ECE 438
Exam 2 Solutions, 11/19/2003.

- This is a closed-book exam, but you are allowed one standard (8.5-by-11) sheet of notes. No calculators are allowed.
- Total number of points: 120. This exam counts for 20% of your final grade.
- You have 75 minutes to complete 6 problems.
- Be sure to fully and clearly explain all your answers.
- There will not be any discussion of grades. All re-grade requests must be submitted in writing, as stated in the course information handout.

Score                   Grader
1________  ________
2________  ________
3________  ________
4________  ________
5________  ________
6________  ________

Total score:_______
Problem 1 (15 points). Let \( X(n) \) be a wide-sense stationary sequence of random variables with zero mean and autocorrelation function \( r_{XX}(n) = \delta(n) \). Suppose this sequence is filtered to generate the output sequence

\[
Y(n) = \frac{2}{3} X(n) + \frac{1}{3} X(n - 1)
\]

a. (5 points) Find the mean of the sequence \( Y(n) \).

b. (5 points) Find the cross-correlation \( c_{XY}(n) \) between the sequences \( X(n) \) and \( Y(n) \).

c. (5 points) Find the autocorrelation \( r_{YY}(n) \) of the sequence \( Y(n) \).

Fully and clearly explain all your answers.

Solution.

a. \( E[Y(n)] = \frac{2}{3} E[X(n)] + \frac{1}{3} E[X(n - 1)] = 0 \)

b. 

\[
c_{XY}(n) = h * r_{XX}(n) = h(n) = \frac{2}{3} \delta(n) + \frac{1}{3} \delta(n - 1).
\]

c. 

\[
r_{YY} = h_\_ * c_{XY}(n)
= \left( \frac{2}{3} \delta(n) + \frac{1}{3} \delta(n + 1) \right) * \left( \frac{2}{3} \delta(n) + \frac{1}{3} \delta(n - 1) \right)
= \frac{4}{9} \delta(n) + \frac{4}{9} \delta(n - 1) + \frac{2}{9} \delta(n + 1) + \frac{1}{9} \delta(n)
= \frac{2}{9} \delta(n + 1) + \frac{5}{9} \delta(n) + \frac{2}{9} \delta(n - 1).
\]
Problem 2 (15 points). Consider the following random sequence:

\[ X(n) = \cos(\omega_0 n + \Theta), \]

where \( \omega_0 \) is a non-random constant, and \( \Theta \) is a random variable, uniformly distributed between 0 and \( 2\pi \):

\[ f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi \\ 0, & \text{otherwise} \end{cases} \]

a. (5 points) Compute the mean sequence, \( E[X(n)] \).

b. (5 points) Compute the autocorrelation sequence, \( r_{XX}(m, n) \). You may find the following formula helpful: \( \cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta)) \).

c. (5 points) Is the sequence \( X(n) \) wide-sense stationary?

Fully and clearly explain all your answers.

Solution.

a.

\[ E[X(n)] = \int_{-\infty}^{\infty} \cos(\omega_0 n + \theta) f_\Theta(\theta) d\theta \\
= \int_{0}^{2\pi} \cos(\omega_0 n + \theta) \frac{1}{2\pi} d\theta \\
= \sin(\omega_0 n + \theta) \bigg|_{\theta=2\pi}^{\theta=0} \\
= \frac{1}{2\pi} (\sin(\omega_0 n + 2\pi) - \sin(\omega_0 n)) = 0. \]

b.

\[ r_{XX}(m, n) = E[X(m)X(n)] \\
= \int_{-\infty}^{\infty} \cos(\omega_0 n + \theta) \cos(\omega_0 m + \theta) f_\Theta(\theta) d\theta \\
= \int_{0}^{2\pi} \frac{1}{2} [\cos(\omega_0 (n + m) + 2\theta) + \cos(\omega_0 (n - m))] \frac{1}{2\pi} d\theta \\
= \frac{1}{8\pi} \sin(\omega_0 (n + m) + 2\theta) \bigg|_{\theta=2\pi}^{\theta=0} + \frac{1}{4\pi} \cos(\omega_0 (n - m)) \bigg|_{\theta=2\pi}^{\theta=0} \\
= \frac{1}{2} \cos(\omega_0 (n - m)). \]

c. The autocorrelation function \( r_{XX}(m, n) \) depends only on \( |n - m| \). Also, \( E[X(n)] = 0 \). Therefore, \( X(n) \) is wide-sense stationary.
Problem 3 (20 points). Consider the following system, S:

\[ y(n) = x(n) + \frac{1}{2} x(n-1) \quad \text{for} \quad -\infty < n < \infty, \]

where \( x(n) \) is the input to the system, and \( y(n) \) is its output.

a. (5 points) Find the transfer function \( H(z) \) of this system.

b. (5 points) Find the transfer function of the inverse system, \( S_1 \). (Recall that if the response of \( S \) to \( x(n) \) is \( y(n) \), then the response of the inverse system \( S_1 \) to \( y(n) \) must be \( x(n) \).)

c. (5 points) Assuming that \( S_1 \) is causal, is \( S_1 \) BIBO stable?

d. (5 points) Find the impulse response of \( S_1 \) assuming that \( S_1 \) is a causal system.

