EE 302 Division 1
Exam 1 Solutions, 2/13/2002.

• This is a closed-book exam. A formula sheet is provided. No calculators are allowed.

• You have one hour to complete THREE problems.

• Be sure to fully and clearly explain all your answers.

• There will not be any discussion of grades. All re-grade requests must be submitted in writing, as stated in the course information handout.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
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<tbody>
<tr>
<td>1</td>
<td>40</td>
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<tr>
<td>2</td>
<td>60</td>
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<td>3</td>
<td>20</td>
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<tr>
<td>TOTAL</td>
<td>120</td>
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</tbody>
</table>
Some random variables and their PMF’s:

<table>
<thead>
<tr>
<th>Random variable</th>
<th>PMF</th>
<th>Mean $\mu$</th>
<th>Variance $\text{Var}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete uniform</td>
<td>$\frac{1}{n}, k = k_0 + 1, k_0 + 2, \ldots, k_0 + n$</td>
<td>$k_0 + \frac{n+1}{2}$</td>
<td>$\frac{n^2-1}{12}$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$(1 - p)^{k-1} p, k = 1, 2, 3, \ldots$</td>
<td>$\frac{1}{p}$</td>
<td>$\frac{1}{p^2} - \frac{1}{p}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\binom{n}{k} (1 - p)^{n-k} p^k, k = 0, 1, \ldots, n$</td>
<td>$pn$</td>
<td>$np(1 - p)$</td>
</tr>
</tbody>
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where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$
Problem 1 (40 points). For three tosses of a fair coin, determine the probability of:

a (10 points). The sequence HHH.

b (10 points). A total result of two heads and one tail.

c (10 points). The event “More heads than tails”.

Determine also the conditional probability for:

d (10 points). “More heads than tails” given “At least one tail”.

Fully and clearly explain all your answers.

Solution. There are $2^3 = 8$ outcomes in the sample space. Since the coin is fair, all the outcomes are equiprobable. The probability of each outcome is therefore equal to $1/8$.


b. The event “two heads and one tail” is $\{HHT, HTH, TTH\}$. Since it consists of three outcomes, its probability is $3/8$.

c. The event “more heads than tails” is $\{HHT, HTH, THH, HHH\}$. Since it consists of four outcomes, its probability is $4/8 = 1/2$.

d. “At least one tail” includes all outcomes but HHH, i.e. seven outcomes. Three of those seven (namely, HHT, HTH, and THH) result in more heads than tails. Therefore, the conditional probability is $3/7$. 


**Problem 2 (60 points).** Consider taking a round trip from city A to city B and back to city A. The probability to lose a suitcase during the outbound flight (i.e. the flight from A to B) is \( \frac{1}{5} \). If it is not lost during the outbound flight, the probability to lose it during the inbound flight (i.e. the flight from B back to A) is \( \frac{1}{16} \). The fates of all suitcases are mutually independent.

**a.** Prof. Pollak takes one round trip and loses both his suitcases in the process. Given this information, determine the conditional probabilities that:

[i] (10 points). One suitcase was lost during the outbound flight and one was lost during the inbound flight.

[ii] (10 points). Both suitcases were lost during the outbound flight.

[iii] (10 points). Both suitcases were lost during the inbound flight.

You may assume that the fates of the two suitcases are conditionally independent given that both of them are lost.

**b** (15 points). On the average, how many round trips (city A - city B - city A) would someone who travels with exactly one suitcase be able to make until he loses his suitcase? (Include the last trip---i.e. the trip during which the suitcase was lost---in your count.)

**c** (15 points). A family traveling with 10 suitcases makes the round trip city A - city B - city A. What is the expectation of the number of suitcases that they will lose?

Fully and clearly explain all your answers.

