ECE 302 Division 1

- This is a closed-book exam. A formula sheet is provided. No calculators are allowed.
- Total number of points: 150. This exam counts for 25% of your final grade.
- You have 75 minutes to complete 6 problems.
- Be sure to **fully and clearly** explain all your answers to all problems except Problem 1.
- There will not be any discussion of grades. All re-grade requests must be submitted in writing, as stated in the course information handout.

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<th>Problem</th>
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<td>1</td>
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<td><strong>TOTAL</strong></td>
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Some random variables, their distributions, and associated transforms:

<table>
<thead>
<tr>
<th>Random variable</th>
<th>PMF or PDF</th>
<th>Mean</th>
<th>Variance</th>
<th>Transform</th>
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<tbody>
<tr>
<td>Bernoulli</td>
<td>$p$ for $k = 1$; $1 - p$ for $k = 0$.</td>
<td>$p$</td>
<td>$p(1 - p)$</td>
<td>$1 - p + pe^s$</td>
</tr>
<tr>
<td>Discrete uniform</td>
<td>$\frac{1}{n}$, $k = k_0 + 1, k_0 + 2, \ldots, k_0 + n$</td>
<td>$k_0 + \frac{n+1}{2}$</td>
<td>$\frac{n^2-1}{12}$</td>
<td>$\frac{e^s(e^{(k_0+n)s} - e^{k_0s})}{n(e^s-1)}$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$(1 - p)^{k-1}p$, $k = 1, 2, 3, \ldots$</td>
<td>$\frac{1}{p}$</td>
<td>$\frac{1}{p^2} - \frac{1}{p}$</td>
<td>$\frac{pe^s}{1-(1-p)e^s}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\binom{n}{k} (1-p)^{n-k}p^k$, $k = 0, 1, \ldots, n$</td>
<td>$pn$</td>
<td>$np(1 - p)$</td>
<td>$(1 - p + pe^s)^n$</td>
</tr>
<tr>
<td>Continuous uniform</td>
<td>$\frac{1}{b-a}$, $a \leq x \leq b$</td>
<td>$\frac{b+a}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
<td>$\frac{e^{sb} - e^{sa}}{(b-a)s}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\lambda e^{-\lambda x}$, $x \geq 0$</td>
<td>$\lambda^{-1}$</td>
<td>$\lambda^{-2}$</td>
<td>$\frac{\lambda}{\lambda - s}$</td>
</tr>
<tr>
<td>Normal (Gaussian)</td>
<td>$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
<td>$\frac{e^{-\frac{s^2}{2}}}{\sqrt{e^{-\frac{s^2}{2}} + \frac{\mu s}{\sqrt{2\pi}\sigma}}}$</td>
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where

$$\binom{n}{k} = \frac{n!}{(n-k)k!}.$$
Problem 1 (40 points). A continuous random variable $X$ has the following probability density:

$$f_X(x) = \begin{cases} C, & 1 \leq x \leq 3, \\ 0, & \text{otherwise}, \end{cases}$$

where $C$ is a positive real number.

a (10 points). Find the constant $C$.

b (10 points). Find $E[X]$.

c (10 points). Find var($X$).

d (10 points). Let $Y = \frac{X-1}{2}$. Find $f_Y(y)$, the probability density of $Y$.

For each part, provide only the answer without explanation. No partial credit will be given.

Solution. Since the probability density must integrate to one, we have: $\int_1^3 C \, dx = 1$ which means that $C = 1/2$. Since $X$ is uniformly distributed between $a = 1$ and $b = 3$, its mean and variance are 2 and 1/3, respectively (these were given on the formula sheet).

Since $X$ is uniform between 1 and 3, $X/2$ is uniform between $1/2$ and $3/2$, and $Y = X/2 - 1/2$ is uniform between 0 and 1. Alternatively, we can write:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y)$$

$$= \frac{d}{dy} P \left( \frac{X - 1}{2} \leq y \right) = \frac{d}{dy} P (X \leq 2y + 1) = \frac{d}{dy} F_X(2y + 1) = 2f_X(2y + 1)$$

$$= \begin{cases} 2C, & 1 \leq 2y + 1 \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$
Problem 2 (40 points). A discrete random variable \( K \) has the following probability mass function:

\[
p_K(k) = \begin{cases} 
1/2, & k = 0,1 \\
0, & \text{otherwise}.
\end{cases}
\]

A continuous random variable \( Z \) has the following conditional densities given \( K \):

\[
f_{Z|K}(z|k = 0) = \begin{cases} 
1, & 0 \leq z \leq 1, \\
0, & \text{otherwise},
\end{cases}
\]

\[
f_{Z|K}(z|k = 1) = \begin{cases} 
1/2, & 1 < z \leq 3, \\
0, & \text{otherwise}.
\end{cases}
\]

a (8 points). Find \( f_Z(z) \), the marginal probability density function for \( Z \).
b (8 points). Find \( M_Z(s) \), the moment generating function for \( Z \).
c (8 points). Find \( E[Z] \).
d (8 points). Find \( \text{var}(Z) \).
e (8 points). Find the following conditional probabilities: \( P(K = 1|Z \geq 2) \) and \( P(K = 2|Z \leq 2) \).

