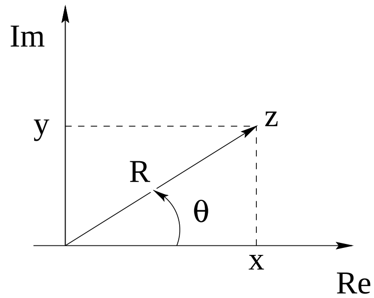

Figure 1: A complex number z can be represented in Cartesian coordinates (x, y) or polar coordinates (R, θ).

1.3 Frequency Analysis.

1.3.1 A Review of Complex Numbers.

A complex number is represented symbolically in the form

$$z = x + jy,$$

where x and y are real numbers satisfying the usual rules of addition, multiplication, and so on, and the symbol j, called the *imaginary unit*, formally has the property

$$j^2 = -1.$$

The numbers x and y are called the real and imaginary part of z, respectively, and are denoted by

$$x = \Re(z), \;\; y = \Im(z).$$

We say that z is real if $y = 0$, while it is purely imaginary if $x = 0$.
Example. The complex number $z = 3 + 2j$ has real part 3 and imaginary part 2, while the real number 5 can be viewed as the complex number $z = 5 + 0j$ whose real part is 5 and imaginary part is 0.

Geometrically, complex numbers can be represented as vectors in the plane. We will call the xy-plane, when viewed in this manner, the *complex plane*, with the x-axis designated as the real axis, and the y-axis as the imaginary axis. We designate the complex number zero as the origin. Thus,

$$x + jy = 0 \text{ means } x = y = 0.$$

In addition, since two points in the plane are the same if and only if both their x- and y-coordinates agree, we can define equality of two complex numbers by

$$x_1 + jy_1 = x_2 + jy_2 \text{ means } x_1 = x_2 \text{ and } y_1 = y_2.$$

Thus, we see that a single equation between two complex quantities actually contains two real equations.

Definition (Complex Arithmetic). Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then we define:

(a) $z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2)$;

(b) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$;

(c) for $z_2 \neq 0$, $w = \frac{z_1}{z_2}$ is the complex number for which $z_1 = z_2 w$.

Note that, instead of the Cartesian coordinates x and y, we could use polar coordinates to represent points in the plane. The polar coordinates are the radial distance R and the angle θ, as illustrated in Fig. 1. The relationship between the two sets of coordinates is:

$$\begin{aligned} x &= R\cos\theta; \\ y &= R\sin\theta; \\ R &= \sqrt{x^2+y^2} = |z|; \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

Note that R is called the modulus, or the absolute value of z. Thus, the polar representation is:

$$z = |z|\cos\theta + j|z|\sin\theta = |z|(\cos\theta + j\sin\theta).$$

Definition (Complex Exponential Function). The complex exponential function, denoted by e^z, or $\exp(z)$, is defined by

$$e^z = e^{x+jy} = e^x(\cos y + j\sin y).$$

In particular, if $x = 0$, we have Euler's equation:

$$e^{jy} = \cos y + j\sin y.$$

Comparing this with the terms in the polar representation of a complex variable, we see that any complex variable can be written as:

$$z = |z|e^{j\theta}.$$

Properties of complex exponentials.

$$\begin{aligned} \cos\theta &= \frac{1}{2}(e^{j\theta} + e^{-j\theta}); \\ \sin\theta &= \frac{1}{2j}(e^{j\theta} - e^{-j\theta}); \\ |e^{j\theta}| &= 1; \\ e^{z_1}e^{z_2} &= e^{z_1+z_2}; \\ e^{-z} &= \frac{1}{e^z}; \\ e^{z+2\pi jn} &= e^x(\cos(y+2\pi n) + j\sin(y+2\pi n)) \\ &= e^x(\cos(y) + j\sin(y)) = e^z, \text{ for any integer } n. \end{aligned}$$

DT complex exponential functions whose frequencies differ by 2π are identical:

$$e^{j(\omega+2\pi)n} = e^{j\omega n + 2\pi jn} = e^{j\omega n}.$$

(We have seen examples of this phenomenon before, when we talked about DT sinusoids.)
It follows from the multiplication rule that

$$z_1 z_2 = |z_1| e^{j\theta_1} |z_2| e^{j\theta_2} = |z_1||z_2| e^{j(\theta_1+\theta_2)}$$

- add the angles;
- multiply the absolute values.

Definition (Complex Conjugate). If $z = x + jy$, then the complex conjugate of z, z^* (sometimes also denoted $\bar{z}$) is:

$$z^* = x - jy.$$

Note that, if $z = |z|e^{j\theta}$, then $z^* = |z|e^{-j\theta}$. Some other useful identities:

$$\begin{aligned}
\Re(z) &= \frac{1}{2}(z + z^*) \\
\Im(z) &= \frac{1}{2j}(z - z^*) \\
|z| &= \sqrt{zz^*} \\
(z^*)^* &= z \\
z^* = z &\Leftrightarrow z \text{ is real} \\
(z_1 + z_2)^* &= z_1^* + z_2^* \\
(z_1 z_2)^* &= z_1^* z_2^*
\end{aligned}$$

Examples. Take $z = 1 + j$, and let's compute various quantities defined above.

1. $z^* = 1 - j$.
2. $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Alternatively,
   $|z| = \sqrt{zz^*} = \sqrt{(1+j)(1-j)} = \sqrt{1 + j - j - j^2} = \sqrt{2}$.
3. $\Re(z) = \Im(z) = 1$.
4. Polar representation: $z = \sqrt{2}(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}) = \sqrt{2}e^{j\frac{\pi}{4}}$
5. To compute z^2, square the absolute value, and double the angle:
   $z^2 = 2(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}) = 2j = 2e^{j\frac{\pi}{2}}$.
   Check: $(1+j)(1+j) = 1 + 2j + j^2 = 1 + 2j - 1 = 2j$.
6. $\frac{1}{z} = \frac{1}{1+j} = \frac{1}{\sqrt{2}e^{j\frac{\pi}{4}}} = \frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right)\right) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{j}{2}$.
   Check: $\left(\frac{1}{2} - \frac{j}{2}\right)(1+j) = \frac{1}{2} - \frac{j}{2} + \frac{j}{2} - \frac{j^2}{2} = 1$.