EE 438  
Final Exam Solutions, 12/10/2002.

- This is a closed-book exam, but you are allowed three standard (8.5-by-11) sheets of notes. No calculators are allowed.
- Total number of points: 200. This exam counts for 20% of your final grade.
- You have two hours to complete SEVEN problems.
- For all problems except Problem 1, be sure to fully and clearly explain all your answers.
- There will not be any discussion of grades. All re-grade requests must be submitted in writing.

1_______
2_______
3_______
4_______
5_______
6_______
7_______

Total score:_______
Problem 1 (60 points). For this problem, you only need to provide the answers. No partial credit will be given except in the situation described after Part f. (However, we reserve the right to deduct points for correct answers accompanied by wrong explanations.)

Consider the following discrete-time system, $S$:

$$y(n) = x(n) + \frac{1}{2} x(n - 1) \text{ for } -\infty < n < \infty,$$

where $x(n)$ is the input to the system, and $y(n)$ is its output.

a. (10 points) Find the unit step response of the system (i.e., the response when the input signal is the unit step).

b. (10 points) Find the transfer function $H(z)$ of this system.

c. (10 points) Find the transfer function of the causal inverse system, $S_1$. (Recall that if the response of $S$ to $x(n)$ is $y(n)$, then the response of the inverse system $S_1$ to $y(n)$ must be $x(n)$.)

d. (10 points) Is $S_1$ BIBO stable?

e. (10 points) Is $S_1$ a low-pass filter, a high-pass filter, or a band-pass filter?

f. (10 points) Find the impulse response of $S_1$. (If you cannot solve Part c., do Parts d.,e.,f. for system $S$. You will then get 70% of the credit for Parts d.,e.,f. If you do solve Part c., then you must do Parts d.,e.,f. for the system you obtain in Part c.)

Solution. a. If $x(n) = u(n)$, then $y(n) = u(n) + \frac{1}{2} u(n - 1) = \delta(n) + \frac{3}{2} u(n - 1)$.

b. $H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2} z^{-1}$.

c. Denote the transfer function of the inverse system by $H_i(z)$. It is

$$H_i(z) = \frac{1}{H(z)} = \frac{1}{1 + \frac{1}{2} z^{-1}}.$$  

d. The transfer function of the inverse system has one pole at $z = -\frac{1}{2}$; it is given that we are considering the causal inverse. The ROC thus includes the unit circle, which means that the system is BIBO stable.

e. The magnitude response of the system $S$ is

$$|1 + \frac{1}{2} e^{-j\omega}| = \sqrt{(1 + \frac{1}{2} e^{-j\omega})(1 + \frac{1}{2} e^{j\omega})} = \sqrt{1.25 + \cos \omega}.$$  

This is a low-pass filter (see Fig. 1,a), therefore, the inverse system, whose magnitude response is

$$\frac{1}{\sqrt{1.25 + \cos \omega}},$$

is a high-pass filter (see Fig 1,b).

f. The impulse response is the inverse z-transform of the transfer function. Since it is given that the system is causal, we have:

$$h(n) = \left(-\frac{1}{2}\right)^n u(n).$$
Problem 2 (20 points). Compute the quantity
\[
\sum_{n=0}^{N-1} x_1(n)x_2(n)
\]
for the following pairs of sequences. For (ii), assume that \(N > 8\).

(i) \(x_1(n) = \cos \left( \frac{2\pi}{N} n \right) \quad 0 \leq n \leq N - 1\)
\(x_2(n) = \sin \left( \frac{2\pi}{N} n \right) \quad 0 \leq n \leq N - 1\)

(ii) \(x_1(n) = \delta(n) + \delta(n - 8) \quad 0 \leq n \leq N - 1\)
\(x_2(n) = u(n) - u(n - N) \quad 0 \leq n \leq N - 1\)

Fully and clearly explain your answers.

