EE 438
Supplementary Notes on Fourier Series and Linear Algebra.

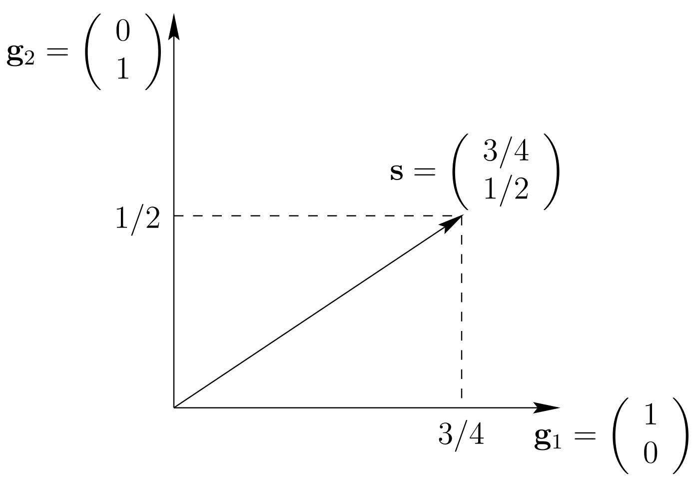
1 Discrete-Time Fourier Series.

How to write a vector $\mathbf{s}$ as a sum

$$\mathbf{s} = \sum_{k=1}^{m} a_k \mathbf{g}_k,$$

where $\{\mathbf{g}_1, \ldots, \mathbf{g}_m\}$ are pairwise orthogonal?

Answer: project $\mathbf{s}$ onto each $\mathbf{g}_k$, and sum the results. For example, in 2-D:



Suppose $\mathbf{g}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{g}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\mathbf{s} = \begin{pmatrix} 3/4 \\ 1/2 \end{pmatrix}$. Then

$$\text{the projection of } \mathbf{s} \text{ onto } \mathbf{g}_1 \text{ is } \begin{pmatrix} 3/4 \\ 0 \end{pmatrix} = 3/4 \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$
$$\text{the projection of } \mathbf{s} \text{ onto } \mathbf{g}_2 \text{ is } \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = 1/2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Their sum:

$$3/4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1/2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/2 \end{pmatrix} = \mathbf{s}.$$

Let us generalize this idea to many dimensions.

Example 1. Let

$$g_1(n) = \exp\left(\frac{j2\pi(n-1)}{4}\right), \quad n = 1, 2, 3, 4;$$
$$\text{and } g_2(n) = \exp\left(\frac{j2\pi 2(n-1)}{4}\right), \quad n = 1, 2, 3, 4.$$

(a) Suppose that

$$s(n) = \begin{cases} 2, & n = 1 \\ -1 + j, & n = 2 \\ 0, & n = 3 \\ -1 - j, & n = 4. \end{cases}$$

Find the coefficients a_1, a_2 in the following Fourier series expansion:

$$s(n) = a_1 g_1(n) + a_2 g_2(n).$$

(b) Suppose that

$$s'(n) = \begin{cases} 0, & n = 1 \\ 0, & n = 2 \\ 1, & n = 3 \\ 0, & n = 4. \end{cases}$$

Find the coefficients a'_1, a'_2 in the following Fourier series expansion:

$$s'(n) = a'_1 g_1(n) + a'_2 g_2(n).$$

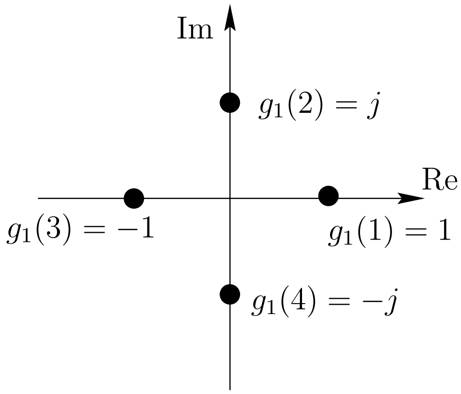
Solution.
(a) Write all three signals as vectors, i.e.,

$$\mathbf{g}_1 = \begin{pmatrix} g_1(1) \\ g_1(2) \\ g_1(3) \\ g_1(4) \end{pmatrix}, \quad \mathbf{g}_2 = \begin{pmatrix} g_2(1) \\ g_2(2) \\ g_2(3) \\ g_2(4) \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} s(1) \\ s(2) \\ s(3) \\ s(4) \end{pmatrix}.$$

