31. *Let*

$$H(z) = \frac{z}{z-a} \quad .$$

(a) Difference Equation *To see how the z-transform is related to the time-domain representation, we expand it :*

$$\begin{aligned} \frac{Y(z)}{X(z)} &= \frac{1}{1-az^{-1}} \\ Y(z) - az^{-1}Y(z) &= X(z) \\ Y(z) &= az^{-1}Y(z) + X(z) \end{aligned}$$

Inverting the z-transform, we have:

$$y(n) = ay(n-1) + x(n).$$

(b) System Diagram *The system diagram is depicted in Fig. 1.72.*

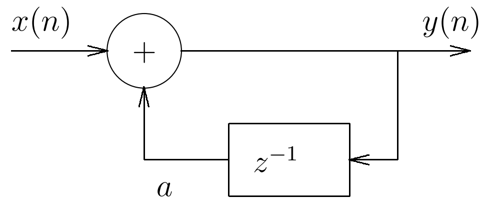


Figure 1.72. The system diagram of Example 1.31. z^{-1} delays a signal by one time unit.

(c) Frequency Response *The frequency response of the system can be obtained by replacing* z *with* $e^{j\omega}$ *in the transfer function:*

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= \frac{1}{1-ae^{-j\omega}} \\
\left|H\left(e^{j\omega}\right)\right| &= \frac{1}{\sqrt{(1-ae^{-j\omega})(1-ae^{j\omega})}} \\
&= \frac{1}{\sqrt{1-2a\cos\omega+a^2}} \\
&= \frac{1}{\sqrt{(1-a)^2+2a(1-\cos\omega)}}
\end{aligned}
$$

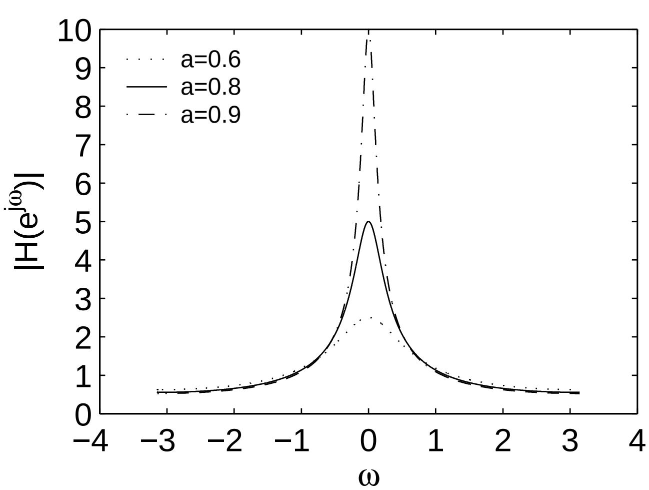


Figure 1.73. Magnitude response of the filter in Example 1.31 for several values of the parameter a.

Fig. 1.73 shows the magnitude response of the filter for different values of a*. In* *particular, note that the height of the peak is determined only by a, since the term with the cosine is removed when $\omega = 0$. The closer a is to the unit circle, the sharper the peak, and the thinner the passband.*

As we have seen from Example 1.31, in general, a pole near the unit circle will cause the frequency response to increase in the neighborhood of that pole; a zero will cause the frequency response to decrease in the neighborhood of that zero (Fig. 1.74).