Solution. (i) It is easily checked that, for \(N = 1\) and \(N = 2\), the answer is 0. For \(N \geq 3\),
\[
\sum_{n=0}^{N-1} x_1(n)x_2(n) = \frac{1}{4j} \sum_{n=0}^{N-1} \left( e^{j\frac{2\pi}{N} n} + e^{-j\frac{2\pi}{N} n} \right) \left( e^{j\frac{2\pi}{N} n} - e^{-j\frac{2\pi}{N} n} \right)
\]
\[
= \frac{1}{4j} \sum_{n=0}^{N-1} \left( e^{j\frac{4\pi}{N} n} - e^{-j\frac{4\pi}{N} n} \right)
\]
\[
= \frac{1}{4j} \left( \frac{1 - e^{j4\pi}}{1 - e^{j2\pi}} - \frac{1 - e^{-j4\pi}}{1 - e^{-j2\pi}} \right)
\]
\[
= 0.
\]

(ii) Note that \(x_2 = 1\) for \(0 \leq n \leq N - 1\). Therefore,
\[
\sum_{n=0}^{N-1} x_1(n)x_2(n) = 1 \cdot 1 + 1 \cdot 1 = 2.
\]
Problem 3 (10 points). Suppose that $X$ and $Y$ are zero-mean real-valued random variables with variances $\sigma_x^2 = \sigma_y^2 = 1$, and correlation $E[XY] = \frac{1}{2}$. Find real numbers $a$ and $b$ such that the following three conditions hold simultaneously:

(i) $X$ and $aX + bY$ are uncorrelated;
(ii) the variance of $aX + bY$ is 1;
(iii) $a > 0$.

Fully and clearly explain your answer.

Solution. Since $X$ and $aX + bY$ are uncorrelated, we must have $E[X(aX+bY)] = E[X]E[aX+bY] = 0$, i.e., $aE[X^2] + bE[XY] = 0$, which means that $a + 0.5b = 0$. Similarly, Condition (ii) results in:

$$a^2 + ab + b^2 = 1.$$ 

Substituting $b = -2a$ into this equation, we get:

$$a^2 - 2a^2 + 4a^2 = 1.$$ 

Since $a > 0$, this means that $a = \frac{1}{\sqrt{3}}$, and therefore $b = -\frac{2}{\sqrt{3}}$. 

\[4\]
**Problem 4 (30 points).** Suppose that $X(n)$ is a wide-sense stationary sequence of random variables each of which has mean zero and variance 1.

a. (15 points) $X(n)$ is quantized to obtain $Y(n)$. Assume that the quantization error may be modeled as a random variable which is uniformly distributed between $-\Delta/2$ and $\Delta/2$. Determine an expression for the average signal-to-noise ratio in dB, defined as

$$\text{SNR} = 10 \log_{10} \frac{E[(X(n))^2]}{E[(Y(n) - X(n))^2]}.$$ 

b. (15 points) Suppose that, if a 10-bit uniform quantizer is used, the SNR is equal to 60. Find the SNR for an 11-bit uniform quantizer. (In both cases, assume the maximum possible number of quantization levels. Assume that the quantizer range is the same for the two quantizers. You may use $\log_{10} 4 \approx 0.6$.)

**Fully and clearly** explain your answers.

**Solution.**

a. Denote $W = X(n) - Y(n)$. It is given that the PDF of $W$ is

$$f_W(w) = \begin{cases} \frac{1}{\Delta}, & -\Delta/2 \leq w \leq \Delta/2 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$E(W^2) = \int_{-\infty}^{\infty} w^2 f_W(w) \, dw = \int_{-\Delta/2}^{\Delta/2} \frac{w^2}{\Delta} \, dw = \frac{w^3}{3\Delta} \bigg|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}.$$ 

Since $X(n)$ has mean 0 and variance 1, $E[(X(n))^2] = 1$. Therefore,

$$\text{SNR} = 10 \log_{10} \frac{12}{\Delta^2}.$$ 

b. For a B-bit quantizer, the maximum possible number of quantization levels is $2^B$. The quantizer range—which is the same in both cases—is equal to the number of quantization levels times the quantization step:

$$2^{10} \Delta_{10} = 2^{11} \Delta_{11},$$

where $\Delta_{10}$ and $\Delta_{11}$ are the quantization steps for the 10-bit and 11-bit quantizers, respectively. Therefore,

