EE 302 Division 1 MWF 3:30-4:20 (Prof. Pollak)

- This is a closed-book exam. A formula sheet is provided. No calculators are allowed.
- You have two hours to complete FOUR problems.
- Be sure to fully and clearly explain all your answers.
- There will not be any discussion of grades. All re-grade requests must be submitted in writing.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
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<tr>
<td>1</td>
<td>65</td>
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<td>2</td>
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<td>3</td>
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<td>4</td>
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<td>TOTAL</td>
<td>210</td>
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Some random variables, their distributions, and associated transforms:

<table>
<thead>
<tr>
<th>Random variable</th>
<th>PMF or PDF</th>
<th>Mean</th>
<th>Variance</th>
<th>Transform</th>
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<tbody>
<tr>
<td>Bernoulli</td>
<td>$p$ for $k = 1; 1 - p$ for $k = 0.$</td>
<td>$p$</td>
<td>$p(1-p)$</td>
<td>$1 - p + pe^s$</td>
</tr>
<tr>
<td>Discrete uniform</td>
<td>$\frac{1}{n}, k = k_0 + 1, k_0 + 2, \ldots, k_0 + n$</td>
<td>$k_0 + \frac{n + 1}{2}$</td>
<td>$\frac{n^2 - 1}{12}$</td>
<td>$\frac{e^s(e^{(k_0+n)s} - e^{k_0s})}{n(e^s-1)}$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$(1 - p)^{k-1}p, k = 1, 2, 3, \ldots$</td>
<td>$\frac{1}{p}$</td>
<td>$\frac{1}{p^2} - \frac{1}{p}$</td>
<td>$\frac{pe^s}{1-(1-p)e^s}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\binom{n}{k}(1-p)^{n-k}p^k, k = 0, 1, \ldots, n$</td>
<td>$pn$</td>
<td>$np(1-p)$</td>
<td>$(1 - p + pe^s)^n$</td>
</tr>
<tr>
<td>Pascal of order $k$</td>
<td>$\binom{t-1}{k-1}p^k(1-p)^{t-k}, t = k, k+1, \ldots$</td>
<td>$k\left(\frac{1}{p^2} - \frac{1}{p}\right)$</td>
<td>$\left(\frac{pe^s}{1-(1-p)e^s}\right)^k$</td>
<td></td>
</tr>
<tr>
<td>Poisson</td>
<td>$e^{-\lambda \frac{\lambda^k}{k!}}, k = 0, 1, 2, \ldots$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$e^{\lambda(e^s-1)}$</td>
</tr>
<tr>
<td>Continuous uniform</td>
<td>$\frac{1}{b-a}, a \leq x \leq b$</td>
<td>$\frac{b+a}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
<td>$\frac{e^{sb} - e^{sa}}{(b-a)s}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\lambda e^{-\lambda x}, x \geq 0$</td>
<td>$\lambda^{-1}$</td>
<td>$\lambda^{-2}$</td>
<td>$\frac{\lambda}{\lambda-s}$</td>
</tr>
<tr>
<td>Normal (Gaussian)</td>
<td>$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
<td>$\frac{e^{\frac{x^2}{2} + \mu s}}{\frac{\sigma}{\lambda-s}}$</td>
</tr>
<tr>
<td>Erlang of order $k$</td>
<td>$\frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, y \geq 0$</td>
<td>$k\frac{1}{\lambda}$</td>
<td>$k\frac{1}{\lambda^2}$</td>
<td>$\left(\frac{\lambda}{\lambda-s}\right)^k$</td>
</tr>
</tbody>
</table>

where

\[
\binom{n}{k} = \frac{n!}{(n-k)!k!}.
\]
Problem 1 (65 points). Two independent discrete random variables $X$ and $Y$ have the following probability mass functions:

$$
p_X(x) = \begin{cases} 
\frac{1}{3} & x = 0, 1, 2 \\
0 & \text{otherwise}
\end{cases}$$

$$
p_Y(y) = \begin{cases} 
\frac{y}{11} & y = 1, 2, 3, 4 \\
0 & \text{otherwise}
\end{cases}$$

a (7 points). Find the expected value of $X$.

