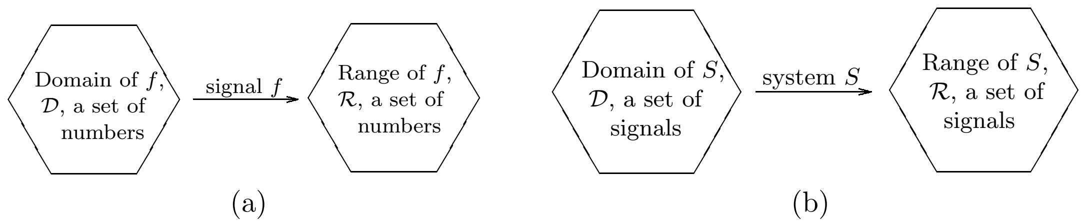

Figure 1.9. (a) A signal is a mapping between two sets of numbers. (b) A system is a mapping between two sets of signals.

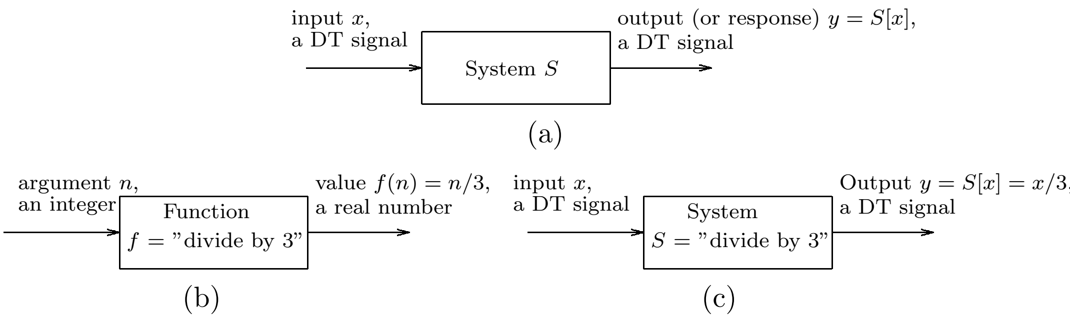

Figure 1.10. (a) A generic block diagram for a system. (b) DT signal "divide by 3". (c) DT system "divide by 3".

1.2 Systems

1.2.1 What is a System?

The concept of a system is very similar to that of a signal. Recall that a signal is a rule for producing a number in its range, given a number from its domain. Both the domain and the range of a signal are thus sets of numbers, as shown in Fig. 1.9(a).

A system is also a mapping between two sets; however, both the domain and the range of a system are sets of signals. A system is thus a rule for producing a signal in its range, given a signal from its domain.[3]

Recall also that we classified signals according to their domains and ranges. For example, a signal whose domain is an interval of integers and whose range is an interval of reals is called a discrete-time signal. We can similarly classify systems. Specifically, we will distinguish two important classes of systems. For a discrete-time (DT) system, both the range and the domain are sets of DT signals. For a continuous-time (CT) system, both the range and the domain are sets of CT signals.

A system can be represented as a block diagram, as in Fig. 1.10(a). It is important to remember that the input and output are not single numbers, they are signals. The

[3] A more general view of systems which is beyond the scope of this course, defines a system as anything that imposes constraints on a set of signals.

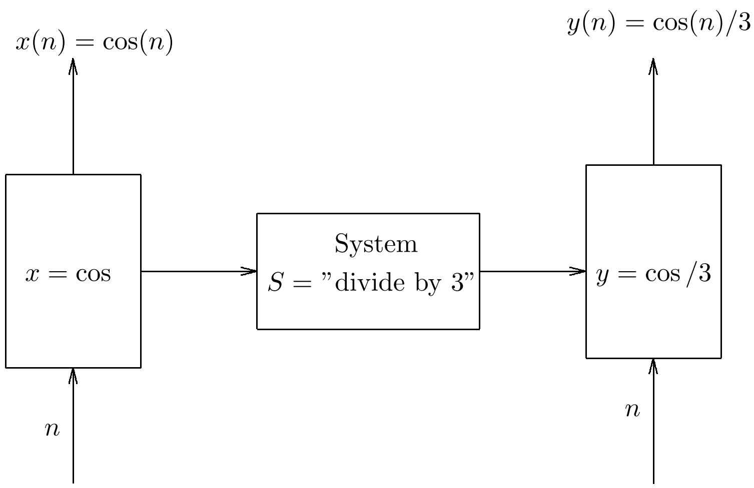

Figure 1.11. System "divide by 3" for the specific case when the input signal is $x = \cos$.