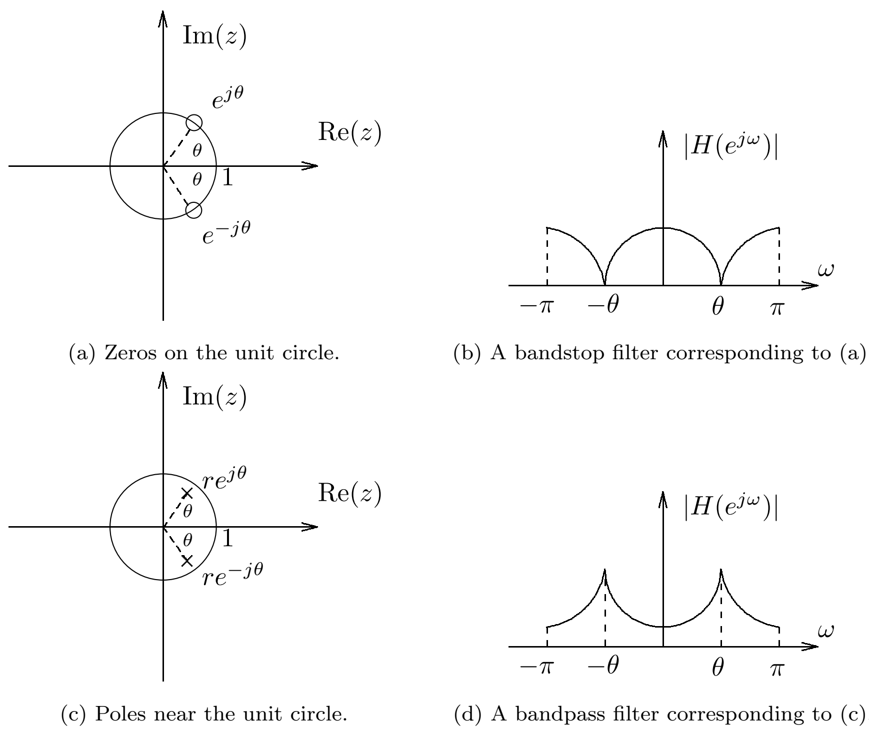


Figure 1.74. The effects of zeros and poles near the unit circle.

■ 1.6.2 Region of Convergence (ROC) of the Z-Transform

Example 1.32. *Let $x(n) = 2^n u(n)$. Then,*

$$X(z) = \sum_{n=0}^{\infty} 2^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n .$$

When can we sum this series? In other words, when does this series converge? Consider these two cases:

1. *If $|z| \le 2$, then $\left|\frac{2}{z}\right| \ge 1$. This means that every term in the series has an absolute value greater than or equal to 1. If it is greater than 1, then every successive terms grows larger. If it is equal to 1, then we are just adding 1 an infinite number of times. In both cases, the series diverges.*

2. *On the other hand, if* $|z| > 2$*, then the geometric series converges because* $\left|\frac{2}{z}\right| < 1$*, and we have*

$$X(z) = \frac{1}{1 - \frac{2}{z}} \quad . \tag{1.50}$$

Thus, the z-transform of $x(n) = 2^n u(n)$ *is*

$$X(z) = \begin{cases} \frac{1}{1-\frac{2}{z}} & |z| > 2 \\ \textit{undefined} & |z| \le 2 \end{cases}$$

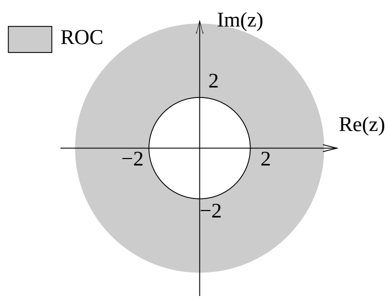


Figure 1.75. The ROC of Example 1.32.

Usually, $|z| > 2$ *is called the "region of convergence" (ROC) of the z-transform, because when z lies in this region, the series actually converges to the function (1.50). A slightly more accurate term would be "the region of definition", since the z-transform is undefined outside of this region.*

Example 1.33. *Let* $y(n) = -2^n u(-n-1)$*. Then*

$$\begin{aligned} Y(z) &= \sum_{n=-1}^{-\infty} -2^n z^{-n} \\ &= -\sum_{m=0}^{\infty} 2^{-m-1} z^{m+1} \qquad \textit{where } m = -n-1 \\ &= -\frac{z}{2} \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^m . \end{aligned}$$

1. *If* $|z| \ge 2$*, the series diverges.*

2. *if* $|z| < 2$*, the series converges, and*

$$\begin{aligned} Y(z) &= -\frac{z}{2}\frac{1}{1-\frac{z}{2}} \\ &= -\frac{z}{2-z} \\ &= \frac{1}{1-\frac{2}{z}} \quad . \end{aligned}$$

Putting it all together,

$$Y(z) = \begin{cases} \text{undefined} & |z| \geq 2 \\ \\ \frac{1}{1-\frac{2}{z}} & |z| < 2 \end{cases}$$

We get the same expression for the z-transform as in Example 1.32 but the ROC is different and in fact does not intersect the ROC from Example 1.32.

