Problem 1. Prove the following $z$-transform properties:

1. **Linearity:**
   \[ a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_1 X_1(z) + a_2 X_2(z). \]

2. **Delay:**
   \[ x(n - n_0) \leftrightarrow z^{-n_0} X(z). \]

3. **Modulation:**
   \[ z_0^n x(n) \leftrightarrow X \left( \frac{z}{z_0} \right) \]

4. **Differentiation in $z$-domain:**
   \[ nx(n) \leftrightarrow -z \frac{dX}{dz} \]
   **Hint.** Differentiate \( \sum x(n) z^{-n} \) termwise with respect to \( z \), and then multiply the result by \(-z\).

5. **Convolution:**
   \[ x * y(n) \leftrightarrow X(z)Y(z) \]
   **Hint.** Assume that you can change the order of summations in
   \[ \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x(k)y(n-k) \right) z^{-n}. \]

Problem 2. For each of the signals below, find the $z$-transform, specify the region of convergence of the $z$-transform, and sketch its poles and zeros in the complex plane.

(a) \( \delta(n) \).

(b) \( a^n u(n), a \neq 0 \).

(c) \(-a^n u(-n - 1), a \neq 0 \).

(d) \( na^n u(n), a \neq 0 \).
   **Hint.** Apply Property 4 form Problem 1 to Part (b).

(e) \(-na^n u(-n - 1), a \neq 0 \).

Problem 3. Consider the $z$-transform
\[ X(z) = \frac{-2.5z}{z^2 - 3.5z + 1.5} \]
Sketch the three different ROC’s that are possible for this $z$-transform. For each ROC, find the corresponding signal \( x(n) \).
**Problem 4.** (a) An LTI, causal system has the following transfer function:

\[
1 - \frac{1}{2}z^{-1} \\
\frac{1}{(1 - 16z^{-1})(1 + 0.3z^{-1})(1 - 7z^{-1})}
\]

Can this system be BIBO stable?

(b) An LTI, BIBO stable system has the following transfer function:

\[
\frac{438 + 2001z^{-1}}{(1 - 438z^{-1})(1 + z^{-1} + 8z^{-2} + 19z^{-3} + 23z^{-4})}
\]

Can this system be causal?

(c) A BIBO stable system has the following impulse response:

\[
h(n) = 1, \quad \text{for } -\infty < n < \infty.
\]

Can this system be LTI?

**Problem 5.** An LTI system has the following transfer function:

\[
\frac{2.96 - z^{-1}}{(1 - 0.2z^{-1})(1 - 0.1z^{-1})} \quad \text{ROC: } |z| > 0.2.
\]

Find the response of this system to the following input signal:

\[
x(n) = \left(\frac{1}{2}\right)^n, \quad -\infty < n < \infty.
\]

**Hint.** The easiest way is to use the fact that exponentials are eigenfunctions of LTI systems.

**Problem 6.** Suppose the z-transform of \(x(n)\) is:

\[
X(z) = e^z; \quad \text{region of convergence is the whole } z\text{-plane}.
\]

Find \(x(n)\).

**Hint.** The function \(X(z)\) possesses a Taylor series about the origin which converges to \(X(z)\) for all \(z\).