3.3 Basic Linear Algebra (continued)

2. Inner Products and Orthogonality

We need the inner products and orthogonality to be able to generalize
notions related to angles to N dimensions. We need norms to be able
to generalize the notion of distance.

Now let's look at how to calculate the coordinates of a vector in an
arbitrary coordinate system in N dimensions.

6. Orthogonal Projections

In a plane, the coordinates of a vector are given by the projections
of the vector onto the coordinate axes.

\[
\begin{pmatrix}
(x_1) \\
(y_1)
\end{pmatrix}
\]

We will soon see that the coordinates of a signal in a Fourier basis -
that is, the Fourier series coefficients - can also be computed from
the projections of the signal onto the individual complex exponentials.

So, what is the orthogonal projection of a vector \( s = (s_1(n), \ldots, s_N(n)) \)
onto a vector \( g = (g_1(n), \ldots, g_N(n)) \)?

**Definition.** The orthogonal projection of a vector \( s = (s_1(n), \ldots, s_N(n)) \)
onto a non-zero vector \( g = (g_1(n), \ldots, g_N(n)) \) is the vector \( s_g \), such that:

1. \( s_g = a g \) for some complex number \( a \)
2. \( (s - s_g) \perp g \)

What is \( a \)?

\[
\begin{align*}
\langle s - s_g, g \rangle &= 0 \\
\langle s - a g, g \rangle &= 0 \\
\langle s, g \rangle - a \langle g, g \rangle &= 0
\end{align*}
\]

\[
a = \frac{\langle s, g \rangle}{\langle g, g \rangle}
\]

Thus,

\[
\begin{pmatrix}
s_g \end{pmatrix} = \frac{\langle s, g \rangle}{\langle g, g \rangle} \begin{pmatrix} g \end{pmatrix}
\]
Example. What's the proj. of (2) onto (1)? Should be (1).
\[
\langle (\frac{1}{2}, 0), (1) \rangle = \frac{1 \cdot 1 + 2 \cdot 0}{1 \cdot 1 + 0 \cdot 0} \cdot (1) = (\frac{1}{2})
\]

How to project a vector \( s \) onto a vector subspace \( G \)?

**Definition.** The set of all \( n \)-point vectors is called \( \mathbb{R}^n \). A subset \( G \) of \( \mathbb{R}^n \) is called a subspace of \( \mathbb{R}^n \) if

- \( a \in G \) for any \( a \in \mathbb{R} \) and \( g \in G \) for any \( g \in G \), and
- \( g_1 + g_2 \in G \) for any \( g_1, g_2 \in G \).

Example. The set of all \( n \)-point real-valued vectors is called \( \mathbb{R}^n \). A subset \( G \) of \( \mathbb{R}^n \) is called a vector subspace of \( \mathbb{R}^n \) if

- \( a \in G \) for any \( a \in \mathbb{R} \) and
- \( g_1, g_2 \in G \) for any \( g_1, g_2 \in G \).

Example. The set of all vectors of the form \((0, 0, a)\) where \( a \in \mathbb{C} \) is a vector subspace of \( \mathbb{C}^3 \) (why?). The set of all vectors of the form \((a, 0, 0)\) is not a vector subspace of \( \mathbb{C}^3 \) (why!).

**Definition.** Vectors \( g_1, \ldots, g_n \) are called linearly independent if

\[ a_1 g_1 + a_2 g_2 + \cdots + a_n g_n = 0 \implies a_1 = a_2 = \cdots = a_n = 0. \]

**Definition.** The space spanned by vectors \( g_1, \ldots, g_n \) is the set of all their linear combinations - i.e., all vectors of the form

\[ \sum a_i g_i, \]

where \( a_i \) are numbers.
Definition: If \( G = \text{span}\{g_1, ..., g_m\} \) and if \( g_1, ..., g_m \) are linearly independent, then \( \{g_1, ..., g_m\} \) is said to be a basis for space \( G \). If, in addition, \( g_1, ..., g_m \) are pairwise orthogonal, then \( \{g_1, ..., g_m\} \) is said to be an orthogonal basis.

Example: \( \text{span}\{ (1), (0) \} = \mathbb{R}^2 \) since \( (x) = x(1) + y(0) \) for any \( (x, y) \in \mathbb{R}^2 \).

![Orthogonal Basis Diagram](image)

**Definition:** The orth projection of \( s \) onto \( G \) is the vector \( s_0 \) s.t.

1. \( s_0 \in G \)
2. \( (s - s_0) \perp G \) (i.e., \( s - s_0 \perp \) to any vector in \( G \)).

Suppose that \( \{g_1, ..., g_m\} \) is an orthonormal basis of \( G \). Then \( s \in G \) means \( s = \sum_{k=1}^{m} a_k g_k \) for some numbers \( a_1, ..., a_m \).

How to compute the projection coefficients in terms of \( s \) and \( g_1, ..., g_m \)?

\( s - s_0 \perp G \) implies \( (s - s_0) \perp g_1, (s - s_0) \perp g_2, ..., (s - s_0) \perp g_m \):

\[
\begin{align*}
\langle s - s_0, g_p \rangle &= 0, \quad \text{for any } p = 1, ..., m, \\
\langle s, g_p \rangle - \langle s_0, g_p \rangle &= 0, \\
\langle s, g_p \rangle &= \langle \sum_{k=1}^{m} a_k g_k, g_p \rangle, \\
\langle s, g_p \rangle &= \sum_{k=1}^{m} a_k \langle g_k, g_p \rangle, \\
\langle g_k, g_p \rangle &= 0 \text{ if } k \neq p, \\
\langle g_k, g_k \rangle &= 1.
\end{align*}
\]

Thus, \( a_p = \frac{\langle s, g_p \rangle}{\langle g_p, g_p \rangle} \) for \( p = 1, ..., m \).

\[
\begin{align*}
s_0 &= \sum_{k=1}^{m} a_k g_k \
&= \sum_{k=1}^{m} \frac{\langle s, g_k \rangle}{\langle g_k, g_k \rangle} g_k
\end{align*}
\]

(Projections onto the individual basis vectors.) Substitute the results.