What are the entries of these vectors?

$$\begin{aligned} g_1(1) &= \exp\left(\frac{j2\pi 0}{4}\right) = \exp(j0) \\ g_1(2) &= \exp\left(\frac{j2\pi 1}{4}\right) = \exp\left(j\frac{\pi}{2}\right) \\ g_1(3) &= \exp\left(\frac{j2\pi 2}{4}\right) = \exp(j\pi) \\ g_1(4) &= \exp\left(\frac{j2\pi 3}{4}\right) = \exp\left(j\frac{3\pi}{2}\right) \end{aligned}$$

Plot these in the complex plane, using the fact that $\exp(j\theta)$ has absolute value 1 and angle θ:



Calculations for $\mathbf{g}_2$ are similar. We obtain:

$$\mathbf{g}_1 = \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix}, \quad \mathbf{g}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Also,

$$\mathbf{s} = \begin{pmatrix} 2 \\ -1+j \\ 0 \\ -1-j \end{pmatrix} = \mathbf{g}_1 + \mathbf{g}_2 \quad \text{(by inspection).}$$

Thus, the Fourier series coefficients are $a_1 = a_2 = 1$:

$$s(n) = g_1(n) + g_2(n).$$

(b) Take $n = 1$:

$$\begin{aligned} s'(1) &= a_1' g_1(1) + a_2' g_2(1) \\ 0 &= a_1' \cdot 1 + a_2' \cdot 1 \\ a_1' &= -a_2'. \end{aligned} \tag{1}$$

Now take $n = 2$:

$$\begin{aligned} s'(2) &= a_1' g_1(2) + a_2' g_2(2) \\ 0 &= a_1' \cdot j + a_2' \cdot (-1). \end{aligned}$$

Substitute $a_1' = -a_2'$ from (1):

$$\begin{aligned} 0 &= -a_2' \cdot j + a_2' \cdot (-1) \\ 0 &= a_2'(-j-1) \\ a_2' &= 0. \end{aligned}$$

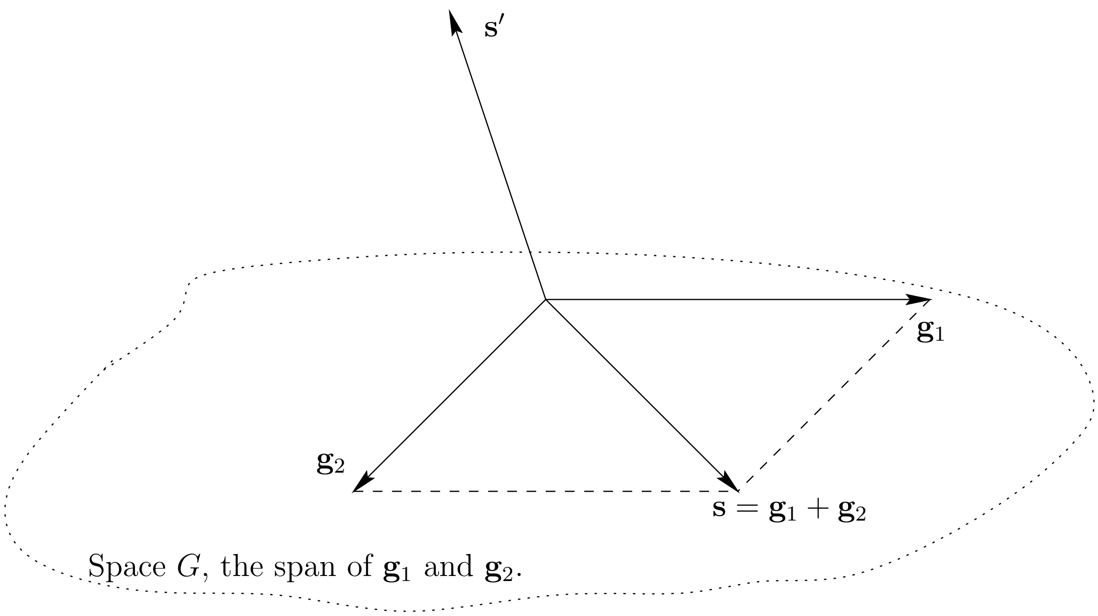
But since $a_1' = -a_2'$, it follows that $a_1' = 0$. But notice that our "solution" $a_1' = a_2' = 0$ does not work for $n = 3$:

