ECE 438
Exam 1 Solutions, 10/08/2003.

• This is a closed-book exam, but you are allowed one standard (8.5-by-11) sheet of notes. No calculators are allowed.

• Total number of points: 120. This exam counts for 20% of your final grade.

• You have 75 minutes to complete 5 problems.

• Be sure to fully and clearly explain all your answers.

• There will not be any discussion of grades. All re-grade requests must be submitted in writing, as stated in the course information handout.

Score          Grader
1_______        _______
2_______        _______
3_______        _______
4_______        _______
5_______        _______

Total score:_______
Problem 1 (50 points). Let
\[ x(n) = \begin{cases} 
1, & \text{if } n = 0, 1 \\
0, & \text{otherwise}
\end{cases} \]

a. (5 points). Find the energy of \( x \).
b. (5 points). Is \( x \) bounded?
c. (5 points). Is \( x \) a finite-duration signal?
d. (5 points). Let \( y \) be the discrete-time convolution of \( x \) and \( x \): 
\[ y(n) = x * x(n). \] Find \( y(n) \).
e. (5 points). Find the discrete-time Fourier transform of \( x \).
f. (5 points). Let \( S \) be a discrete-time LTI system whose response to the input signal \( \delta(n) \) is the signal \( x(n) \) given above. Decide whether the system \( S \) cannot, possibly could, or must be BIBO stable.
g. (5 points). Let \( x_u(n) \) be the result of upsampling \( x(n) \) by a factor of 2. Find \( x_u(n) \).
h. (5 points). Let \( x_d(n) \) be the result of downsampling \( x(n) \) by a factor of 3. Find \( x_d(n) \).
i. (5 points). Let \( w(n) = x(n) - x(n - 1) \). Find \( w(n) \).
j. (5 points). Let \( v(n) = x(n)x(n - 1) \). Find \( v(n) \).

Fully and clearly explain all your answers.

Solution.

a. The energy of \( x \) is \( 1^2 + 1^2 = 2 \).
b. Since \( |x(n)| < 2 \) for all \( n \), \( x \) is bounded.
c. Since \( x(n) = 0 \) outside of the interval \([0, 1]\), \( x \) is a finite-duration signal.
d. Since \( x(n) = \delta(n) + \delta(n - 1) \), its convolution with itself is:
\[ y(n) = x(n) * (\delta(n) + \delta(n - 1)) = x(n) + x(n - 1) = \delta(n) + 2\delta(n - 1) + \delta(n - 2). \]
e. The DTFT of \( x \) is \( 1 + e^{-j\omega} \).
f. The signal \( x \) is absolutely summable. Since it is the impulse response of the system \( S \), and since \( S \) is LTI, \( S \) must be BIBO stable.
g. Inserting a zero after each sample of \( x \) results in: \( x_u(n) = \delta(n) + \delta(n - 2) \).
h. Retaining only the points which are integer multiples of 3 results in: \( x_d(n) = \delta(n) \).
i. \( w(n) = \delta(n) - \delta(n - 2) \).
j. This product is zero for all values of \( n \) except \( n = 1 \): \( v(n) = \delta(n - 1) \).
Problem 2 (30 points) Consider two discrete-time filters with impulse responses $h_1$ and $h_2$ defined as follows:

$$h_1(n) = \begin{cases} \frac{1}{2}, & \text{if } n = -1, 1 \\ 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$h_2(n) = \begin{cases} \frac{1}{2}, & \text{if } n = -1, 1 \\ -1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

a. (5 points) Find the frequency response $H_1(e^{j\omega})$ of the first filter.
b. (5 points) Plot the magnitude response $|H_1(e^{j\omega})|$ of the first filter.
c. (5 points) Is the first filter a lowpass, bandpass, or highpass filter?
d. (5 points) Find the frequency response $H_2(e^{j\omega})$ of the second filter.
e. (5 points) Plot the magnitude response $|H_2(e^{j\omega})|$ of the second filter.
f. (5 points) Is the second filter a lowpass, bandpass, or highpass filter?