In particular, if \( s \in G \), then \( s = s_0 = \sum_{k=1}^{m} \frac{\langle s, g_k \rangle}{\langle g_k, g_k \rangle} g_k \).
1.3.4 DT Fourier Series, Revisited.

Example 4. \( g_k(n) = e^{i\frac{2\pi}{4}nk} \), for \( n = 0, 1, 2, 3; k = 0, 1, 2, 3 \).

(a) Prove that \( g_k + g_{k+p} \) for \( k \neq p \).

Find \( \|g_k\|_2 = 4 \).

(b) Let \( s(n) = \sum_{k=0}^{3} q_k e^{i\frac{2\pi}{4}nk} \). Find the \( a_0, a_1, a_2, a_3 \) in:

\[
 s(n) = a_0 g_0(n) + a_1 g_1(n) + a_2 g_2(n) + a_3 g_3(n)
\]

Solution (a) HW 4, Prob. 3. \( \|g_k\|_2 = 4 \).

(b) From linear algebra:

\( N \) pairwise orthogonal non-zero vectors in \( \mathbb{C}^N \)

from an orthonormal basis for \( \mathbb{C}^N \)

\[
\Rightarrow \{ g_0, g_1, g_2, g_3 \} \text{ is a basis for } \mathbb{C}^4
\]

\Rightarrow representation (\ref{eq:1}) is indeed possible, and

\[
a_0 = \frac{\langle s, g_0 \rangle}{\|g_0\|^2} = \frac{1}{4}
\]

\[
a_1 = \frac{\langle s, g_1 \rangle}{\|g_1\|^2} = \frac{1}{4}
\]

\[
a_2 = \frac{\langle s, g_2 \rangle}{\|g_2\|^2} = \frac{1}{4}
\]

\[
a_3 = \frac{\langle s, g_3 \rangle}{\|g_3\|^2} = \frac{1}{4}
\]

Let us generalize this to \( N \) dimensions.

Example 3. Consider the following DT complex exponential fourier:

\[
g_k(n) = e^{i\frac{2\pi}{N}kn} \quad n = 0, \ldots, N-1; k = 0, \ldots, N-1.
\]

\[
g_0 = \begin{pmatrix} g_0(0) \\ g_0(1) \\ \vdots \\ g_0(N-1) \end{pmatrix}, \quad g_1 = \begin{pmatrix} g_1(0) \\ g_1(1) \\ \vdots \\ g_1(N-1) \end{pmatrix}, \quad \ldots
\]

\[
g_{N-1} = \begin{pmatrix} g_{N-1}(0) \\ g_{N-1}(1) \\ \vdots \\ g_{N-1}(N-1) \end{pmatrix}
\]

(a) Prove that \( g_k \perp g_p \) for \( k \neq p \).

And \( \|g_k\|_2 = 1 \).

(b) Find a formula for the fourier series coefficients \( a_0, \ldots, a_N \) of a \( N \)-length complex-valued signal \( s(n) \):

\[
s(n) = \sum_{n=0}^{N-1} a_n g_n(n)
\]

Solution (a) HW 4, Prob. 3. \( \|g_k\|_2^2 = N \).
The complex exponentials could be redefined for a slightly different set of indexes (as in HW 4), or with a different normalization. However, the Fourier series formulas would then look somewhat different. However, there is no need to memorize the formulas. As long as you remember their meaning you can re-derive them quite easily. And the message is that we are computing the coefficients in the decomposition of a signal into an orthogonal basis.

\[ s(t) = \sum_{k=-\infty}^{\infty} a_k g_k(t) \]

\[ a_k = \frac{<s, g_k>}{|g_k|^2} \]

**Summary:** how to write \( s = \sum_{k=0}^{\infty} a_k g_k \), where \( g_0, g_1, \ldots \) are \( \text{basis elements} \).

**Answer:** \( a_k = \text{projection of } s \text{ onto } g_k \),

\[ a_k = \frac{<s, g_k>}{<g_k, g_k>} \]

The importance of this is that we've reduced rather complicated manipulations with complicated signals to a very intuitive geometric interpretation: namely, calculating the coordinates of a vector.

Another important consequence is that this is a general framework, which we can (and will) apply to other kinds of orthogonal bases.

### 13.5 CT Fourier Series

The notion of orthogonal bases and projections can be extended to spaces of CT signals. Determining whether a series represents some converges is an ill-defined process, since CT signals are more complicated than the finite-dimension DT signals. We therefore will restrict ourselves to just one example: CT Fourier series, which you've already seen in 301 and Lab 3, and which is very well understood.

**Periodic CT signals, \( s(t) \) and \( g(t) \), with period \( T_0 \):**

\[ <s, g> = \frac{1}{T_0} \int_{T_0} s(t) g(t) \, dt \]

It turns out that if we define the inner product this way, we can still use our magic projection formula.

\[ g_k(t) = e^{\frac{j2\pi}{T_0} k t} \quad k = 0, \pm 1, \pm 2, \ldots \]

This is a basis for all \( T_0 \)-periodic \( s(t) \) for which \( \int_{T_0} s(t) \, dt < \infty \).