$$s'(3) \neq 0g_1(3) + 0g_2(3).$$

Thus, the system of equations

$$s'(n) = a'_1 g_1(n) + a'_2 g_2(n), \quad n = 1, 2, 3, 4,$$

does not have any solutions a'_1, a'_2 which satisfy all four equations. This means that $\mathbf{s}'$ cannot be represented as a linear combination of $\mathbf{g}_1$ and $\mathbf{g}_2$. Geometrically, vector $\mathbf{s}'$ does not lie in the space spanned by $\mathbf{g}_1$ and $\mathbf{g}_2$:



[*End of Example 1.*]

We need a systematic way of determining:

(a) whether a signal is representable as a linear combination of a set of "basis signals";

(b) if it is, what are the coefficients in this representation?

A sufficient condition for (a) to hold is:
if $\mathbf{g}_1, \ldots, \mathbf{g}_N$ *are* N *pairwise orthogonal,* N*-dimensional, non-zero complex-valued vectors, then any* N*-dimensional complex-valued vector can be uniquely represented as their linear combination.*
In this case, the answer to (b) is:
the coefficients in the representation

$$\mathbf{s} = \sum_{k=1}^{N} a_k \mathbf{g}_k$$

are then computed as:

$$a_k = \frac{\langle \mathbf{s}, \mathbf{g}_k \rangle}{\langle \mathbf{g}_k, \mathbf{g}_k \rangle}. \tag{2}$$

Example 2. In addition to signals $g_1(n)$ and $g_2(n)$ defined in Example 1 above, define $g_0(n)$ and $g_3(n)$ as follows:

$$\begin{aligned} g_0(n) &= \exp\left(\frac{j2\pi 0(n-1)}{4}\right), \quad n = 1,2,3,4; \\ \text{and } g_3(n) &= \exp\left(\frac{j2\pi 3(n-1)}{4}\right), \quad n = 1,2,3,4. \end{aligned}$$

In other words, we now have four signals, $g_k(n)$, $k = 0,1,2,3$, defined for $n = 1,2,3,4$ by:

$$g_k(n) = \exp\left(\frac{j2\pi k(n-1)}{4}\right).$$

It is given that these four signals are pairwise orthogonal, which means that formula (2) is applicable.

(a) Using formula (2), find coefficients a_0, a_1, a_2, a_3 in the following Fourier series expansion:

$$s(n) = a_0 g_0(n) + a_1 g_1(n) + a_2 g_2(n) + a_3 g_3(n),$$

for $s(n)$ defined in Example 1, part (a).

(b) Using formula (2), find coefficients a'_0, a'_1, a'_2, a'_3 in the following Fourier series expansion:

$$s'(n) = a'_0 g_0(n) + a'_1 g_1(n) + a'_2 g_2(n) + a'_3 g_3(n),$$

for $s'(n)$ defined in Example 1, part (b).