**Solution.**

![Sequential model for the fate of a single suitcase](image)

\[
P(\text{lost during the inbound flight}) = \frac{4}{5} \cdot \frac{1}{16} = \frac{1}{20}.
\]

\[
P(\text{lost during the outbound flight}) = \frac{1}{5}
\]

\[
P(\text{lost during the outbound flight}) = \frac{1}{5}
\]

**a.** A sequential model for the fate of a single suitcase is depicted in the figure: the first stage of the figure corresponds to the outbound flight, and the second stage corresponds to the inbound flight. The probability that a suitcase is lost during the inbound flight is the probability that it is not lost during the outbound flight and then is lost on the inbound flight, which is \( \frac{4}{5} \cdot \frac{1}{16} = \frac{1}{20} \).
The total probability that a suitcase is lost during the trip is therefore $1/5 + 1/20 = 1/4$. The conditional probability that the suitcase is lost on the outbound flight, given that it is lost, is, using the definition of conditional probability, $(1/5)/(1/4) = 4/5$, and the conditional probability that it is lost on the inbound flight given that it is lost, is $(1/20)/(1/4) = 1/5$. Since the losses of the two suitcases are conditionally independent, the conditional probability that both were lost on the outbound flight is $(4/5)^2 = 16/25$, the conditional probability that both were lost on the inbound flight is $(1/5)^2 = 1/25$, and the conditional probability that one was lost on the outbound flight and one was lost on the inbound flight is $1 - 16/25 - 1/25 = 8/25$:

[i] $\frac{8}{25}$.
[ii] $\frac{16}{25}$.
[iii] $\frac{1}{25}$.

b. For this part, we are only interested in whether or not a suitcase was lost during a series of round trips, i.e., we are conducting independent Bernoulli trials with probability of “success” (i.e., probability of losing the suitcase during a trip) $p = 1/4$ which was calculated in Part a. If $X$ is the number of trips until the suitcase is lost—including the trip during which the suitcase was lost—then $X$ is a geometric random variable with parameter $p = 1/4$, and therefore $E[X] = 4$.

c. We are now conducting 10 independent Bernoulli trials with probability of “success” (i.e., probability of losing the suitcase) $p = 1/4$, and are looking for the number of “successes” (i.e. the number of lost suitcases). This is a binomial random variable with parameters $n = 10$ and $p = 1/4$ which therefore has expectation $np = 10/4 = 2.5$. 


Problem 3 (20 points). In this communication network, link failures are independent, and each link has a probability of failure of \( p \). Consider the physical situation before you write anything. \( A \) can communicate with \( B \) as long as they are connected by at least one path which contains only in-service links. (Note: \( a, b, c, d, e, f, g, \) and \( h \) are just the names of the links, NOT probabilities.)

(a) (10 points). Given that exactly five links have failed, determine the conditional probability that \( A \) can communicate with \( B \).

(b) (10 points). Given that \( a, d, \) and \( h \) have failed (but no information about the condition of other links), determine the probability that \( A \) can communicate with \( B \).

Fully and clearly explain all your answers.

Solution.

a. Since the failures of all groups of exactly five links are equally likely (namely, the probability that any specific five links have failed and the other three are working, is \( p^5(1-p)^3 \)), our conditional probability law is uniform. Therefore, we need to simply count the number of outcomes in the event “\( A \) can communicate with \( B \)” given that five links have failed, and divide it by the number of the outcomes in the conditioning event “five links have failed”. The total number of groups of five links out of eight that could have failed is \( \binom{8}{5} = \frac{8!}{3!5!} = \frac{8 	imes 7 	imes 6}{3 	imes 2 	imes 1} = 56 \). The only situations when five links are out but \( A \) can still communicate with \( B \) are when \( a, d, \) and \( g \) are up, or when \( b, c, \) and \( h \) are up. The conditional probability is therefore \( \frac{2}{56} = \frac{1}{28} \).

b. The conditional probability that \( A \) can communicate with \( B \) is the probability that \( b, c, f, \) and \( g \) are working (and it does not matter whether \( e \) has failed or not), which is \( (1-p)^4 \).