Fully and clearly explain all your answers.

Solution.

a. \( H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1}. \)

b. The transfer function of the inverse system is

\[ \frac{1}{H(z)} = \frac{1}{1 + \frac{1}{2}z^{-1}}. \]

c. The transfer function of the inverse system has one pole at \( z = -1/2 \). Since it is causal, the ROC includes the unit circle, which means that the system is BIBO stable.

d. The impulse response is the inverse \( z \)-transform of the transfer function. Since it is given that the system is causal, the impulse response is

\[ \left( -\frac{1}{2} \right)^n u(n). \]
Problem 4 (30 points). Consider the $z$-transform

$$H(z) = \frac{2 - \frac{5}{6}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}.$$  

a. (6 points) Find the poles of $H(z)$.

b. (6 points) Sketch the three different ROC’s that are possible for this $z$-transform.

c. (18 points) For each ROC,

   i find the corresponding signal $h(n)$,
   ii determine whether or not the LTI system with impulse response $h(n)$ is causal,
   iii determine whether or not the LTI system with impulse response $h(n)$ is BIBO stable.

Fully and clearly explain all your answers.

Solution.

a. The denominator is $(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})$, and therefore the poles are $\frac{1}{2}$ and $\frac{1}{3}$.

b. The three possible ROCs are: the inside of the circle with radius 1/3, the ring with radii 1/3 and 1/2, and the outside of the circle with radius 1/2, see Fig. 1.

c. From

$$H(z) = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}} = \frac{(A + B)(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})},$$

we get: $A = B = 1$.

For the ROC in Fig. 1(a), the parts corresponding to both poles are left-sided since the ROC is inside both poles:

$$h(n) = -u(-n - 1) \left( \left( \frac{1}{3} \right)^n + \left( \frac{1}{2} \right)^n \right).$$
The corresponding system is noncausal since \( h(n) \neq 0 \) for \( n < 0 \); it is not BIBO stable since the ROC does not contain the unit circle.

For the ROC in Fig. 1(b), the part corresponding to the pole at 1/3 is right-sided, and the part corresponding to the pole at 1/2 is left-sided:

\[
h(n) = \left( \frac{1}{3} \right)^n u(n) - \left( \frac{1}{2} \right)^n u(-n - 1).
\]

The system is noncausal and not BIBO stable for the same reasons as in the first case.

For the ROC in Fig. 1(c), both parts are right-sided,

\[
h(n) = \left( \frac{1}{3} \right)^n + \left( \frac{1}{2} \right)^n u(n).
\]

In this case, \( h(n) = 0 \) for \( n < 0 \), and therefore the system is causal. Moreover, the unit circle is in the ROC, and therefore the system is BIBO stable.
Problem 5 (25 points). Consider the following AR process $S(n)$:

$$S(n) = \frac{1}{3}S(n - 1) + \frac{1}{6}S(n - 2) + X(n),$$

where $X(n)$ is a zero-mean white noise process with variance $\sigma_x^2 = 1$.

a. (5 points) Find $c_{XS}(-1)$.

b. (5 points) Find $c_{XS}(0)$.

c. (5 points) Prove that

$$r_{ss}(0) = \frac{1}{3}r_{ss}(1) + \frac{1}{6}r_{ss}(2) + 1. \quad (1)$$

d. (10 points) Recall that, if $W(n)$ is a $p$-th order AR process with coefficients $a_k$, then

$$r_{WW}(m) = \sum_{k=1}^{p} a_k r_{WW}(m - k), \quad \text{for any integer } m > 0.$$

Using this fact together with Eq. (1) for the process $S(n)$ given above, find $r_{ss}(0)$, $r_{ss}(1)$, and $r_{ss}(2)$.

Fully and clearly explain all your answers.

Solution.

a. Since $X$ is white and therefore uncorrelated with past values of $S$, $c_{XS}(-1) = E[X(n)S(n-1)] = 0$.

b. $c_{XS}(0) = E[X(n)S(n)] = \frac{1}{3}c_{XS}(-1) + \frac{1}{6}c_{XS}(-2) + \sigma_x^2 = \sigma_x^2 = 1$.

c. Multiplying both sides of the system equation by $S(n)$ and taking expectations, we get:

$$E[(S(n))^2] = \frac{1}{3}E[S(n - 1)S(n)] + \frac{1}{6}E[S(n - 2)S(n)] + E[X(n)S(n)]$$

$$r_{ss}(0) = \frac{1}{3}r_{ss}(1) + \frac{1}{6}r_{ss}(2) + c_{XS}(0),$$

which is the same as Eq. (1) since $c_{XS}(0) = 1$.