Solution.

a \( f_Z(z) = f_{Z|K}(z|k = 0)p_K(0) + f_{Z|K}(z|k = 1)p_K(1) = \begin{cases} 
1/2, & 0 \leq z \leq 1 \\
1/4, & 1 < z \leq 3 \\
0, & \text{otherwise}
\end{cases} \)

b Since the conditional distribution of \( Z \) given \( k = 0 \) is uniform between 0 and 1, the corresponding moment generating function is \( (e^s - 1)/s \). Similarly, the moment generating function associated with \( f_{Z|K}(z|k = 1) \) is \( (e^{3s} - e^s)/(2s) \). Combining these, we get:

\[
M_Z(s) = \frac{e^s - 1}{s} \cdot \frac{1}{2} + \frac{e^{3s} - e^s}{2s} \cdot \frac{1}{4} = \frac{2e^s - 2 + e^{3s} - e^s}{4s} = \frac{e^s + e^{3s} - 2}{4s}.
\]

c Method 1:

\[
E[Z] = \int_{-\infty}^{\infty} z f_{Z}(z) \, dz = \int_{0}^{1} \frac{z}{2} \, dz + \int_{1}^{3} \frac{z}{4} \, dz = \frac{z^2}{4} \bigg|_{0}^{1} + \frac{z^2}{8} \bigg|_{1}^{3} = \frac{1}{4} + \frac{9}{8} - \frac{1}{8} = \frac{5}{4}.
\]

Method 2: using the total expectation theorem,

\[
E[Z] = E[Z|k = 0]p_K(0) + E[Z|k = 1]p_K(1) = \frac{1}{2} \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{5}{4}.
\]

Method 3: using the properties of the moment generating function,

\[
E[Z] = \lim_{s \to 0} \frac{d}{ds} M_Z(s) = \lim_{s \to 0} \frac{(e^{s} + 3e^{3s})4s - 4(e^s + e^{3s} - 2)}{16s^2} = \lim_{s \to 0} \frac{se^s + 3se^{3s} - e^s - e^{3s} + 2}{4s^2} = \frac{5}{4}.
\]
d

\[ E[Z^2] = \int_{-\infty}^{\infty} z^2 f_Z(z) dz = \int_{0}^{1} \frac{z^2}{2} dz + \int_{1}^{3} \frac{z^2}{4} dz = \left[ \frac{z^3}{6} \right]_{0}^{1} + \left[ \frac{z^3}{12} \right]_{1}^{3} = \frac{1}{6} + \frac{27}{12} - \frac{1}{12} = \frac{28}{12} = \frac{7}{3}. \]

Therefore, \( \text{var}(Z) = E[Z^2] - (E[Z])^2 = \frac{7}{3} - \frac{25}{16} = \frac{37}{48}. \)

e Since \( K = 0 \) corresponds to \( 0 \leq Z \leq 1 \) and \( K = 1 \) corresponds to \( 1 < Z \leq 3 \), we have that \( Z \geq 2 \) implies \( K = 1 \): \( P(K = 1 | Z \geq 2) = 1 \). \( K \) can only be 0 or 1 with nonzero probability, and therefore \( P(K = 2 | Z \leq 2) = 0. \)
Problem 3 (10 points). Let \( Z \) be a continuous random variable, uniformly distributed between 0 and \( 2\pi \). Are the random variables \( X = \sin Z \) and \( Y = \cos Z \) uncorrelated? Are they independent? (Recall that \( X \) and \( Y \) are said to be uncorrelated if \( E[(X - E[X])(Y - E[Y])]) = 0 \).