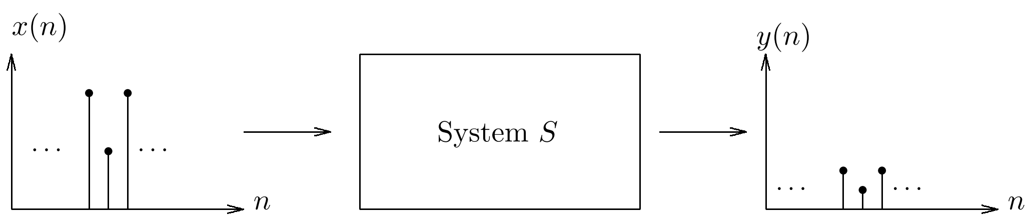

Figure 1.12. Another view of what a system is.

specification of a range and a domain are crucial for defining both signals and systems. In fact, the actual mapping may be identical for a signal f and a system S; the two will however always have different ranges and domains: the range and the domain for f are sets of numbers while the range and the domain for S are sets of signals. To illustrate this point, let us consider the example of the discrete-time function "divide by 3", shown in Fig. 1.10(b). We can also define the *system* "divide by 3", shown in Fig. 1.10(c). The two objects are completely different: the function "divide by 3" takes in a single integer number n and produces a single real number $n/3$, whereas the system "divide by 3" takes in a DT signal x and produces another DT signal y such that $y(n) = x(n)/3$ for all integer values of n.

A specific example of this is given in Fig. 1.11: supposing that the input signal is $x(n) = cos(n)$, the output is another signal, y(n) $= cos(n)/3$. In other words, x is a rule for transforming a single number into another number; the system changes this rule into y.

Another way of thinking about what a system does is that the whole graph of the input signal x is fed into S, and it produces the whole graph of the output signal

y, as depicted in Fig. 1.12. To emphasize that the input of S is the whole signal x, we will be using $S[x]$ to denote the output signal y, rather then $S[x(n)]$. The latter notation is also acceptable, provided you keep in mind that what it really stands for is: $S[x(n)$ for all $n]$, i.e. that S operates on all the samples of x. Once the system's response is known, it can be evaluated at a particular n: $S[x](n)$ is synonymous with $y(n)$ and means the n-th sample of y, where y is the response of system S to input x.

Very often, systems are specified by input-output relationships. As we saw above, the expression

$$y(n) = \frac{x(n)}{3}, -\infty < n < \infty$$

specifies a system.

■ 1.2.2 Properties of Systems: Linearity and Time-Invariance

(a) Linearity. A system S is linear if, for any two input signals x_1 and x_2 from the domain of S and for any two numbers a_1 and a_2, it satisfies

$$S[a_1x_1 + a_2x_2] = a_1S[x_1] + a_2S[x_2]. \tag{1.3}$$

That is, if the response to any linear combination of any two inputs is the same linear combination of the responses to these two inputs, then the system is linear. This is illustrated in Fig. 1.13.

If a system is linear we can therefore compute the response to a complicated signal as the sum of responses to simpler signals. We will exploit this property many times in our treatment of linear systems.

Example 1.2. *Let us consider the system specified by the following input-output relationship:*

$$y(n) = \begin{cases} \sum_{k=0}^{n} x(k), & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Before trying to determine whether the system is linear, it is useful to try to guess the answer. Since every sample of the output is just a sum of several samples of the input, we guess that the system is linear. In order to prove our conjecture, we have to show that this system satisfies our definition of linearity. Suppose that x_1 and x_2 are two arbitrary input signals, and a_1 and a_2 are two arbitrary numbers. Let $x_3 = a_1x_1 + a_2x_2$. Then the responses of the system to to x_1, x_2, and x_3 are, respectively, as follows:

$$y_1(n) = \begin{cases} \sum_{k=0}^{n} x_1(k), & n \geq 0, \\ 0, & n < 0, \end{cases}$$

$$y_2(n) = \begin{cases} \sum_{k=0}^{n} x_2(k), & n \geq 0, \\ 0, & n < 0, \end{cases}$$