\[ \text{SNR}_{11} = 10 \log_{10} \frac{12}{\Delta_{11}^2} = 10 \log_{10} \frac{12}{\frac{2^{10}}{2\pi} \Delta_{10}^2} = 10 \log_{10} \frac{12}{\frac{2^{10}}{2\pi} \Delta_{10}^2} = 10 \log_{10} \frac{12}{\Delta_{10}^2} + 10 \log_{10} 4 \approx 10 \log_{10} 4 + 66. \]
Problem 5 (30 points). In the above picture, all the images are $20 \times 20$ discrete-space, continuous-valued, grayscale images. Each pixel can have values from 0 (black) to 3 (white). The image in the top row depicts a light square on a dark background. Within the square, the image has a constant value of 2; within the background, it has a constant value of 1. (The image looks somewhat grainy and non-constant on your printout because of imperfections of my printing and copying devices.) Each of the six images in the second and third rows was obtained by modifying the square image in the top row. The objective of this problem is to determine what operations resulted in the six images.

a. (15 points) One of the images in the second row was obtained from the square image in the top row by adding white zero-mean noise with marginal density which is uniform between -1 and 1. One of the images in the second row was obtained from the square image in the top row by adding white zero-mean Gaussian noise with standard deviation 0.2. One of the images in the second row was obtained from the square image in the top row by “occlusive” noise: each pixel stayed the same with probability 0.9, was replaced by 0 with probability 0.05, and was replaced by 3 with probability 0.05. Clearly indicate which image in the second row (left, center, right) corresponds to each type of noise (uniform, Gaussian, occlusive), and fully explain your answers.

b. (15 points) The three images in the third row were obtained by applying a median filter with a $3 \times 3$ window to the three images in the second row. Clearly indicate which image in the third row corresponds to each image in the second row, and fully explain your answers.
Solution. a. Occlusive noise must correspond to the image where about 90% of the pixels are the same as in the original image, and the remaining pixels are black and white. This is the image in the center of the second row. A Gaussian random variable is very unlikely to take on values which are more than three standard deviations away from its mean. We would therefore expect the image corrupted by Gaussian noise with standard deviation 0.2 to have values in the range between 0.4 and 2.6, and most of the values to be fairly close to the original 1 and 2. Since uniform noise ranges from -1 to 1, we would expect many more very dark and very light pixels in the resulting image. Conclusion: the left image was obtained by adding Gaussian noise, the right image was obtained by adding uniform noise.

b. As shown in class, median filtering is very effective in removing isolated impulses, and therefore it will clean up occlusive noise. The right image in the third row corresponds to applying the median filter to the center image of the second row. Since median filtering cannot increase local maxima (i.e. produce very bright pixels where there were none), the left image in the third row could not possibly result from applying a median filter to the left image in the second row. Therefore, the left image in the third row corresponds to the right image in the second row, and the center image in the third row corresponds to the left image in the second row.
Problem 6 (30 points). In a binary optical communication system, the receiver counts the number of photoelectrons ejected by the light incident on the photocell during a time interval $(0, T)$. When no light signal has been transmitted toward the photocell (hypothesis $H_0$), the probability that $k$ photoelectrons are counted is

$$P(k|H_0) = A_0 v_0^k, \quad k = 0, 1, 2, \ldots$$

However, when a signal has been transmitted (hypothesis $H_1$),

$$P(k|H_1) = A_1 v_1^k, \quad k = 0, 1, 2, \ldots$$

with $0 < v_0 < v_1 < 1$. The prior probabilities of the two hypotheses are $P(H_0) = P(H_1) = \frac{1}{2}$.

a. (10 points) Determine the two constants $A_0$ and $A_1$ (i.e., express them in terms of $v_0$ and $v_1$).

b. (10 points) Determine the conditional probability that a signal was sent, given that exactly $m$ photoelectrons were counted. (Find an expression in terms of $v_0$, $v_1$, and $m$.)

c. (10 points) When $k \geq n_0$ the receiver decides that a signal was indeed sent; when $0 \leq k < n_0$, it decides that no signal was sent, where $n_0$ is some positive integer. For this decision rule, calculate the probability $P_e$ of error incurred by the receiver in terms of $n_0$, $v_0$, and $v_1$. For what value of $n_0$ is the probability $P_e$ of error minimum?

Fully and clearly explain all your answers.

