Problem 1. (a) Prove that the following DT sinusoid is not periodic: \( x(n) = \sin(2\pi^2 n) \).
(b) Consider the following DT sinusoid: \( x(n) = \sin(3\pi n/4) + 0.3\cos(11\pi n/19) \). Is it periodic? If so, find a period of this sinusoid. Do not use Matlab.

Problem 2. Each system below is defined by an input-output relation (the response to the input signal \( x \) is signal \( y \)). For each system, determine whether or not it is:

(i) linear,
(ii) time-invariant,
(iii) causal,
(iv) BIBO stable.

For each of the above properties, if you think it holds, prove it. Otherwise, find a counter-example. In addition, find the unit impulse response.

(a) \( y(n) = x(n - 10)x(n - 9) \cdots x(n - 2)x(n - 1) \) for all integer \( n \).
(b) \( y(n) = 0.25x(n - 1) + 0.5x(n) + 0.25x(n + 1) \) for all integer \( n \).
(c) \( y(n) = x(n) + 2(n + 2) \) for all integer \( n \).

Problem 3. For a fixed integer \( N \geq 1 \), we let \( \mathbb{R}^N \) be the space of all \( N \)-dimensional real vectors. In other words, the elements of \( \mathbb{R}^N \) are all vectors \( x \) of the form:

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix},
\]

where \( x_1, x_2, \ldots, x_N \) are real numbers. As mentioned in class, we will be using \( \mathbb{R}^N \) to model signals of duration \( N \) and also periodic signals with period \( N \). This will help develop intuition about signals by viewing them as geometric objects. In this problem, you will investigate different ways of introducing the notion of length in \( \mathbb{R}^N \).

A function \( \ell \) from \( \mathbb{R}^N \) to \( \mathbb{R} \) is called a length function (or a norm) if it satisfies the following four properties:
A. \( \ell(x) \geq 0 \) for every vector \( x \) in \( \mathbb{R}^N \).

B. \( \ell(x) = 0 \) if and only if \( x \) is the zero vector, i.e. \( x = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \).

C. \( \ell(ax) = |a|\ell(x) \) for every real number \( a \) and every vector \( x \) in \( \mathbb{R}^N \).

D. \( \ell(x + y) \leq \ell(x) + \ell(y) \) for any two vectors \( x \) and \( y \) in \( \mathbb{R}^N \).

(a) Prove that the function \( A \) defined by
\[
A(x) = \sum_{n=1}^{N} |x_n|
\]
is a norm. In order to do this, you need to show that \( A \) satisfies the four properties A,B,C,D listed above.

(b) Prove that the magnitude function \( M \) defined by
\[
M(x) = \max_{1 \leq n \leq N} |x_n|
\]
is a norm.

**Problem 4.** The following pair of discrete-time difference equations has been used as a model for certain types of population growth:
\[
\begin{align*}
p(n + 1) &= 2q(n) \\
q(n + 1) &= \mu q(n)(1 - p(n)),
\end{align*}
\]
for \( n \geq 0 \). A model in this form, namely one that comprises a set of coupled first-order equations, is said to be in *state-space form*, and the values of the *state variables* \( p(n) \) and \( q(n) \) constitute the *state* of the model at time \( n \).

(a) Is this system linear? Time-invariant?

(b) Find the *equilibrium points* of the model, i.e. values \( \bar{p} \) and \( \bar{q} \) of \( p(n) \) and \( q(n) \) respectively such that \( p(n + 1) = p(n) = \bar{p} \) and \( q(n + 1) = q(n) = \bar{q} \).

(c) Assume \( p(0) = \frac{1}{2} \) and \( q(0) = \frac{1}{4} \). Compute and plot the state vector \( (p(n), q(n)) \) for \( 0 \leq n \leq 500 \) for \( \mu = 1.8, 1.9, 2.0, 2.1, \) and \( 2.2 \) using Matlab (five separate plots). Describe the behavior of the state trajectories as \( n \) increases, for each of these values of \( \mu \). How do these trajectories relate to the equilibrium points found in (b)?

**Note:** For plotting, you can store the computed \( p(n) \) and \( q(n) \) in vectors \( (p \) and \( q) \) and use the following Matlab command:
\[
\text{plot}(p, q, 'b--', p, q, 'ro', p(501), q(501), 'y*');
\]