b (7 points). Find the variance of $X$.

c (8 points). Find the transform associated with $X$.

d (7 points). Find the correlation coefficient of $X$ and $Y$.

e (8 points). Find the joint probability mass function $p_{X,Y}(x, y)$, for all $x$ and $y$.

f (8 points). Given $Y = 3$, find the conditional probability mass function of $X$, i.e. find $p_{X|Y}(x|3)$.

g (8 points). Let $Z = (X - 1)^{200}$. Find the probability mass function of the random variable $Z$.

h (12 points). Let $W = X + Y$. Find the probability mass function of the random variable $W$.

Fully and clearly explain all your answers.

Solution.

a. $E[X] = \frac{1}{3}(0 + 1 + 2) = 1$.

b. $\text{var}(X) = \frac{1}{3}((0 - 1)^2 + (1 - 1)^2 + (2 - 1)^2) = \frac{2}{3}$.

c. Using the definition, $M_X(s) = E[e^{sX}] = \frac{1}{3}(1 + e^s + e^{2s})$.

d. Since $X$ and $Y$ are independent, they are uncorrelated, and so the correlation coefficient is zero.

e. Since $X$ and $Y$ are independent, the joint PMF is the product of the marginal PMF’s:

$$
p_{X,Y}(x, y) = \begin{cases} 
\frac{y}{30} & x = 0, 1, 2; \ y = 1, 2, 3, 4; \\
0 & \text{otherwise}
\end{cases}$$

f. Since $X$ and $Y$ are independent, $p_{X|Y}(x|3) = p_X(x)$ which is given above.

g. Note that if $X = 0, 2$, then $Z = 1$; if $X = 1$, then $Z = 0$. Therefore,

$$
p_Z(z) = \begin{cases} 
\frac{1}{2} & z = 0 \\
\frac{2}{3} & z = 1 \\
0 & \text{otherwise}
\end{cases}$$
h. Since $X$ and $Y$ are independent, the PMF of their sum is the convolution of $p_X$ and $p_Y$:

$$p_W(w) = \sum_{k=-\infty}^{\infty} p_X(k)p_Y(w-k)$$

$$= p_X(0)p_Y(w) + p_X(1)p_Y(w-1) + p_X(2)p_Y(w-2)$$

$$= \frac{1}{3}(p_Y(w) + p_Y(w-1) + p_Y(w-2))$$

$$= \begin{cases} 
\frac{1}{30}, & w = 1 \\
\frac{3}{30}, & w = 2 \\
\frac{3}{30}, & w = 3 \\
\frac{3}{30}, & w = 4 \\
\frac{4}{30}, & w = 5 \\
\frac{3}{30}, & w = 6 
\end{cases}$$
Problem 2 (50 points). Consider the Markov chain below. For all parts of this problem, the process is in state 0 immediately before the first transition.

\[
\begin{array}{ccc}
2 & & 3 \\
1/2 & 3/4 & 1/2 \\
1/4 & & 1/4 \\
& 0 & \\
& & 1
\end{array}
\]

a (8 points). Find the variance for \(J\), the number of transitions up to and including the transition on which the process leaves state 0.

b (8 points). Find the probability to go to state 3 for the first time on the 5-th transition.

c (10 points). Find the probability to be at state 3 after 5 transitions. (Note: this event includes getting to 3 before the 5-th transition.)

d (7 points). Find the probability to never get to state 1.

e (10 points). Find \(\pi_i\) for \(i = 0, 1, 2, 3\), the probability that the process is in state \(i\) after 10 transitions or explain why these probabilities cannot be found.

f (7 points). Given that the process never enters state 1, find the \(\pi_i\)’s as defined in part (e) or explain why they cannot be found.