Solution. We first find the expectations of \( X \) and \( Y \) and then show that \( \text{cov}(X,Y) = 0 \).

\[
E[X] = \int_0^{2\pi} \frac{1}{2\pi} \sin z \, dz = 0,
\]
\[
E[Y] = \int_0^{2\pi} \frac{1}{2\pi} \cos z \, dz = 0,
\]
\[
\text{cov}(X,Y) = E[XY] = E[\sin Z \cos Z] = \frac{1}{2} E[\sin 2Z] = \frac{1}{2} \int_0^{2\pi} \frac{1}{2\pi} \sin 2z \, dz = 0,
\]

therefore, \( X \) and \( Y \) are uncorrelated. However, note that the knowledge of \( Y \) gives us information about \( X \). For example, given that \( Y = -1 \), we know that \( Z \) must be equal to \( \pi \), and therefore \( X = 0 \): \( P(X = 0|Y = -1) = 1 \). However, given that \( Y = 0 \), \( Z \) is either \( \pi/2 \) or \( 3\pi/2 \), and therefore \( X \neq 0 \): \( P(X = 0|Y = 0) = 0 \). Thus, the conditional distribution for \( X \) given \( Y = y \) depends on \( y \), which means that \( X \) and \( Y \) are not independent.
Problem 4 (30 points). Recall that the least squares estimator of $X$ based on $Y$ is given by $\hat{X}(Y) = E[X|Y]$.

a (10 points). Suppose that $X$ and $Y$ are independent, identically distributed exponential random variables with parameter $\lambda = 3$.

(i) What is the least squares estimate of $X$ based on the observation $Y = 10$?
(ii) What is the linear least squares estimate of $X$ based on the same observation $Y = 10$?

Solution. Since $X$ and $Y$ are independent, $E[X|Y = 10] = E[X]$. Since $X$ is exponential with $\lambda = 3$, $E[X] = 1/3$. The least squares estimator is $\hat{X}(Y) = E[X]$ which is linear, and therefore the linear least squares estimator is the same, and the linear least squares estimate of $X$ based on $Y = 10$ is again $1/3$.

b (10 points). Now suppose that $Y = 2X + 3$, and that $X$ is a continuous uniform random variable, uniformly distributed between 0 and 10.

(i) What is the least squares estimate of $X$ based on the observation $Y = 13$?
(ii) What is the linear least squares estimate of $X$ based on the same observation $Y = 13$?

Solution. The least squares estimator in this case is $\hat{X}(Y) = (Y - 3)/2$ which achieves zero mean squared error since $X = \hat{X}(Y)$. Since $\hat{X}(Y)$ is a linear function of $Y$, it is also the linear least squares estimator. Given $Y = 13$, both estimates are therefore $(13 - 3)/2 = 5$.

c (10 points). Now suppose that the joint probability density function of $X$ and $Y$ is:

$$f_{X,Y}(x, y) = \begin{cases} \frac{6}{(1-y)^2}x^2 + \frac{6}{(1-y)^3}x, & 0 < x \leq 1 - y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) What is the least squares estimate of $X$ based on the observation $Y = y$ where $y$ is any fixed number between 0 and 1?
(ii) What is the \textbf{linear} least squares estimate of $X$ based on the same observation $Y = y$?

\textbf{Solution.} For a fixed $y = y_0$, the density $f_{X,Y}(x, y_0)$ is a parabola as a function of $x$. Note that $f_{X,Y}(0, y_0) = 0$ and $f_{X,Y}(1 - y_0, y_0) = 0$, and therefore $f_{X,Y}(x, y_0)$ is symmetric about $x = \frac{1 - y_0}{2}$, as shown in Fig. 1. The conditional density $f_{X|Y}(x|y_0)$, for a fixed $y = y_0$, is related to the joint density as follows:

$$f_{X|Y}(x|y_0) = \frac{f_{X,Y}(x, y_0)}{f_Y(y_0)},$$

i.e., it is the joint density normalized by a constant $f_Y(y_0)$. Therefore, $f_{X|Y}(x|y_0)$ must also be a parabola symmetric about $x = \frac{1 - y_0}{2}$. Therefore, $E[X|Y = y_0] = \frac{1 - y_0}{2}$, and $E[X|Y] = \frac{1 - Y}{2}$. This is the least squares estimator, and, since it is a linear function of $Y$, it is also the linear least squares estimator.

We may also obtain the same result by computing everything explicitly.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_{0}^{1-y} \left( -\frac{6}{(1-y)^3} x^2 + \frac{6}{(1-y)^2} x \right) dx$$

$$= \frac{6}{(1-y)^3} \left( -\frac{x^3}{3} \bigg|_{0}^{1-y} + (1-y) \frac{x^2}{2} \bigg|_{0}^{1-y} \right)$$

$$= \frac{6}{(1-y)^3} \left( -\frac{(1-y)^3}{3} + \frac{(1-y)^3}{2} \right) = 1, \text{ for } 0 < y \leq 1.$$

In other words, $Y$ is uniform between 0 and 1, and therefore

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = f_{X,Y}(x, y) \text{ for } 0 < x \leq 1 - y \leq 1.$$ 