d. In our case, $p = 2$, $a_1 = \frac{1}{3}$, and $a_2 = \frac{1}{6}$. Writing out the given equation for $m = 1$, we get:

$$r_{ss}(1) = \frac{1}{3}r_{ss}(0) + \frac{1}{6}r_{ss}(-1) = \frac{1}{3}r_{ss}(0) + \frac{1}{6}r_{ss}(1),$$

i.e., $r_{ss}(1) = \frac{2}{5}r_{ss}(0)$. Writing out the given equation for $m = 2$, we get:

$$r_{ss}(2) = \frac{1}{3}r_{ss}(1) + \frac{1}{6}r_{ss}(0) = \frac{1}{3} \cdot \frac{2}{5}r_{ss}(0) + \frac{1}{6}r_{ss}(0) = \left( \frac{4}{30} + \frac{5}{30} \right) r_{ss}(0) = \frac{3}{10}r_{ss}(0).$$
Finally, using all this in Eq. (1), we have:

\[
\begin{align*}
    r_{ss}(0) &= \frac{1}{3} \cdot \frac{2}{5} r_{ss}(0) + \frac{1}{6} \cdot \frac{3}{10} r_{ss}(0) + 1 \\
    &= \left( \frac{2}{15} + \frac{3}{60} \right) r_{ss}(0) + 1 \\
    &= \frac{8 + 3}{60} r_{ss}(0) + 1 \\
    49 \cdot \frac{r_{ss}(0)}{60} &= 1 \\
    r_{ss}(0) &= \frac{60}{49} \\
    r_{ss}(1) &= \frac{2}{5} r_{ss}(0) = \frac{24}{49} \\
    r_{ss}(2) &= \frac{3}{10} r_{ss}(0) = \frac{18}{49}
\end{align*}
\]
Problem 6 (15 points). Suppose that $Y_1, Y_2, \ldots, Y_N$ are iid Gaussian random variables with mean $\mu$ and variance 1. It is known that $\mu$ is 0, 1, or 2 (but whatever $\mu$ is, it is the same for all $N$ random variables $Y_1, Y_2, \ldots, Y_N$). Let

$$W = \frac{1}{N} \sum_{n=1}^{N} Y_n.$$

a. (7 points) Suppose that the individual random variables $Y_1, Y_2, \ldots, Y_N$ are not observed, and that $W$ is observed. Find the maximum likelihood estimate of $\mu$ based on observing $W$.

b. (8 points) Now suppose that $Y_1, Y_2, \ldots, Y_N$ are observed. Find the maximum likelihood estimate of $\mu$ based on these observations.

Fully and clearly explain all your answers.

Solution.

a. The random variable $W$ is Gaussian with mean $\mu$ and variance $1/N$, and therefore we maximize the following quantity:

$$\frac{\sqrt{N}}{\sqrt{2\pi}} e^{-\frac{(w-\mu)^2}{2/N}}$$

with respect to $\mu$, which is the same as minimizing $|w - \mu|$. The answer is therefore that we should choose the value for $\mu$ among the three possible values (0,1,2) which is the closest to the observation $w$:

$$\hat{\mu}(w) = \begin{cases} 
0 & w < 1/2 \\
1 & 1/2 \leq w \leq 3/2 \\
2 & w > 3/2 
\end{cases} \tag{2}$$

b. We now need to maximize

$$\prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_n-\mu)^2}{2/N}},$$

which is the same as minimizing

$$\sum_{n=1}^{N} (y_n - \mu)^2 = \sum_{n=1}^{N} y_n^2 - 2\mu \sum_{n=1}^{N} y_n + N\mu^2,$$

or

$$\mu^2 - \mu \frac{2}{N} \sum_{n=1}^{N} y_n = \mu^2 - 2\mu w,$$

where $w$ is as in Part a. Thus, if

$$0^2 - 2 \cdot 0 \cdot w \geq 1^2 - 2 \cdot 1 \cdot w,$$

i.e. $w \geq 1/2$, 

$$w \geq 1/2,$$
then we prefer $\mu = 1$ to $\mu = 0$; otherwise, we prefer $\mu = 0$ to $\mu = 1$. Similarly, we prefer $\mu = 1$ to $\mu = 2$ if

\[
2^2 - 2 \cdot 2 \cdot w \geq 1^2 - 2 \cdot 1 \cdot w,
\]

\[
3 \geq 2w,
\]

\[
3/2 \geq w,
\]

and prefer $\mu = 2$ to $\mu = 1$ otherwise. Finally, to decide between 0 and 2, we compare $0^2 - 2 \cdot 0 \cdot w = 0$ and $2^2 - 2 \cdot 2 \cdot w = 4 - 4w$, and so if $w > 1$ we prefer $\mu = 2$ and otherwise we prefer $\mu = 1$. Putting these together, we obtain exactly the same decision rule (2) as in Part a.