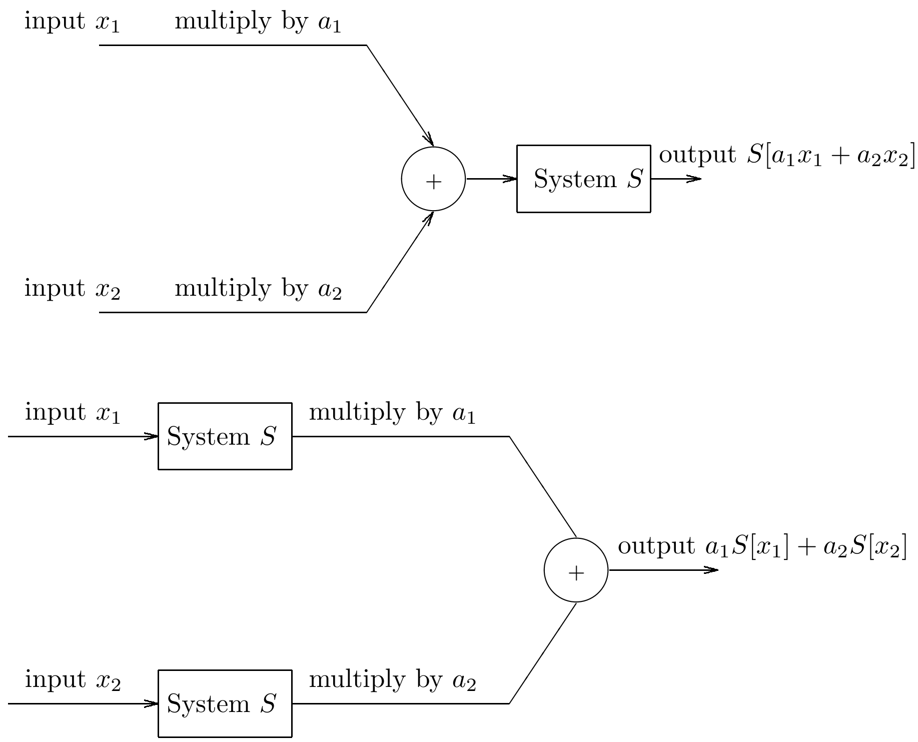

Figure 1.13. System S is linear if and only if the outputs of the two systems above are identical for any pair of signals x_1 and x_2 and any pair of numbers a_1 and a_2.

$$\begin{aligned} y_3(n) &= \begin{cases} \sum_{k=0}^{n} x_3(k), & n \geq 0 \\ 0, & n < 0 \end{cases} = \begin{cases} \sum_{k=0}^{n} [a_1x_1(k) + a_2x_2(k)], & n \geq 0 \\ 0, & n < 0 \end{cases} \\ &= \begin{cases} a_1\sum_{k=0}^{n} x_1(k) + a_2\sum_{k=0}^{n} x_2(k), & n \geq 0 \\ 0, & n < 0 \end{cases} = a_1y_1(n) + a_2y_2(n) \quad \textit{for all integer } n. \end{aligned}$$

Since this holds for any inputs x_1, x_2 and any numbers a_1, a_2, the system is indeed linear. ■

Example 1.3. *The system specified by the following input-output relationship:*

$$y(n) = 2x(n) + 3, \;\; \textit{for all integer } n,$$

is actually nonlinear. To show this, we need just one example that violates the definition of linearity. If, in that definition, we set $a_2 = 0$ and $a_1 = 2$, we see that if a system is linear then multiplying the input signal x_1 by 2 must produce the response $2y_1$ where y_1 is the response to x_1. This does not necessarily happen here: supposing that $x_1(n) = 1$

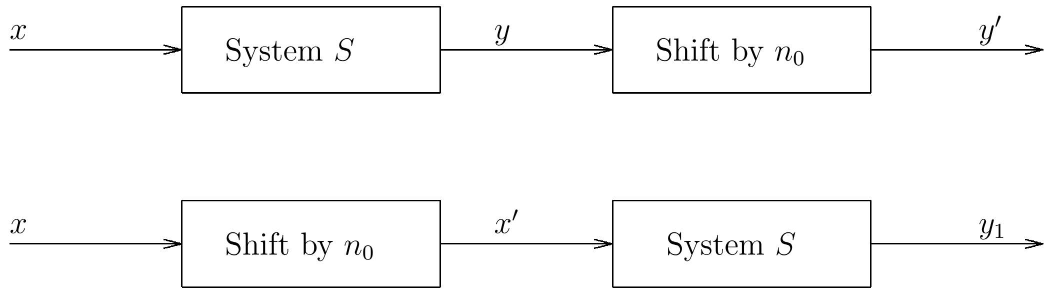

Figure 1.14. System S is time-invariant if it commutes with any shift operator, in other words, if the outputs of the two composite systems above are the same for any input x and any shift n_0.

for all integer n*, we get* $y_1(n) = 2 \cdot 1 + 3 = 5$ *for all integer* n*, but the response to* $2x_1$ *is* $2 \cdot 2 + 3 = 7$ *for all integer* n *which is not the same as* $2y_1(n) = 10$*. Therefore, the system is nonlinear.* ■

Note again that, in order to prove that a system is linear, we need to prove the condition stated in the definition of linearity for *every* possible pair of inputs. In order to show that a system is non-linear, however, one counterexample to that statement is enough.

(b) Time-Invariance. A system S is time-invariant if shifting the input results in only an identical shift of the output. Otherwise, S is called time-varying.

Suppose that y is the response of S to x, and that y' and x' are shifted versions of y and x, respectively, with integer shift n_0:

$$\begin{aligned} y'(n) &= y(n-n_0), \\ x'(n) &= x(n-n_0). \end{aligned}$$

The definition above says that, if $S[x'] = y'$ for any input signal x and any integer shift n_0, then system S is time-invariant. This is illustrated in Fig. 1.14.

