These problems have been constructed over many years, using many different sources. If you find that some problem or solution is incorrectly attributed, please let me know at ipollak@ecn.purdue.edu.

Suggested reading: Sections 7.1-7.4 in the recommended text [1]. Equivalently, Sections 8.1-8.4 (discrete-time only) in the Leon-Garcia text [3].

Problem 1. Random walk in 1-D. (Ilya Pollak and Bin Ni.)
Random walks and the resulting difference and differential equations are used in a variety of fields, for example, physics (to model particle and heat diffusion processes), finance (to model the prices of financial instruments, to manage market risk, and to develop automated trade execution algorithms), and image processing (to design noise-suppressing filters and image reconstruction procedures). This problem explores a 1-D random walk model.

Prof. Pollak likes to take a stroll every night. Since he is completely absorbed in thinking about his research, his steps are somewhat random. Each step he takes is in one of two directions, east or west, with respective probabilities \( \frac{1}{2} \) and \( \frac{1}{2} \). The size of each step is exactly 1 meter. The direction of each step is independent of the past steps. Let \( X_n \) be his coordinate (in meters) after \( n \) steps, for \( n \geq 0 \).

(a) Let \( p(n, k) \) be the probability that \( X_n = k \), i.e., that he is at location \( k \) after \( n \) steps. Express \( p(n, k) \) in terms of \( p(n - 1, k - 1) \) and \( p(n - 1, k + 1) \).

(b) Suppose that \( X_0 = 0 \). Without using Matlab, find \( p(10, 5) \), \( p(10, 10) \), and \( p(10, 12) \). Using Part (a) and Matlab, find \( p(10, 4) \).

(c) Again, suppose \( X_0 = 0 \). Experimentally estimate (in Matlab) the probability that Prof. Pollak will eventually come home (i.e., that there exists \( n > 0 \) for which \( X_n = 0 \)).

(d) Let \( P_{i,k} \) be the conditional probability to eventually get to \( k \) having started at \( i \), i.e., \( P_{i,k} = P(\text{there exists } n > 0 \text{ for which } X_n = k \mid X_0 = i) \).

(i) Argue that \( P_{1,0} = P_{2,1} \).

(ii) Show that \( P_{2,0} = P_{2,1} P_{1,0} \), and therefore, by (i), that \( P_{2,0} = P_{1,0}^2 \).

(iii) Show that \( P_{1,0} = 0.5 + 0.5P_{2,0} \).

(iv) Find \( P_{1,0} \) by combining (iii) and (ii).

(v) Argue that \( P_{-1,0} = P_{1,0} \), and find \( P_{0,0} \). Make sure that your result agrees with your experimental finding of Part (c).

Solution.
(a) By the total probability theorem,

\[ p(n, k) = \mathbb{P}(X_n = k) = \sum_{i=-\infty}^{\infty} \mathbb{P}(X_n = k | X_{n-1} = i)\mathbb{P}(X_{n-1} = i) \]

\[ = \mathbb{P}(X_n = k | X_{n-1} = k - 1)\mathbb{P}(X_{n-1} = k - 1) + \mathbb{P}(X_n = k | X_{n-1} = k + 1)\mathbb{P}(X_{n-1} = k + 1) \]

\[ = \frac{1}{2}p(n - 1, k - 1) + \frac{1}{2}p(n - 1, k + 1). \]

(b) First, for \(|k| > n, p(n, k) = 0\), because Prof. Pollak cannot go farther than the total number of steps. For \(|k| \leq n\), suppose Prof. Pollak reached point \(k\) with \(n\) steps, having made \(n_1\) steps east and \(n_2\) steps west. Then \(n_1\) and \(n_2\) must be integer solutions to the following linear equations:

\[ n_1 + n_2 = n, \]

\[ n_1 - n_2 = k. \]

Therefore

\[ n_1 = (n + k)/2 \text{ and } n_2 = (n - k)/2. \]

Note that there are \( \binom{n}{n_1} = \frac{n!}{(n - n_1)!n_1!} \) possible ways to choose the \(n_1\) eastward steps out of the total of \(n\) steps, resulting in \( \binom{n}{n_1} \) distinct \(n\)-step paths that lead from 0 to \(k\). The probability of each path is \((1/2)^n\), which means that, when \(n_1\) is an integer, the overall probability to reach \(k\) in \(n\) steps is

\[ p(n, k) = \binom{n}{n_1} \left(\frac{1}{2}\right)^n, \]

which is a binomial distribution.