Example 1.34. *Let* $w(n) = 2^n$ *for* $-\infty < n < \infty$*. Then*

$$2^n u(n) + 2^n u(-n-1) = x(n) - y(n),$$

where $x(n)$ *and* $y(n)$ *are from Examples 1.32 and 1.33, respectively. Hence,*

$$\begin{aligned} W(z) &= X(z) - Y(z) \\ &= \frac{1}{1-\frac{2}{z}} - \frac{1}{1-\frac{2}{z}} \\ &= 0??? \end{aligned}$$

What is wrong with this derivation? As we saw in the two previous examples, $X(z)$ *and* $Y(z)$ *have no common ROC:*

- $X(z)$ *is undefined for* $|z| \leq 2$*.*
- $Y(z)$ *is undefined for* $|z| \geq 2$*.*

This means that $W(z)$ *is not defined for any* z*.*

Definition 1.12. *The region of convergence of*

$$\sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

is the set of all z *for which this series is absolutely convergent, i.e.,*

$$\sum_{n=-\infty}^{\infty} \left| x(n) z^{-n} \right| < \infty.$$

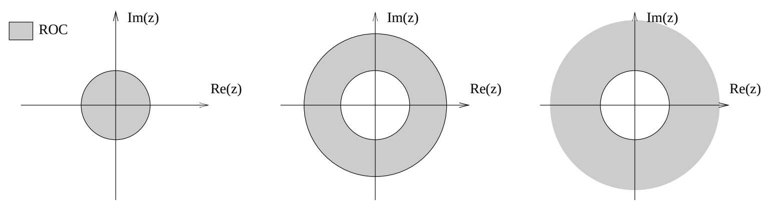
We use absolute convergence to avoid certain pathological series which converge, but do not converge absolutely. An example of this is

$$\sum_{n=-\infty}^{\infty} (-1)^n u(n) \frac{1}{n} z^{-n} \qquad \text{at } z = 1.$$

1.6.3 Properties of ROC, Poles and Zeros

The following is a list of several important properties of the z-transform.

1. The ROC is a ring or a disc centered at the origin: $r_1 < z < r_2$. Note that r_1 or r_2 could be 0 or ∞.



(a) $x(n)$ left-sided. (b) Generic $x(n)$. (c) $x(n)$ right-sided.

Figure 1.76. The geometries of the ROC.

2. The ROC cannot contain any poles.
3. If $x(n)$ is a finite-duration sequence (i.e., $x(n) = 0$ except for $N_1 \leq n \leq N_2$), then the ROC is the whole z-plane except possibly $z = 0$.
4. If $x(n)$ is a right-sided sequence ($x(n) = 0$ for $n < N_1$), then the ROC is
$$|z| > |p_{max}|,$$
where p_{max} is the outermost finite pole of $X(z)$.
5. If $x(n)$ is a left-sided sequence ($x(n) = 0$ for $n \geq N_2$), then the ROC is
$$|z| < |p_{min}|,$$
where p_{min} is the innermost non-zero pole of $X(z)$.
6. Generalizing Example 1.32 and Example 1.33, we have:
$$a^n u(n) \leftrightarrow \frac{1}{1 - az^{-1}}, \quad \text{where the ROC is } |z| > |a|, \tag{1.51}$$
$$-a^n u(-n-1) \leftrightarrow \frac{1}{1 - az^{-1}}, \quad \text{where the ROC is } |z| < |a|. \tag{1.52}$$

7. An *LTI system* is BIBO stable if and only if the ROC of its transfer function $H(z)$ includes $z = 1$, (i.e., includes the unit circle):

$$\left[\sum_n \left|h(n)z^{-n}\right|\right]_{z=1} < \infty$$
$$\Updownarrow$$
$$\sum_n |h(n)| < \infty \Leftrightarrow \text{BIBO stability.}$$

Note that this stability criterion is applicable to LTI systems only.

Example 1.35. *Find all sequences whose z-transform is*

$$X(z) = \frac{1-4z^{-1}}{1-3z^{-1}+2z^{-2}} \ .$$

Solution. *We decompose $X(z)$ into partial fractions,*

$$X(z) = \frac{1-4z^{-1}}{(1-z^{-1})(1-2z^{-1})} = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-2z^{-1}}. \tag{1.53}$$

Then we solve for A_1 and A_2.

Method 1. *Rearranging the equation, we have*

$$X(z) = \frac{A_1(1-2z^{-1})+A_2(1-z^{-1})}{(1-z^{-1})(1-2z^{-1})} = \frac{A_1+A_2-(2A_1+A_2)z^{-1}}{(1-z^{-1})(1-2z^{-1})} \ .$$

Comparing terms, we solve for A_1 and A_2 :

$$\begin{cases} A_1 + A_2 &= 1 \\ 2A_1 + A_2 &= 4 \end{cases} \Rightarrow \begin{cases} A_1 &= 3 \\ A_2 &= -2 \end{cases}$$

Method 2. *Using (1.53),*

$$[X(z)(1-z^{-1})]_{z=1} = \left[\frac{1-4z^{-1}}{1-2z^{-1}}\right]_{z=1} = \left[A_1 + \frac{A_2(1-z^{-1})}{1-2z^{-1}}\right]_{z=1}$$
$$\Rightarrow 3 = A_1.$$

$$[X(z)(1-2z^{-1})]_{z=2} = \left[\frac{1-4z^{-1}}{1-1z^{-1}}\right]_{z=2} = \left[A_1 + \frac{1-4z^{-1})}{1-z^{-1}}\right]_{z=2}$$
$$\Rightarrow -2 = A_2.$$

Thus, we have

$$X(z) = \frac{3}{1 - z^{-1}} - \frac{2}{1 - 2z^{-1}} \ .$$

We now consider the three possible ROC's that this z-transform can have.