Let us now look at the two systems of Examples 1.2 and 1.3, and determine whether they are time-invariant. A rule of thumb for determining this is to look for the explicit occurrence of the time variable n in the system specification. If it does occur explicitly then the system is usually time-varying; if n only occurs as an argument of signals (such as $x(n)$, $y(n)$, etc.) but does not occur by itself, the system is usually time-invariant. This is not a hard and fast rule, however: it is only useful for guessing the answer. Once we guess the answer, we still have to rigorously prove it using the definition of time-invariance.

Example 1.4. *In the following system specification:*

$$y(n) = \begin{cases} \displaystyle\sum_{k=0}^{n} x(k), & n \geq 0, \\ 0, & n < 0, \end{cases}$$

the time variable n appears as the upper limit of the summation. We therefore guess that the system is time-varying. In order to show that the system is time-varying, we need to come up with a signal x and a shift n_0 for which shifting $S[x]$ by n_0 is not the same thing as applying S to a shifted version of x. Since we are just looking for one example, a reasonable strategy is to try a very simple signal first. For example, let $x(n) = 1$ for all integer n, and let the system's response to x be y. Shifting x by $n_0 = 1$ results in x' defined by $x'(n) = x(n-1)$ for all integer n. Clearly the signal x' is in this case the same as x, and is identically equal to one for all integer n. Let y and y_1 be the responses of the system to the inputs x and x', respectively, and let y' be defined by $y'(n) = y(n-n_0) = y(n-1)$. Note that, since the inputs x and x' are identical, we also have $y = y_1$:

$$\begin{aligned} y'(n) &= y(n-1), \\ y_1(n) &= y(n). \end{aligned}$$

For the system to be time-invariant, it must be that $y'(n) = y_1(n)$, i.e. that $y(n-1) = y(n)$. Taking $n = 1$, we must have $y(0) = y(1)$. But in fact $y(0) = x(0) = 1$ while $y(1) = x(0) + x(1) = 2 \neq y(0)$. We therefore have come up with an input signal x, a shift n_0, and a time instant n for which the statement in the definition of time-invariance is violated. The system is therefore time-varying. ■

Example 1.5. *The system $y(n) = 2x(n) + 3$ is easily seen to be time-invariant since shifting the response to x produces $y(n-n_0) = 2x(n-n_0) + 3$ which is the same signal as the response of the system to x' defined by $x'(n) = x(n-n_0)$.* ■

It is important to emphasize once again the following part in the definition of time-invariance: "for any x and any shift n_0". This means, that, in order to prove that a system is time-invariant, we have to prove it for all possible input signals, and all possible time shifts. On the other hand, in order to prove that a system is not time-invariant, we only need to come up with one specific counter-example.

■ 1.2.3 Impulse Response and Convolution

We now take a closer look at LTI systems in the input-output form and develop a method to compute the output of an LTI system, given its input. Specifically, we will see that the output is the convolution of the input with the impulse response. Our plan for deriving this fact is:

1. Write the input signal as a linear combination (weighted sum) of shifted unit impulse signals.

2. Use linearity to write the response as the sum of responses to shifted impulses.

3. Use time-invariance to find the response to a shifted impulse. Specifically, the response of a time-invariant system to signal $\delta(n-k)$ is $h(n-k)$ where $h(n)$ is the unit impulse response.

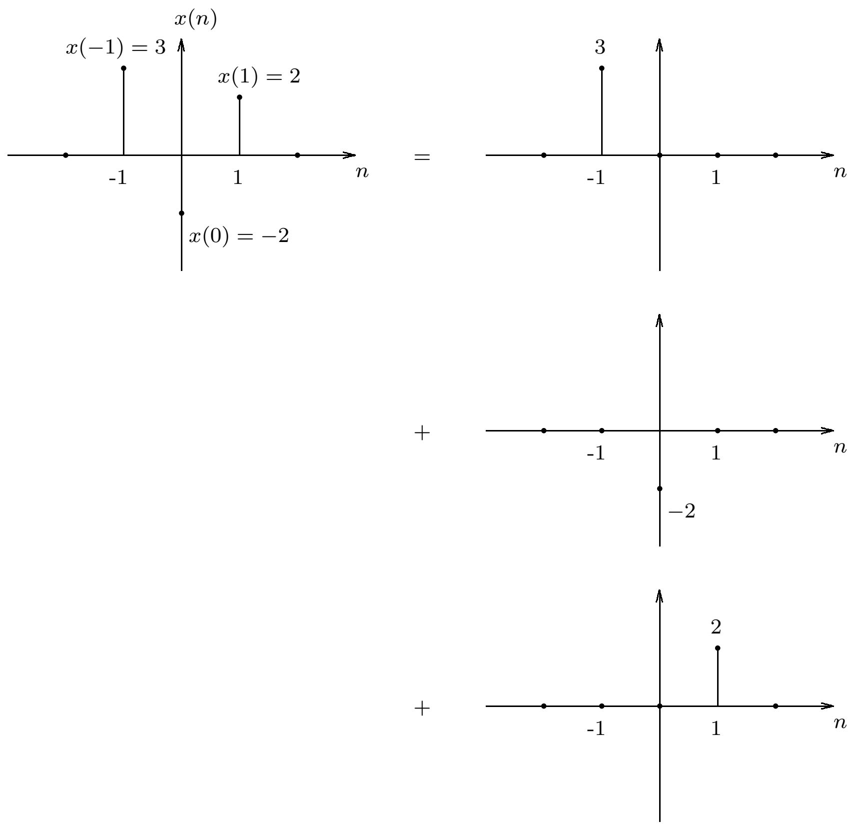

Figure 1.15. The signal $x(n)$ is represented as a sum of impulse signals.