Therefore,

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx = \int_{-\infty}^{\infty} x f_{X,Y}(x, y) dx = \int_{0}^{1-y} \left( -\frac{6}{(1-y)^3} x^3 + \frac{6}{(1-y)^2} x^2 \right) dx$$

$$= \frac{6}{(1-y)^3} \left( -\frac{x^4}{4} \bigg|_{0}^{1-y} + (1-y) \frac{x^3}{3} \bigg|_{0}^{1-y} \right)$$

$$= \frac{6}{(1-y)^3} \left( -\frac{(1-y)^4}{4} + \frac{(1-y)^4}{3} \right) = \frac{6}{(1-y)^3} \frac{(1-y)^4}{12} = \frac{1 - y}{2},$$

which is the same answer as we obtained before.
Problem 5 (10 points). The random variables $B$ and $C$ are jointly uniform over a $2\ell \times 2\ell$ square centered at the origin, i.e., $B$ and $C$ have the following joint probability density function:

$$f_{B,C}(b, c) = \begin{cases} \frac{1}{4\ell^2}, & -\ell \leq b \leq \ell \text{ and } -\ell \leq c \leq \ell, \\ 0, & \text{otherwise}. \end{cases}$$

It is given that $\ell \geq 1$. Find the probability that the quadratic equation $x^2 + 2Bx + C = 0$ has real roots. (Your answer will be an expression involving $\ell$.) What is the limit of this probability as $\ell \to \infty$?

Solution. The quadratic equation has real roots if and only if $B^2 - C \geq 0$. To find the probability of this event, we integrate the joint density over all points $(b, c)$ of the $2\ell \times 2\ell$ square for which $b^2 - c \geq 0$, i.e., over the gray set in Fig. 2. To do this, it is easier to integrate over the white portion of the square and subtract the result from 1:

$$1 - \int_{-\sqrt{\ell}}^{\sqrt{\ell}} \int_{-\sqrt{\ell}}^{\sqrt{\ell}} \frac{1}{4\ell^2} \, dc \, db = 1 - \int_{-\sqrt{\ell}}^{\sqrt{\ell}} \int_{-\sqrt{\ell}}^{\sqrt{\ell}} \frac{1}{4\ell^2} \, db \, dc - \frac{b^3}{12\ell^2} \bigg|_{-\sqrt{\ell}}^{\sqrt{\ell}}$$

$$= 1 - \frac{2\sqrt{\ell}}{4\ell} + \frac{2\sqrt{\ell}}{12\ell^2} = 1 - \frac{1}{3\sqrt{\ell}}.$$

As $\ell \to \infty$, this tends to 1.
Problem 6 (20 points). A stick of unit length is broken into two at random, i.e., the location of the breakpoint is uniformly distributed between the two ends of the stick.

a (10 points). What is the expected length of the smaller piece?

b (10 points). What is the expected value of the ratio \( \frac{\text{length of the smaller piece}}{\text{length of the larger piece}} \)? (You can use \( \ln 2 \approx 0.69 \).)

Solution.

a Let the length of the left piece be \( X \), and the length of the smaller piece be \( Y \). For \( X \leq 1/2 \), \( Y = X \), and for \( X > 1/2 \), \( Y = 1 - X \). In other words, \( f_{Y|X \leq 1/2}(y) = f_{X|X \leq 1/2}(y) \) which is uniform between 0 and 1/2, resulting in \( E[Y|X \leq 1/2] = 1/4 \), and \( f_{Y|X > 1/2}(y) = f_{1-X|X > 1/2}(y) \) which is also uniform between 0 and 1/2, resulting in \( E[Y|X > 1/2] = 1/4 \). Therefore, using the total expectation theorem,

\[
E[Y] = E[Y|X \leq 1/2]P(X \leq 1/2) + E[Y|X > 1/2]P(X > 1/2) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4}.
\]

b An application of the total probability theorem, instead of the total expectation theorem, in Part a, shows that

\[
f_Y(y) = f_{Y|X \leq 1/2}(y)P(X \leq 1/2) + f_{Y|X > 1/2}(y)P(X > 1/2) = \text{uniform between 0 and 1/2}.
\]

Therefore,

\[
E\left[ \frac{Y}{1-Y} \right] = \int_0^{1/2} 2 \cdot \frac{y}{1-y} \, dy = \int_{1/2}^1 2 \cdot \frac{1-v}{v} \, dv = 2 \ln v \bigg|_{1/2}^1 - 2 \ln v \bigg|_{1/2}^1 = 2 \ln 2 - 1 \approx 0.38,
\]

where we used a change of variable \( v = 1 - y \).