Solution.
(a) First, write all signals as vectors:

$$\mathbf{g}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{g}_1 = \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix}, \quad \mathbf{g}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{g}_3 = \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 2 \\ -1+j \\ 0 \\ -1-j \end{pmatrix}.$$

Calculate the inner products used in (2), and compute the coefficients:

$$\begin{aligned} \langle \mathbf{s}, \mathbf{g}_0 \rangle &= 2 \cdot 1^* + (-1+j) \cdot 1^* + 0 \cdot 1^* + (-1-j) \cdot 1^* = 0. \\ \langle \mathbf{g}_0, \mathbf{g}_0 \rangle &= 1 \cdot 1^* + 1 \cdot 1^* + 1 \cdot 1^* + 1 \cdot 1^* = 4. \\ a_0 &= \frac{\langle \mathbf{s}, \mathbf{g}_0 \rangle}{\langle \mathbf{g}_0, \mathbf{g}_0 \rangle} = \frac{0}{4} = 0. \\ \langle \mathbf{s}, \mathbf{g}_1 \rangle &= 2 \cdot 1^* + (-1+j) \cdot j^* + 0 \cdot (-1)^* + (-1-j) \cdot (-j)^* \\ &= 2 + (-1+j)(-j) + (-1-j)j \\ &= 2 + j - j^2 - j - j^2 = 4. \\ \langle \mathbf{g}_1, \mathbf{g}_1 \rangle &= 1 \cdot 1^* + j \cdot j^* + (-1) \cdot (-1)^* + (-j) \cdot (-j)^* \\ &= |1|^2 + |j|^2 + |-1|^2 + |-j|^2 = 4. \\ a_1 &= \frac{\langle \mathbf{s}, \mathbf{g}_1 \rangle}{\langle \mathbf{g}_1, \mathbf{g}_1 \rangle} = \frac{4}{4} = 1. \end{aligned}$$

$$
\begin{aligned}
\langle \mathbf{s}, \mathbf{g}_2 \rangle &= 2 \cdot 1^* + (-1+j) \cdot (-1)^* + 0 \cdot 1^* + (-1-j) \cdot (-1)^* \\
&= 2 + (1-j) + (1+j) = 4. \\
\langle \mathbf{g}_2, \mathbf{g}_2 \rangle &= 4. \\
a_2 &= \frac{\langle \mathbf{s}, \mathbf{g}_2 \rangle}{\langle \mathbf{g}_2, \mathbf{g}_2 \rangle} = \frac{4}{4} = 1. \\
\langle \mathbf{s}, \mathbf{g}_3 \rangle &= 2 \cdot 1^* + (-1+j) \cdot (-j)^* + 0 \cdot (-1)^* + (-1-j) \cdot j^* \\
&= 2 + (-1+j)j + (-1-j)(-j) \\
&= 2 - j + j^2 + j + j^2 = 0. \\
\langle \mathbf{g}_3, \mathbf{g}_3 \rangle &= 4. \\
a_3 &= \frac{\langle \mathbf{s}, \mathbf{g}_3 \rangle}{\langle \mathbf{g}_3, \mathbf{g}_3 \rangle} = \frac{0}{4} = 0.
\end{aligned}
$$

So, $a_0 = a_3 = 0$ and $a_1 = a_2 = 1$:

$$s(n) = g_1(n) + g_2(n).$$

This is the same result we got in Example 1, part (a).