Let us begin with signal $x(n) = 3\delta(n+1) - 2\delta(n) + 2\delta(n-1)$, defined for all integer n. As shown in Fig. 1.15, this signal can be represented as follows:

$$x(n) = x(-1)\delta_{-1}(n) + x(0)\delta_0(n) + x(1)\delta_1(n),$$

where δ_k is the unit impulse shifted by k, i.e. $\delta_k(n) = \delta(n-k)$ for all integer n and k. Similarly, any arbitrary signal can be represented as the following weighted sum of shifted impulse signals:

$$\begin{aligned} x(n) &= \ldots + x(-2)\delta_{-2}(n) + x(-1)\delta_{-1}(n) + x(0)\delta_0(n) + x(1)\delta_1(n) + x(2)\delta_2(n) + \ldots \\ &= \sum_{k=-\infty}^{\infty} x(k)\delta_k(n), \quad \text{for all integer} \ \ n. \end{aligned}$$

If signal x is put through a linear system S, we can use the above equation and the linearity of the system to write the response y of the system as follows:

$$\begin{aligned}
y(n) &= S[x](n) \\
&= S[\ldots + x(-2)\delta_{-2} + x(-1)\delta_{-1} + x(0)\delta_0 + x(1)\delta_1 + x(2)\delta_2 + \ldots](n) \\
&\overset{\text{linearity}}{=} \ldots + x(-2)S[\delta_{-2}](n) + x(-1)S[\delta_{-1}](n) + x(0)S[\delta_0](n) \\
&\quad + x(1)S[\delta_1](n) + x(2)S[\delta_2](n) + \ldots \\
&= \sum_{k=-\infty}^{\infty} x(k)S[\delta_k](n) \\
&= \sum_{k=-\infty}^{\infty} x(k)h_k(n), \qquad (1.4)
\end{aligned}$$

where we denoted by $h_k = S[\delta_k]$ the system's response to the shifted impulse δ_k.

If system S, in addition to being linear, is time-invariant, then

$$h_k(n) = h(n-k) \text{ for all integer } n \text{ and } k,$$

where h is the response to the unit impulse δ. Substituting this into Eq. (1.4) yields:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k),$$

which is the formula for the discrete-time convolution. We will use the following notation to indicate that signal y is the convolution of signal x with signal h: $y = x * h$. The n-th sample of y is then $y(n) = x * h(n)$.

We have thus shown that the output of a discrete-time LTI system is the discrete-time convolution of the input and the impulse response.

Example 1.6. *Consider the following input-output specification of a system:*

$$y(n) = x(n) + \frac{1}{2}x(n-1), \text{ for all integer } n.$$

Let us find the response to $x(n) = \begin{cases} \frac{1}{3}, & n = -1, \\ 1, & n = 0, \\ \frac{2}{3}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$

The impulse response h of the system is the response to the unit impulse:

$$h(n) = \delta(n) + \frac{1}{2}\delta(n-1) = \begin{cases} 1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