Case 1. *ROC* $|z| > 2$. *Using (1.51),*

$$\begin{aligned} x(n) &= 3 \cdot 1^n u(n) - 2 \cdot 2^n u(n) \\ &= (3 - 2^{n+1}) u(n). \end{aligned}$$

Case 2. *ROC* $1 < |z| < 2$. *Using (1.51) and (1.52),*

$$x(n) = 3 \cdot 1^n u(n) + 2 \cdot 2^n u(-n-1).$$

Case 3. *ROC* $|z| < 1$. *Using (1.52),*

$$x(n) = (-3 + 2^{n+1}) u(-n-1).$$

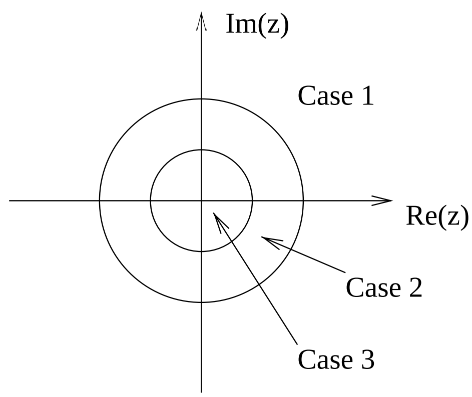


Figure 1.77. ROC for the cases in Example 1.35.

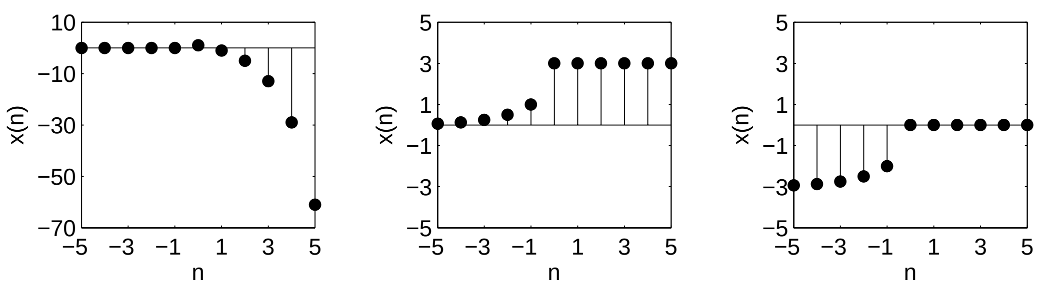


Figure 1.78. The inverse z-transforms for the 3 possible ROC's of Example 1.35.

■ 1.6.4 Discrete-Time Exponentials z_0^n are Eigenfunctions of Discrete-Time LTI systems

Suppose that a discrete-time complex exponential $x(n) = z_0^n$ is the input signal to an LTI system with impulse response $h(n)$. Then the output is:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=-\infty}^{\infty} h(k)z_0^{n-k} \\ &= z_0^n \sum_{k=-\infty}^{\infty} h(k)z_0^{-k} \end{aligned}$$

If z_0 is in the ROC of $H(z)$, we can write

$$y(n) = z_0^n \cdot \mathrm{ZT}\left\{h(k)\right\}|_{z=z_0} = z_0^n H(z_0),$$

as shown in Fig. 1.79.

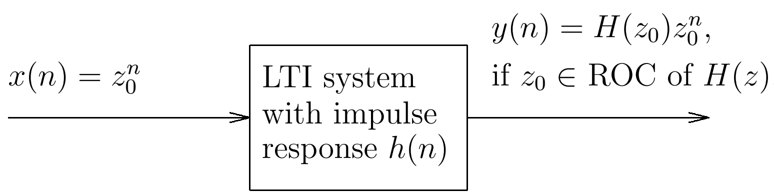


Figure 1.79. An LTI system with z_0^n as the input.

The transfer function $H(z)$ is the z-transform of the impulse response $h(n)$. It is also the scaling factor of z^n when z^n goes through the system.