(b) We can use $\langle \mathbf{g}_k, \mathbf{g}_k \rangle = 4$, computed in part (a) above. Recall that

$$\mathbf{s}' = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

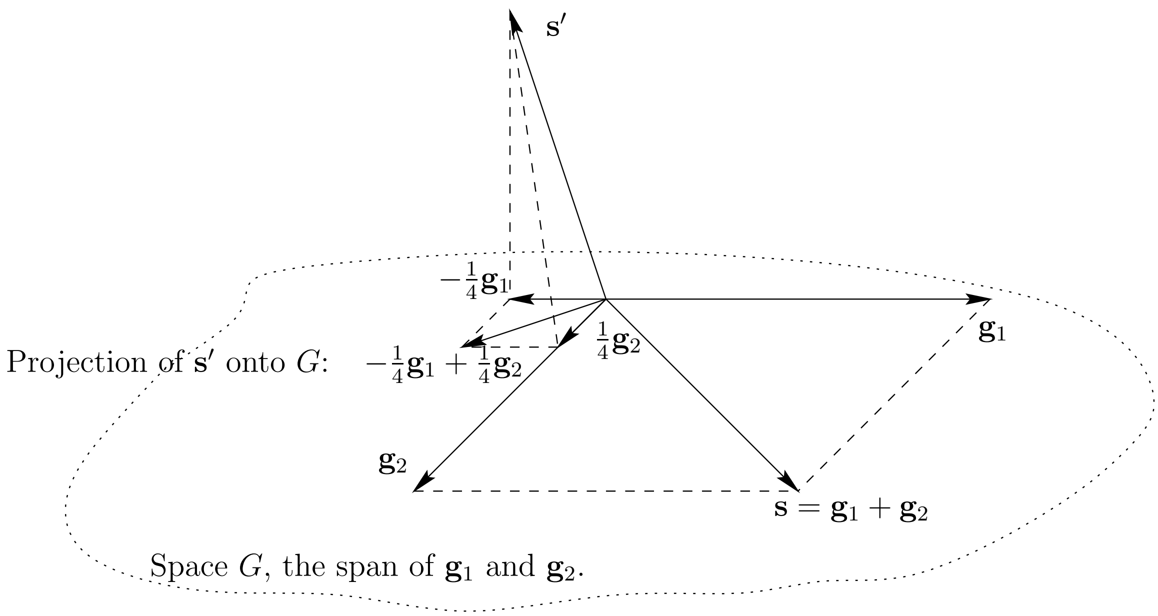
This makes its inner products with the basis vectors very simple:

$$
\begin{aligned}
\langle \mathbf{s}', \mathbf{g}_0 \rangle &= 1. \\
a'_0 &= \frac{\langle \mathbf{s}', \mathbf{g}_0 \rangle}{\langle \mathbf{g}_0, \mathbf{g}_0 \rangle} = \frac{1}{4}. \\
\langle \mathbf{s}', \mathbf{g}_1 \rangle &= -1. \\
a'_1 &= \frac{\langle \mathbf{s}', \mathbf{g}_1 \rangle}{\langle \mathbf{g}_1, \mathbf{g}_1 \rangle} = -\frac{1}{4}. \\
\langle \mathbf{s}', \mathbf{g}_2 \rangle &= 1. \\
a'_2 &= \frac{\langle \mathbf{s}', \mathbf{g}_2 \rangle}{\langle \mathbf{g}_2, \mathbf{g}_2 \rangle} = \frac{1}{4}. \\
\langle \mathbf{s}', \mathbf{g}_3 \rangle &= -1. \\
a'_3 &= \frac{\langle \mathbf{s}', \mathbf{g}_3 \rangle}{\langle \mathbf{g}_3, \mathbf{g}_3 \rangle} = -\frac{1}{4}.
\end{aligned}
$$

Therefore,

$$s'(n) = \frac{1}{4} g_0(n) - \frac{1}{4} g_1(n) + \frac{1}{4} g_2(n) - \frac{1}{4} g_3(n).$$

This is consistent with what we saw in Example 1, part (b): $\mathbf{s}'$ cannot be represented as a linear combination of only $\mathbf{g}_1$ and $\mathbf{g}_2$. Thus, $\mathbf{s}$ belongs to space G spanned by $\mathbf{g}_1$ and $\mathbf{g}_2$, while $\mathbf{s}'$ does not:



Note, however, that if in the expansion

$$\mathbf{s}' = \frac{1}{4}\mathbf{g}_0 - \frac{1}{4}\mathbf{g}_1 + \frac{1}{4}\mathbf{g}_2 - \frac{1}{4}\mathbf{g}_3,$$

we drop the terms which do not contain $\mathbf{g}_1$ and $\mathbf{g}_2$, we will get the following vector:

$$-\frac{1}{4}\mathbf{g}_1 + \frac{1}{4}\mathbf{g}_2.$$

This vector does belong to space G, and is, in a sense, the *closest approximation* to $\mathbf{s}'$ among all the vectors in G. It is the projection of $\mathbf{s}'$ onto G.
[*End of Example 2.*]

To summarize, the series

$$\sum_{k=1}^{m} \frac{\langle \mathbf{s}, \mathbf{g}_k \rangle}{\langle \mathbf{g}_k, \mathbf{g}_k \rangle} \mathbf{g}_k$$

is equal to

- vector $\mathbf{s}$, if $\mathbf{s}$ lies in the space $G = \text{span}(\mathbf{g}_1, \ldots, \mathbf{g}_m)$;
- the projection of $\mathbf{s}$ onto G, if $\mathbf{s}$ does not belong to G.

Now let us generalize Example 2 from four dimensions to N.

Example 3. Consider the following DT complex exponential functions:

$$g_k(n) = \exp\left(\frac{j2\pi kn}{N}\right), \quad n = 0, \ldots, N-1; \quad k = 0, \ldots, N-1. \tag{3}$$

In other words, there are N functions, $g_0(n), g_1(n), \ldots, g_{N-1}(n)$, and each of them is defined for $n = 0, 1, \ldots, N-1$.
(a) Prove that these N signals are pairwise orthogonal, and find their energies.

(b) Find a formula for the Fourier series coefficients $a_0, \ldots, a_{N-1}$ of an N-point complex-valued signal $s(n)$,

$$s(n) = \sum_{k=0}^{N-1} a_k g_k(n).$$

Solution.
(a) To show orthogonality and compute the energies, we need to calculate all inner products $\langle \mathbf{g}_k, \mathbf{g}_i \rangle$, for all $k = 0, \ldots, N-1$ and $i = 0, \ldots, N-1$. If we can show that these inner products for $k \neq i$ are zero, we will show that the signals are pairwise orthogonal. Moreover, the inner products for $k = i$ will give us the energies.

$$\begin{aligned}
\langle \mathbf{g}_k, \mathbf{g}_i \rangle &= \sum_{n=0}^{N-1} g_k(n)(g_i(n))^* \\
&= \sum_{n=0}^{N-1} \exp\left(\frac{j2\pi kn}{N}\right) \exp\left(-\frac{j2\pi in}{N}\right) \\
&= \sum_{n=0}^{N-1} \exp\left(\frac{j2\pi(k-i)n}{N}\right) \\
&= \sum_{n=0}^{N-1} \left[\exp\left(\frac{j2\pi(k-i)}{N}\right)\right]^n
\end{aligned}$$

When $k = i$, each term of the summation is equal to 1, and therefore the sum is N. The energy of each $\mathbf{g}_k$ is therefore N. When $k \neq i$, the sum is zero (why?).

(b) Since the basis is orthogonal, we can use formula (2). The denominator of that formula is the energy of $\mathbf{g}_k$, which was computed above.

$$\begin{aligned}
a_k &= \frac{\langle \mathbf{s}, \mathbf{g}_k \rangle}{\langle \mathbf{g}_k, \mathbf{g}_k \rangle} \\
&= \frac{1}{N} \langle \mathbf{s}, \mathbf{g}_k \rangle \\
&= \frac{1}{N} \sum_{n=0}^{N-1} s(n)(g_k(n))^* \\
&= \frac{1}{N} \sum_{n=0}^{N-1} s(n) \exp\left(-\frac{j2\pi kn}{N}\right).
\end{aligned}$$

[*End of Example 3.*]

2 Continuous-Time Fourier Series.

The notion of bases and projections can be extended to spaces of continuous-time signals. Determining whether a series representation converges (and what it converges to) is much more complicated than for finite-duration DT signals. We therefore will restrict ourselves to just one example–CT Fourier series–whose behavior is well understood.