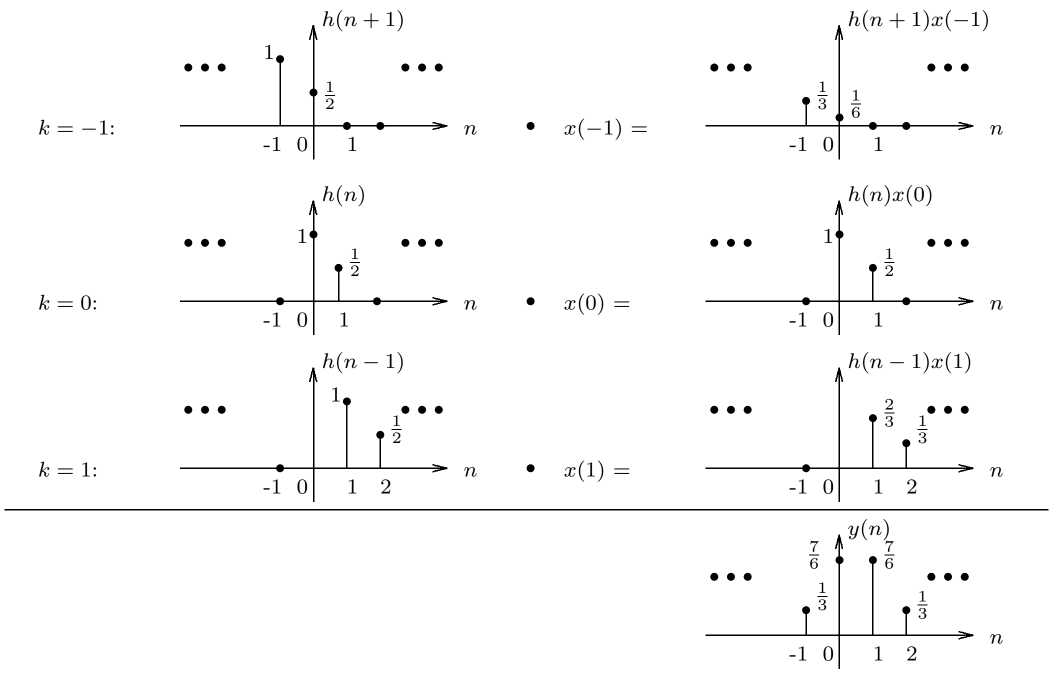

Figure 1.16. Illustration to Example 1.6: the convolution of h and x is a weighted linear combination of shifted versions of h, with the weights given by the samples of x.

Therefore, the response to x is:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1), \text{ for all integer } n.$$

We can evaluate this convolution by directly calculating the linear combination of shifted versions of h. We start by plotting $h(n-k)$ as a function of n, for each k, and proceed as shown in Fig. 1.16.

A more convenient method is illustrated in Fig. 1.17. It involves plotting signal $x(k)$ as a function of k, and plotting signals $h(n-k)$ as a function of k, for each n. Here is the basic procedure for calculating the n-th sample of y:

(1) flip h;

(2) for a fixed n, shift h by n;

(3) for the same fixed n, multiply $x(k)$ by $h(n-k)$, for each k;

(4) Sum the products over k: $\sum_{k=-\infty}^{\infty} x(k)h(n-k) = y(n)$.