Fully and clearly explain all your answers.

Solution.

a. We start at zero and, during each transition we either stay at zero (with probability 1/2) or leave 0 (also with probability 1/2). Since the transitions are independent of each other, this is a Bernoulli process with parameter \(p = 1/2\), and \(J\) is its first arrival time, which is geometric with parameter \(p = 1/2\) and therefore has variance \(p^{-2} - p^{-1} = 4 - 2 = 2\).

b. This is equivalent to staying at zero after each of the first four transitions, and then going to 3: \((1/2)^4 \cdot (1/4) = 1/64\).

c. Now we can stay at zero after each of the first \(n\) transitions, and then go to 3, for \(n = 0, 1, 2, 3, 4\):

\[
\sum_{n=0}^{4} \left(\frac{1}{2}\right)^n \cdot \frac{1}{4} = \frac{1 - \left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}} \cdot \frac{1}{4} = \frac{31}{64}.
\]

d. The probability to stay at 0 forever is 0, and therefore we will eventually be absorbed into 1 or 3, with equal probabilities. Therefore, the probability to never get to 1 is the probability to get absorbed into 3, which is 1/2.
e. This question is similar to Homework 11 Problem 3(c)—see the homework solutions for detailed explanations. The steady-state probability for the transient state 0 is 0; the steady-state probabilities for the three recurrent states are equal to the absorption probabilities multiplied by the conditional steady-state probabilities:

\[
\begin{align*}
\pi_0 &= 0 \\
\pi_1 &= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \\
\pi_2 &= \frac{1}{2} \cdot 1 = \frac{1}{6} \\
\pi_3 &= \frac{1}{2} \cdot 1 = \frac{1}{2}
\end{align*}
\]

f. Given that the process never enters state 1, we immediately know that \(\pi_0 = \pi_1 = \pi_2 = 0\), and therefore (because of the normalization property) \(\pi_3 = 1\).
Problem 3 (40 points). The interarrival times for cars passing a checkpoint are independent random variables with PDF
\[ f_T(t) = \begin{cases} 2e^{-2t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \]
where the interarrival times are measured in minutes. The successive experimental values of the durations of interarrival times are automatically recorded on small cards. The recording operation occupies a negligible time period following each arrival. Each card has space for three entries. As soon as a card is filled, it is replaced by the next card.

a (5 points). Find the expected number of minutes between two consecutive car arrivals.

b (9 points). Given that no car has arrived in the last four minutes, determine the PMF for random variable \( K \), the number of cars to arrive in the next six minutes.

c (9 points.) Find the PDF, the expected value, and the transform, for the total time required to use up the first dozen cards.

d (9 points.) We pick an arbitrary card and note the total time, \( T_i \), the card was in service. Find \( E[T_i] \) and \( \text{var}(T_i) \).

e (8 points.) Given that the card presently in use contains exactly two entries and also that it has been in service for exactly 0.5 minutes, determine and sketch the PDF for the remaining time until the card is completed.

Fully and clearly explain all your answers.

Solution.

a. This is just the mean of an exponential random variable with parameter 2: \( \frac{1}{\lambda} = \frac{1}{2} \).

b. Since Poisson process is memoryless, the number of cars to arrive during any fixed six-minute interval is Poisson with parameter \( 6\lambda = 12 \), regardless of what happened in the past:
\[ p_K(k) = e^{-12} \frac{12^k}{k!}, \quad k = 0, 1, 2, \ldots \]

c. Since each card corresponds to three arrivals, we are asked for the 36-th arrival time whose distribution is Erlang of order 36:
\[
\text{PDF} = \frac{2^{36}t^{35}e^{-2t}}{35!}, \quad t \geq 0 \\
\text{mean} = \frac{36}{2} = 18 \\
\text{transform} = \left( \frac{2}{2 - s} \right)^{36}
\]
d. \( T_i \) is a third-order interarrival time, and its PDF is therefore Erlang of order 3: \( E[T_i] = \frac{3}{2}, \quad \text{var}(T_i) = \frac{3}{4}. \)