We will consider all CT complex-valued periodic signals with period T_0. The projection formula (2) can still be used, if the inner product of two such signals, $s(t)$ and $g(t)$, is defined as follows:

$$\langle s, g\rangle = \int_{\tau}^{\tau+T_0} s(t)\,(g(t))^*\;dt,$$

where τ is an arbitrary number–i.e., the integral is taken over any period.

The CT trigonometric Fourier basis is the following infinite collection of functions:

$$\begin{aligned} c_0(t) &= 1, \\ c_k(t) &= \cos\left(2\pi k \frac{t}{T_0}\right), \quad k = 1, 2, \ldots \\ s_k(t) &= \sin\left(2\pi k \frac{t}{T_0}\right), \quad k = 1, 2, \ldots \end{aligned}$$

As we did with the DT Fourier basis, let us first prove that these functions are pairwise orthogonal, and find their energies. Just as in the DT case, we need to consider all pairwise inner products–which will now be integrals of products of trigonometric functions. We will therefore need the following formulas:

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) \tag{4}$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta)) \tag{5}$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta)) \tag{6}$$

Compute the inner products, keeping in mind that $s_k(t)$ is defined for $k \geq 1$ while $c_k(t)$ is defined for $k \geq 0$:

$$\begin{aligned} \langle s_k, s_i\rangle &= \int_{\tau}^{\tau+T_0} \sin\left(2\pi k \frac{t}{T_0}\right) \sin\left(2\pi i \frac{t}{T_0}\right)\;dt \\ &\overset{\text{Eq. (4)}}{=} \frac{1}{2}\int_{\tau}^{\tau+T_0} \left[\cos\left(2\pi(k-i)\frac{t}{T_0}\right) - \cos\left(2\pi(k+i)\frac{t}{T_0}\right)\right]\;dt \\ &= \begin{cases} \frac{T_0}{2}, & k = i \\ 0, & k \neq i. \end{cases} \end{aligned}$$

$$
\begin{aligned}
\langle s_k, c_i \rangle &= \int_{\tau}^{\tau+T_0} \sin\left(2\pi k \frac{t}{T_0}\right) \cos\left(2\pi i \frac{t}{T_0}\right)\, dt \\
&\overset{\text{Eq. (5)}}{=} \frac{1}{2} \int_{\tau}^{\tau+T_0} \left[\sin\left(2\pi (k-i) \frac{t}{T_0}\right) + \sin\left(2\pi (k+i) \frac{t}{T_0}\right)\right] dt = 0.
\end{aligned}
$$

$$
\begin{aligned}
\langle c_k, c_i \rangle &= \int_{\tau}^{\tau+T_0} \cos\left(2\pi k \frac{t}{T_0}\right) \cos\left(2\pi i \frac{t}{T_0}\right)\, dt \\
&\overset{\text{Eq. (6)}}{=} \frac{1}{2} \int_{\tau}^{\tau+T_0} \left[\cos\left(2\pi (k-i) \frac{t}{T_0}\right) + \cos\left(2\pi (k+i) \frac{t}{T_0}\right)\right] dt \\
&= \begin{cases} T_0, & k = i = 0 \\ \frac{T_0}{2}, & k = i \neq 0 \\ 0, & k \neq i. \end{cases}
\end{aligned}
$$