Fully and clearly explain all your answers.

Solution.

a,b,c. $H_1(e^{j\omega}) = \frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega} = 1 + \cos \omega$. The magnitude response is shown above. This filter is lowpass since it passes the low frequencies (near $\omega = 0$) and stops high frequencies (near $\omega = \pm \pi$).

d,e,f. $H_2(e^{j\omega}) = \frac{1}{2}e^{j\omega} - 1 + \frac{1}{2}e^{-j\omega} = \cos \omega - 1$. The magnitude response is shown above. This filter is highpass since it passes the high frequencies and stops low frequencies.
Problem 3 (15 points). Consider the ideal sampling system studied in class and depicted above. Viewing $x_c(t)$ as the input to this continuous-time system and $x_s(t)$ as the output, answer the following questions.

a. (5 points). Is this system linear?

b. (5 points). Is this system time-invariant?

c. (5 points). Is this system causal?

Fully and clearly explain all your answers.

Solution.

a. Since both the low-pass filter and the multiplication by a fixed signal $s(t)$ are linear, the overall system is also linear.

b. Sampling is inherently a time-varying procedure: if you shift a signal by some amount which is smaller than the sampling period, the samples of the shifted signal may be completely different from the samples of the original signal. The overall system is therefore time-varying.

To verify this, we can take $f_s = 1$ and $x_c(t) = \sin(\pi t/2)$. This signal will go unchanged through the lowpass filter since its frequency, $1/4$, is below the cutoff frequency of the filter. The samples will be $\ldots, 0, 1, 0, -1, \ldots$. If we now take $x'_c(t) = x_c(t - \pi) = \sin(\pi t/2 - \pi^2/2)$, the value of a sample of this signal can never be zero since $\pi n/2 - \pi^2/2$ can never be an integer multiple of $\pi$. Therefore, the samples of the shifted signal are not equal to a shift of the samples of the original signal, and so the system is time-varying.

c. As shown in the class notes, the impulse response of the ideal low-pass filter is a sinc, which is not equal to zero for $t < 0$. Therefore, the low-pass filter alone is not causal, and the overall system is not causal either.
Problem 4 (15 points). The CT signal $x_c(t)$ depicted above is periodic with fundamental period 0.04. This signal is ideally sampled at 100 Hz (without prefiltering) to obtain a DT signal $x(n)$.

a. (5 points) Sketch $x(n)$. Carefully label the axes. Do not forget that $x(n)$ is a discrete-time signal.

b. (5 points) Find such coefficients $X_0, X_1, X_2,$ and $X_3$ that

$$x(n) = X_0 + X_1 e^{j\pi n/2} + X_2 e^{j\pi n} + X_3 e^{j3\pi n/2}$$

for all integer $n$.

c. (5 points) Suppose that a CT signal $y(t)$ is reconstructed from $x(n)$ using the ideal scheme considered in class. Specifically, $x(n)$ is converted into a CT sequence of impulses,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n) \delta \left( t - \frac{n}{f_s} \right),$$

and then $x_s(t)$ is filtered with an ideal low-pass filter whose frequency response is:

$$H(f) = \begin{cases} A, & -\frac{f_s}{2} \leq f \leq \frac{f_s}{2} \\ 0, & \text{otherwise}, \end{cases}$$

where $f_s = 100$ Hz and $A$ is a constant such that $y(0.01) = 1$. Find $y(t)$ and sketch it.
Fully and clearly explain all your answers.