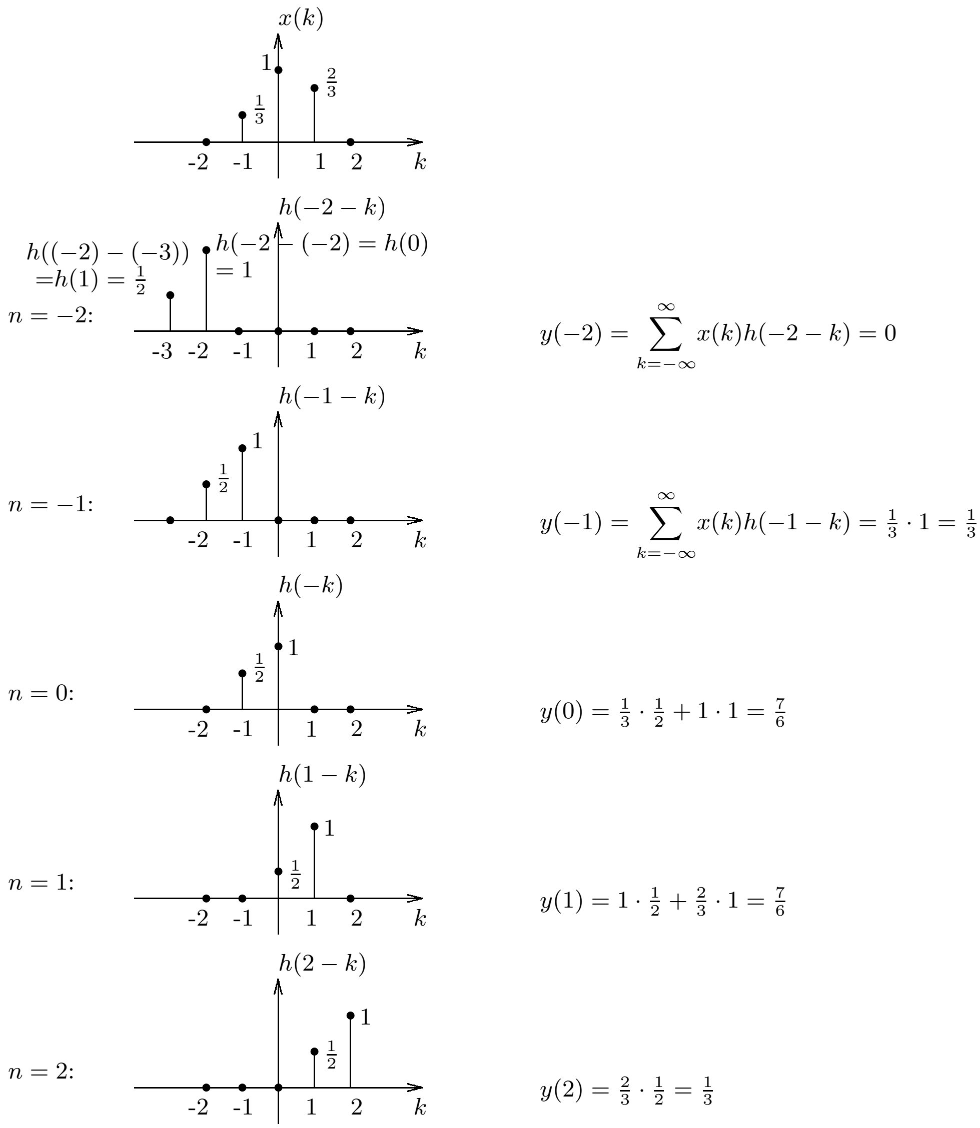

Figure 1.17. Evaluating the convolution sum of Example 1.6.

Both methods of course lead to the same result:

$$y(n) = \begin{cases} \frac{1}{3}, & n = -1, 2, \\ \frac{7}{6}, & n = 0, 1, \\ 0, & \textit{otherwise.} \end{cases} \qquad \blacksquare$$

Example 1.7. *To evaluate the convolution of signals* $x(n) = 2^{-|n|}$ *and* $h(n) = u(n)$, *we substitute the two expressions into the definition of convolution:*

$$\begin{aligned} y(n) &= x * h(n) = \sum_{k=-\infty}^{\infty} 2^{-|k|} u(n-k) \\ &= \sum_{k=-\infty}^{n} 2^{-|k|}. \end{aligned}$$

For $n \le 0$, the summation is only over nonpositive values of k and therefore $|k|$ can be replaced with $-k$:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{n} 2^{-(-k)} = \sum_{k=-\infty}^{n} 2^{k} \\ &= \sum_{m=-n}^{\infty} 2^{-m} = \sum_{m=-n}^{\infty} \left(\frac{1}{2}\right)^m = \frac{(1/2)^{-n}}{1-1/2} \\ &= 2^{n+1}, \text{ for any integer } n \le 0, \end{aligned}$$

where we substituted $m = -k$. When $n > 0$, the summation can be broken into two pieces: one for nonpositive values of k (i.e. for k from $-\infty$ to 0) and the other for positive values of k (i.e. for k from 1 to n):

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{0} 2^{-(-k)} + \sum_{k=1}^{n} 2^{-k} \\ &= \sum_{m=0}^{\infty} 2^{-m} + \sum_{k=1}^{n} 2^{-k} \\ &= \frac{1}{1-1/2} + \frac{(1/2)^1 - (1/2)^{n+1}}{1-1/2} \\ &= 3 - 2^{-n}, \text{ for any integer } n > 0. \end{aligned}$$

Putting together the two cases,

$$y(n) = \begin{cases} 2^{n+1}, & n \le 0, \\ 3 - 2^{-n}, & n > 0. \end{cases}$$ ■

■ 1.2.4 Further Properties of Systems

(a) Causality. A system is causal if the output at any time does not depend on the future values of the input, i.e., if $y(n)$ does not depend on $x(k)$ for $k > n$, for any input x and any time n.

This is equivalent to saying that, whenever two input signals are identical up to some time instant n_0, the system's responses to them must also be identical up to n_0.

It is easily seen that both systems in Examples 1.2 and 1.3 are causal.

For LTI systems, there is a simple criterion which allows you to determine whether the system is causal or not. Recall that, if x is the input to an LTI system with impulse response h, then the output is $y(n) = \sum_{k=\infty}^{\infty} h(k)x(n-k)$. Therefore, for an LTI system to be causal, the portion of this summation involving all the terms $x(n-k)$ for $k < 0$, must be zero for any input x. This can only happen if $h(k) = 0$ for $k < 0$. Conversely, if $h(k) = 0$ for $k < 0$ then the system is clearly causal. We therefore have the following result.

Causality for LTI Systems. An LTI system is causal if and only if its impulse response h satisfies $h(n) = 0$ for $n < 0$.

It is important to remember that this criterion is applicable only to LTI systems. For example, we cannot use this criterion to determine whether systems of Examples 1.2 and 1.3 are causal since neither of these two systems is LTI. It is purely coincidental that in both these cases the impulse response satisfies $h(n) = 0$ for $n < 0$. Consider a system given by the following input-output relationship:

$$y(n) = x(n+1) - \delta(n+1), \text{ for all integer } n.$$

The impulse response of this system is identically zero: $h(n) = 0$ for all integer n. However, the system is clearly noncausal. The causality criterion derived above is not applicable since the system is not LTI. On the other hand, consider the following system:

$$y(n) = x(n) + \delta(n+1), \text{ for all integer } n.$$

The impulse response of this system is $h(n) = \delta(n) + \delta(n+1)$ and therefore $h(-1) = 1$. Despite this fact, the system is causal. Again, the causality criterion is not applicable since the system is not LTI. In order to determine whether a non-LTI system is causal we have to use the definition of causality.