We are now ready to proceed similarly to Example 3, and derive formulas for the coefficients $a_1, a_2 \ldots$ and $b_0, b_1, b_2 \ldots$ of the expansion of a CT periodic (with period T_0) signal $s(t)$:

$$
s(t) = b_0 + \sum_{k=1}^{\infty} a_k s_k(t) + \sum_{k=1}^{\infty} b_k c_k(t).
$$

$$
\begin{aligned}
b_0 &= \frac{\langle s, c_0 \rangle}{\langle c_0, c_0 \rangle} \\
&= \frac{1}{T_0} \int_{\tau}^{\tau+T_0} s(t)\, dt \\
b_k &= \frac{\langle s, c_k \rangle}{\langle c_k, c_k \rangle} \\
&= \frac{2}{T_0} \int_{\tau}^{\tau+T_0} s(t) \cos\left(2\pi k \frac{t}{T_0}\right) dt, \quad k = 1, 2, \ldots \\
a_k &= \frac{\langle s, s_k \rangle}{\langle s_k, s_k \rangle} \\
&= \frac{2}{T_0} \int_{\tau}^{\tau+T_0} s(t) \sin\left(2\pi k \frac{t}{T_0}\right) dt, \quad k = 1, 2, \ldots
\end{aligned}
$$

Example 4. Suppose that the period is $T_0 = 2$, and that $s(t)$ is defined by:

$$
s(t) = \begin{cases} 1, & -1 \leq t < 0 \\ 0, & 0 \leq t < 1. \end{cases}
$$

Compute the Fourier series coefficients.

Solution.

From the formulas above,

$$
\begin{aligned}
b_0 &= \frac{1}{T_0}\int_{\tau}^{\tau+T_0} s(t)\ dt \\
&= \frac{1}{2}\int_{-1}^{0} 1\ dt = \frac{1}{2} \\
\text{For } k \geq 1, \qquad b_k &= \frac{2}{T_0}\int_{\tau}^{\tau+T_0} s(t)\cos\left(2\pi k\frac{t}{T_0}\right)\ dt \\
&= \frac{2}{2}\int_{-1}^{0} \cos\left(2\pi k\frac{t}{2}\right)\ dt \\
&= \int_{-1}^{0} \cos(\pi k t)\ dt \\
&= \frac{1}{\pi k}\sin(\pi k t)\Big|_{t=-1}^{t=0} = 0 \\
a_k &= \frac{2}{T_0}\int_{\tau}^{\tau+T_0} s(t)\sin\left(2\pi k\frac{t}{T_0}\right)\ dt \\
&= \frac{2}{2}\int_{-1}^{0} \sin\left(2\pi k\frac{t}{2}\right)\ dt \\
&= \int_{-1}^{0} \sin(\pi k t)\ dt \\
&= -\frac{1}{\pi k}\cos(\pi k t)\Big|_{t=-1}^{t=0} \\
&= \begin{cases} -\frac{2}{\pi k}, & k \text{ is odd} \\ 0, & k \text{ is even.} \end{cases}
\end{aligned}
$$

[*End of Example 4.*]

3 Summary of Basic Definitions.

1) Suppose $s(n)$ and $g(n)$ are two complex-valued discrete-time signals, defined only for $n = 1, \ldots, N$. They can be identified with N-dimensional vectors,

$$
\mathbf{s} = \begin{pmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} g(1) \\ g(2) \\ \vdots \\ g(N) \end{pmatrix}.
$$

The **inner product** of these two signals is defined by:

$$
\langle \mathbf{s}, \mathbf{g} \rangle = \sum_{n=1}^{N} s(n)\,(g(n))^*.
$$

2) The inner product of two complex-valued, periodic continuous-time signals $s(t)$ and $g(t)$, with period T_0:

$$\langle s, g \rangle = \int_{\tau}^{\tau+T_0} s(t)\,(g(t))^* \;dt,$$

where τ is an arbitrary number–i.e., the integral is taken over any period.

3) The **energy** of a signal is the inner product of the signal with itself. The square root of the energy is called the **norm**. More precisely, it is called the ℓ_2 norm for discrete-time signals, and L_2 norm for continuous-time signals.

4) Two signals are **orthogonal** if their inner product is zero.