Solution. a. We require that

$$\sum_{k=0}^{\infty} P(k|H_0) = A_0 \frac{1}{1-v_0} = 1,$$

which means that $A_0 = 1 - v_0$. Similarly, $A_1 = 1 - v_1$.

b. We have:

$$P(H_1|k = m) = \frac{P(k = m|H_1)P(H_1)}{P(k = m)} = \frac{1}{2} (1 - v_1) v_1^m \frac{1}{2} (1 - v_1) v_1^m + \frac{1}{2} (1 - v_0) v_0^m$$

$$= \frac{1}{1 + \frac{1 - v_0}{1 - v_1} \left( \frac{v_0}{v_1} \right)^m}$$

c. There is an error when a signal was not sent but there were at least $n_0$ photoelectrons counted, or when a signal was sent but fewer than $n_0$ photoelectrons were counted:

$$P_e = \sum_{k=n_0}^{\infty} P(k|H_0)P(H_0) + \sum_{k=0}^{n_0-1} P(k|H_1)P(H_1)$$

$$= \frac{1}{2} \sum_{k=n_0}^{\infty} (1 - v_0) v_0^k + \frac{1}{2} \sum_{k=0}^{n_0-1} (1 - v_1) v_1^k$$

$$= \frac{1}{2} (1 - v_0) \frac{v_0^{n_0}}{1 - v_0} + \frac{1}{2} (1 - v_1) \frac{1 - v_1^{n_0}}{1 - v_1}$$

$$= \frac{1}{2} (1 + v_0^{n_0} - v_1^{n_0})$$
To find the value of $n_0$ which minimizes $P_e$, we temporarily treat $n_0$ as a continuous variable and differentiate $P_e$ with respect to $n_0$. Setting this derivative to zero gives

$$\frac{\partial P_e}{\partial n_0} = 0$$

$$= \frac{1}{2} \left( \ln(v_0)v_0^{n_0} - \ln(v_1)v_1^{n_0} \right)$$

$$\Rightarrow n_0 = \frac{\ln(\ln(v_0)/\ln(v_1))}{\ln(v_1) - \ln(v_0)} = n^*.$$

Therefore, $P_e$ has one local extremum at $n_0 = n^*$, and we must verify that this is a minimum. For $n_0 = 0$, $P_e = \frac{1}{2}$ and $\frac{\partial P_e}{\partial n_0} = \frac{1}{2}(\ln(v_0) - \ln(v_1)) < 0$. Also, $\lim_{n_0 \to \infty} P_e = \frac{1}{2}$ and $\lim_{n_0 \to \infty} \frac{\partial P_e}{\partial n_0} = 0$. Therefore, $P_e$ has a minimum value at $n_0 = n^*$. Note, however, $n_0$ in our problem can only take on integer values. Therefore, the integer $n_0$ that minimizes $P_e$ is either $\lfloor n^* \rfloor$ or $\lceil n^* \rceil$, whichever one gives the lower $P_e$. 


**Problem 7 (20 points).** a. (5 points) Suppose that the input signal to a BIBO stable LTI system with a known transfer function $H(z)$ is constant at some value $\alpha$ for all time $n$. What is the corresponding output at each $n$?

b. (15 points). Denote by $y(n)$ the response of the system in Part a. to the input signal $x(n) = n$. Express $y(n)$, for each $n$, in terms of $H(z)$ and $y(0)$. (Hint. Consider the response of the system to $x_1(n) = n - k$, where $k$ is an arbitrary integer.)

**Fully and clearly** explain your answers.

**Solution.** a. Since an everlasting exponential $z_0^n$ is an eigenfunction of any BIBO stable LTI system, with corresponding eigenvalue $H(z_0)$, the response to $\alpha \cdot 1^n$ is $\alpha H(1)$, for all time.

b. By the time-invariance of the system, the response of the system to $x_1(n) = n - k$ is $y(n - k)$. On the other hand, by the linearity of the system, its response to $x_1(n) = n - k$ is $y(n) - w(n)$, where $w(n)$ is the response to the constant input $k$. From Part a., $w(n) = kH(1)$. The response to $x_1(n) = n - k$ is therefore $y(n) - kH(1)$. The two expressions we found for the response to $x_1(n) = n - k$ must be equal to each other:

$$y(n - k) = y(n) - kH(1)$$

This holds for all integer $n$; since $k$ is an arbitrary integer, this also holds for all integer $k$. Setting $k = n$ and rearranging, we get

$$y(n) = y(0) + nH(1).$$