e. Again, because of the memorylessness of the Poisson process, this is just a first-order interarrival time which is exponential with parameter 2. The formula is given above in the statement of the problem, and a sketch is below.
Problem 4 (55 points). Continuous random variable $Y$ is uniformly distributed between 0 and 1. Given $Y = y$, continuous random variable $X$ is conditionally uniform between 0 and $1 + y$.

a (10 points). Find $E[X]$.

b (15 points). Find $\text{var}(X)$.

c (10 points). Find $f_{X,Y}(x, y)$, the joint PDF of $X$ and $Y$, and sketch the region of the $xy$ plane where $f_{X,Y}(x, y) \neq 0$.

d (10 points). Find the probability of the following event: $2X - 1 \geq Y$.

e (10 points). Find the linear least mean squares estimator $g(Y)$ for estimating $X$ based on $Y$.

Fully and clearly explain all your answers.

Solution.

a. Using the iterated expectation formula, we have:

\[
E[X] = E[E[X|Y]] = E \left[ \frac{1 + Y}{2} \right] = \frac{1 + 1/2}{2} = \frac{3}{4}.
\]

b. Using the law of total variance, we have:

\[
\text{var}(X) = E[\text{var}(X|Y)] + \text{var}(E[X|Y]) = E \left[ \frac{(1 + Y)^2}{12} \right] + \text{var} \left( \frac{1 + Y}{2} \right)
\]

\[
= \frac{1}{12} E[1 + 2Y + Y^2] + \frac{1}{4} \text{var}(Y) = \frac{1}{12} (1 + 1 + \frac{1}{3}) + \frac{1}{4} \cdot \frac{1}{12} \cdot \frac{31}{12} = \frac{31}{144}.
\]

(Note: both Part (a) and Part (b) can be done using different methods, such as, for example, first calculating the joint PDF of $X$ and $Y$ and then integrating to obtain the mean and the variance of $X$.)

c. The support of $f_{X,Y}(x, y)$ is shown with thick lines in the figure:

\[
f_{X,Y}(x, y) = f_Y(y) f_{X|Y}(x|y) = \begin{cases} 
\frac{1}{1+y}, & 0 \leq y \leq 1, \ 0 \leq x \leq y + 1 \\
0 & \text{otherwise}
\end{cases}
\]
d. In the sketch of the support of the joint PDF, the intersection of the support with the event $2X - 1 \geq Y$ is shaded. It is clear from the picture that, for any given $Y = y$, the conditional probability of the event $2X - 1 \geq y$ is $1/2$ since the conditional density of $X$ is uniform. Integrating with respect to $y$ shows that the overall unconditional probability must also be $1/2$. More formally,

$$P(2X - 1 \geq Y) = \int_0^1 \int_{-y/2}^{y+1} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1 \left( \int_{-y/2}^{y+1} \frac{1}{1+y} \,dx \right) \,dy$$

$$= \int_0^1 \frac{1}{2} \,dy = \frac{1}{2}. \tag{10}$$

Note that it is wrong to say that the answer is $1/2$ simply because the shaded area is one-half of the area of the whole trapezoid: since the joint PDF is not uniform, probabilities cannot be obtained by dividing the areas. Those students who did this—despite the fact that their answer in part (c) was not a uniform density—got 0 points for this part.

e. To solve this part, it was not necessary to remember the formula for the linear least mean squares estimator. (However, we applaud the one student in the class who did!) Note that the least mean squares estimator, $E[X|Y] = \frac{1}{2}Y + \frac{1}{2}$ happens to be a linear function of $Y$ in this case. Therefore, it must also be the linear least mean squares estimator. (Our applause also goes to the single student in the class who came up with this argument.)