(b) BIBO Stability. A system S specified by an input-output relationship is said to be bounded-input-bounded-output (BIBO) stable if every bounded input produces a bounded output, i.e., if the fact that $\mathcal{M}(x)$ is finite implies that $\mathcal{M}(S[x])$ is finite.

Example 1.8. *To show that the system of Example 1.2 is not BIBO stable, let $x(n) = 1$ for all integer n. In this case,*

$$y(n) = \begin{cases} \sum_{k=0}^{n} 1 = n+1, & n \geq 0 \\ 0, & n < 0, \end{cases}$$

which is an unbounded signal. Recall that a signal is bounded if a fixed number L can be found such that all the sample values of the signal are between $-L$ and L. Clearly no such number exists for $y(n)$: as $n \to \infty$, $y(n) \to \infty$. We thus found a bounded input signal x which produced an unbounded response y. Therefore the system is not BIBO stable. ■

Example 1.9. *To show that the system of Example 1.3 is BIBO stable, note that* $|y(n)| = |2x(n)+3| \le 2|x(n)|+3$ *for all* n. *(Here we used the fact that* $|a+b| \le |a|+|b|$.*) Maximizing both sides of this inequality over* n, *we get:*

$$\max_{-\infty<n<\infty} |y(n)| \le 2 \max_{-\infty<n<\infty} |x(n)| + 3 = 2\mathcal{M}(x) + 3.$$

So, if x *is bounded (i.e., if* $\mathcal{M}(x)$ *is finite), then* y *is also bounded. Therefore, the system is BIBO stable.* ■

Theorem 1.1 (BIBO Stability for LTI Systems). *An LTI system is BIBO stable if and only if its impulse response is absolutely summable, i.e., if*

$$\sum_{k=-\infty}^{\infty} |h(k)| \text{ is finite.} \tag{1.5}$$

Proof. Let us first show the "only if" part. Suppose that an LTI system with impulse response h is BIBO stable, and let us show that its impulse response must be absolutely summable. Note that

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k).$$

Now consider the following input signal:

$$x(k) = \begin{cases} 1, & h(-k) \ge 0, \\ -1, & h(-k) < 0. \end{cases}$$

Substituting this particular input signal into the above expression for $y(0)$, we get that, in this case, the zeroth sample of the output is:

$$y(0) = \sum_{k=-\infty}^{\infty} |h(k)|.$$

Since x is bounded, and since our system is, by assumption, BIBO stable, y must be bounded, and, in particular, $y(0)$ must be a finite number. This implies Eq. (1.5).

Now let us prove the "if" part. Suppose that the impulse response h of an LTI system is absolutely summable, and let us show that the system is BIBO stable. By assumption, the sum of the absolute values of $h(n)$ is a finite number. Let us call this number L:

$$\sum_{k=-\infty}^{\infty} |h(k)| = L.$$

Consider an arbitrary bounded input x to the system. We have:

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(k)h(n-k) \right|$$

$$
\begin{aligned}
&\leq \sum_{k=-\infty}^{\infty} |x(k)| \cdot |h(n-k)| \\
&\leq \sum_{k=-\infty}^{\infty} \mathcal{M}(x)|h(n-k)| \\
&= \mathcal{M}(x) \sum_{k=-\infty}^{\infty} |h(n-k)| \\
&= \mathcal{M}(x)L.
\end{aligned}
$$

The absolute value of each output sample is therefore bounded from above by $\mathcal{M}(x)L$ which means that the output signal y is bounded. Since this holds for any bounded input, the system is BIBO stable. ■

This stability criterion, just like the causality criterion discussed before, is only applicable to LTI systems. In particular, we cannot use this criterion to judge whether the systems of Examples 1.2 and 1.3 are BIBO stable since neither of these two systems is LTI. It is purely coincidental that the impulse response of the first system (which is BIBO unstable) is not absolutely summable. The second system, specified by

$$y(n) = 2x(n) + 3, \quad \text{for all integer } n,$$

was shown to be stable in Example 1.9. Yet its impulse response, $h(n) = 2\delta(n) + 3$, is clearly not absolutely summable since $\sum_{n=-\infty}^{\infty} |h(n)| = 2 + \sum_{n=-\infty}^{\infty} 3 = \infty$. The BIBO stability criterion derived above is not applicable since the system is not LTI.

Moreover, consider a system specified by the following input-output relationship:

$$y(n) = \begin{cases} \prod_{k=0}^{n} x(k), & n > 0, \\ 0, & n \leq 0. \end{cases}$$

The impulse response of this system is identically zero: $h(n) = 0$ for all integer n, and therefore h is absolutely summable. However, the system is clearly not BIBO stable since the bounded constant input $x(n) = 2$ produces an unbounded response. The BIBO stability criterion derived above is not applicable since the system is not LTI. In order to determine whether a non-LTI system is BIBO stable, we must use the definition of BIBO stability.