Recall that we have already considered the case of $z_0 = e^{j\omega_0}$. The frequency response $H(e^{j\omega_0})$ is:

- the DTFT of the impulse response $h(n)$;
- the scaling factor when $e^{j\omega_0 n}$ is the input, as shown in Fig. 1.80.

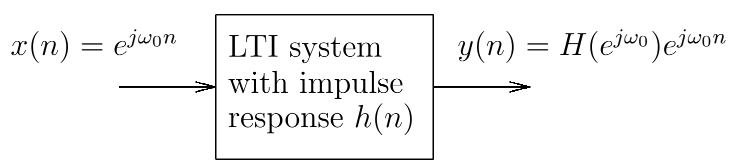
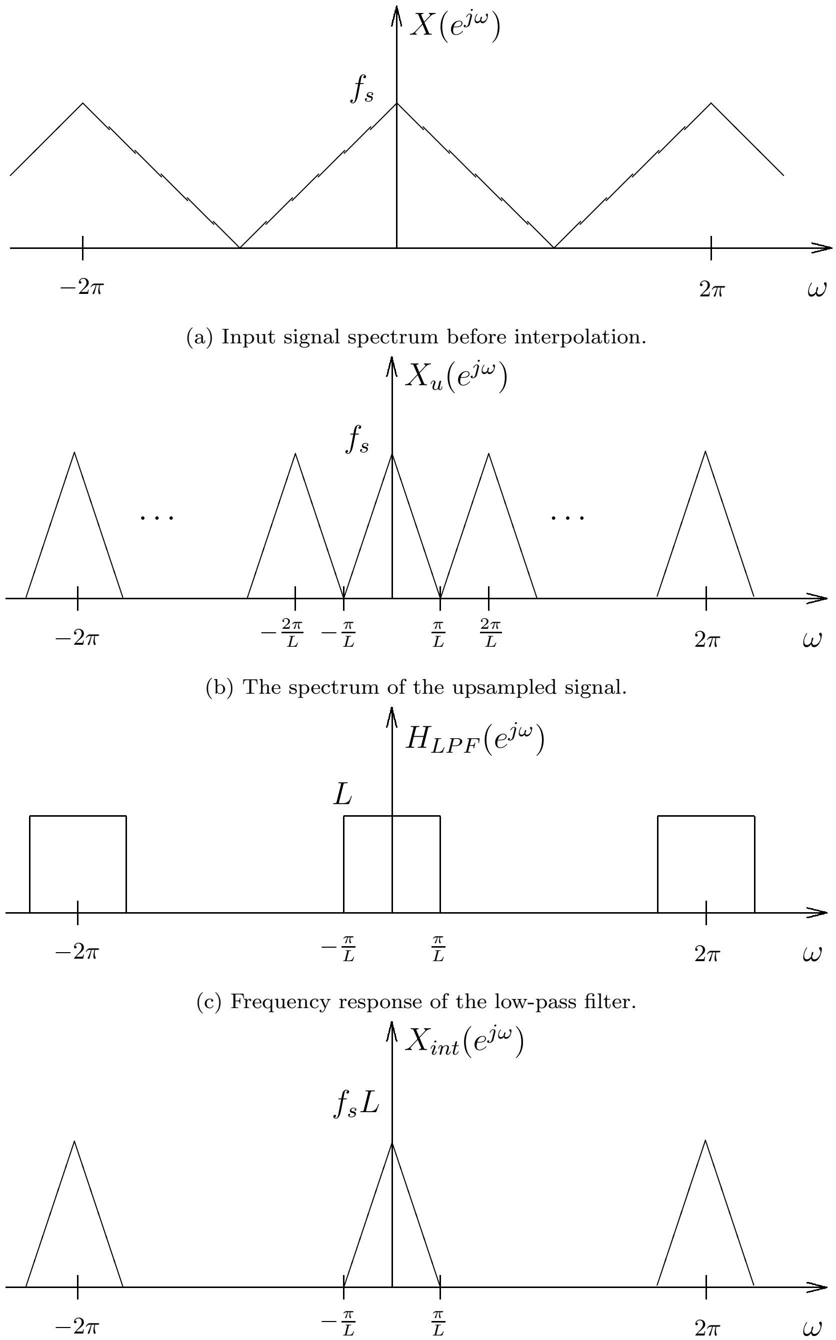


Figure 1.80. An LTI system with $e^{j\omega_0 n}$ as the input.



(a) Input signal spectrum before interpolation.

(b) The spectrum of the upsampled signal.

(c) Frequency response of the low-pass filter.

(d) The spectrum of the interpolated signal.

Figure 1.65. The effect of interpolation on the spectrum of the input signal x.