Solution.

a. The sampled signal is shown above.

b. Note that

\[ x(n) = \sin\left(\frac{\pi n}{2}\right) = \frac{e^{j\pi n/2} - e^{-j\pi n/2}}{2j} = \frac{e^{j\pi n/2} - e^{(2\pi - \pi/2)n}}{2j} = \frac{1}{2j} e^{j\pi n/2} - \frac{1}{2j} e^{j\pi n/2}, \]

and therefore, \( X_1 = \frac{1}{2j}, \) \( X_3 = -\frac{1}{2j}, \) and \( X_0 = X_2 = 0. \)

c. As shown in class, the single DT sinusoid of frequency \( \omega = \pi/2 \) will be reconstructed as a single CT sinusoid of frequency \( f = f_s/4 = 25 \text{ Hz}. \) Since we are given that \( y(0.01) = 1, \) we have:

\[ y(t) = \sin(50\pi t), \]

as shown above.
Problem 5 (10 points). A CT signal $y(t)$ is the CT convolution of CT signals $x(t)$ and $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau.$$ 

a. (5 points). Is it necessarily true that

$$y(n) = \sum_{k=\infty}^{-\infty} x(k) h(n - k) \text{ for all integer } n?$$

If your answer is yes, prove it; otherwise, provide a counterexample.

b. (5 points). Will your answer to Part a be different if it is given, in addition, that both $x(t)$ and $h(t)$ are bandlimited, with the highest frequency $f = 0.4$ Hz? Explain.

Fully and clearly explain all your answers. For this problem, no credit will be given for answers without correct explanations.

Solution.

a. The answer is no. Let

$$x(t) = h(t) = \begin{cases} 
1, & 1/4 \leq t \leq 3/4 \\
0, & \text{otherwise.}
\end{cases}$$

Then

$$y(1) = \int_{-\infty}^{\infty} x(\tau) h(1 - \tau) \, d\tau = \int_{1/4}^{3/4} d\tau = 1/2.$$ 

But $x(n) = h(n) = 0$ for all integer $n$, and therefore the

$$\sum_{k=\infty}^{-\infty} x(k) h(n - k) = 0 \text{ for all integer } n.$$

b. The reason for the negative answer in Part a is aliasing. Let us call the samplings of $x$, $h$, and $y$ at integer points $x_d$, $h_d$, and $y_d$, respectively:

$$x_d(n) = x(n), \quad h_d(n) = h(n), \quad y_d(n) = y(n).$$

Let us denote $w(n) = x_d * h_d(n)$. Note that since $y$ is a CT convolution of $x$ and $h$,

$$Y(f) = X(f)H(f). \quad (1)$$

On the other hand, using what was derived in class,

$$Y_d(e^{j\omega}) = \sum_{n} Y\left(\frac{\omega}{2\pi} - n\right) \quad \text{Eq. (1)} \quad \sum_{n} X\left(\frac{\omega}{2\pi} - n\right) \quad H\left(\frac{\omega}{2\pi} - n\right). \quad (2)$$
Using
\[ X_d(e^{j\omega}) = \sum_n X \left( \frac{\omega}{2\pi} - n \right) \quad \text{and} \quad H_d(e^{j\omega}) = \sum_n H \left( \frac{\omega}{2\pi} - n \right), \]
we have:
\[ W(e^{j\omega}) = X_d(e^{j\omega})H_d(e^{j\omega}) = \sum_n X \left( \frac{\omega}{2\pi} - n \right) \sum_n H \left( \frac{\omega}{2\pi} - n \right). \tag{3} \]

In order for (2) and (3) to be equal, the product in (3) must not have any cross-terms, i.e., the \( n \)-th replica of \( X \) must only overlap with the \( n \)-th replica of \( H \) and not with any others. This may not happen if the signals are sampled at below the Nyquist rate, as we saw in Part a. However, this will always happen if the signals are sampled at above the Nyquist rate. If both \( x \) and \( h \) only have frequencies below 0.4 Hz, the Nyquist rate is 0.8 Hz which is smaller than the sampling rate of 1 Hz.

Answer: the answer to Part a is yes if both signals do not have frequencies above 0.4 Hz.