If \(n_1, n_2\) we get from Eq. (1) are not integers, then it’s impossible for Prof. Pollak to reach point \(k\) with \(n\) steps, and therefore

\[ p(n, k) = 0. \]

In fact, Prof. Pollak can only reach odd coordinates with an odd number of steps and even coordinates with an even number of steps. According to the analysis above, we get:

(i) \(n = 10, k = 5\).

Since he cannot reach \(k = 5\) with an even number of steps, \(p(10, 5) = 0\);

(ii) \(n = 10, k = 10\).

From Eq. (1) we get \(n_1 = (10 + 10)/2 = 10, n_2 = 0\). All of the 10 steps are towards east.

The probability for this to happen is

\[ \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}. \]

(iii) Since \(k = 12 > n = 10, p(10, 12) = 0. \)
(iv) \( n = 10, k = 4 \).
From Eq. (1), \( n_1 = (10 + 4)/2 = 7 \), and therefore, from Eq. (2),
\[
p(10, 4) = \frac{10!}{7!3!} \cdot \frac{1}{1024} = \frac{8 \cdot 9 \cdot 10}{6 \cdot 1024} = \frac{15}{128} \approx 0.1172.
\]
Alternatively, can use recursion from Part (a) to calculate \( p(10, 4) \) in Matlab. The code is given below:

```matlab
function p = prob(n,k) % To calculate p(n,k)
    if n>1
        p = (prob(n-1, k-1)+prob(n-1, k+1))/2; % Recursively calculate it according to
                                                % the equation we get in Part (a)
    else if (k==1|k==-1) % If n=1, that’s easy!
        p = 0.5;
    else
        p = 0;
    end
end
```

(c) One way to estimate this probability in Matlab is to simulate the strolling of Prof. Pollak. We can increase/decrease a variable by 1 randomly at each step with equal probability and observe whether it comes back to its initial value after some finite number of steps \( N \). For each \( N \), we can run many experiments (2000 in the code below) to estimate the probability of coming back, and then see whether our estimate appears to converge to some number for large \( N \). The Matlab code is given below:

```matlab
TestNum=2000; % Total number of simulations.
MaxN=1000; % Maximum number of steps.
ReturnSteps=zeros(1,MaxN); % Allocate memory for ReturnSteps.
    % ReturnSteps(n) records how many simulations
    % return to origin at the n-th step.
ProbN=zeros(1,MaxN); % Allocate memory for ProbN.
    % ProbN(n) records the probability that
    % Prof. Pollak returns home within n steps
for sim_i=1:TestNum
    k=0; % Initialize the variable k to 0
    % k is used to store the position of Prof.
    for N=1:MaxN
        d=rand(1,1); % Generate a random variable to make a decision
        if d<0.5
            k=k-1; % Half of the time, k is decreased by 1.
        else
            k=k+1; % Half of the time, k is increased by 1.
        end
        if k==0
            ReturnSteps(N)=ReturnSteps(N)+1;
            ProbN(N)=ProbN(N)+1/2000; % Update the probability
        else
            ReturnSteps(N)=ReturnSteps(N)+0;
            ProbN(N)=ProbN(N)+0/2000;
        end
    end
end
```
end
if k==0 % Check if Prof. Pollak returns home.
    ReturnSteps(N)=ReturnSteps(N)+1; % Record the number of steps he has gone.
    break % Start a new simulation
end
end
ProbN=cumsum(ReturnSteps, 2); % ProbN(n) now stores how many simulations % return home within n steps
ProbN=ProbN/TestNum; % To get the probability
set(gca, 'fontsize', 16); plot(ProbN); xlabel('N');
ylabel('Prob. of return within N steps');

Fig. 1 shows the estimate of the probability to return as a function of the total number of steps N. The figure suggests that perhaps this probability approaches 1 as $N \to \infty$.

(d) (i) If we just relabel the coordinate axes by adding 1 to each coordinate, Prof. Pollak’s behavior will not change. The probability for him to get back to where he started remains the same if he starts 1 meter east of his home. Therefore,

$$ P_{1,0} = P_{2,1}. $$

(ii) If Prof. Pollak wants to go from 2 to 0, he must reach 1 first. The probability of that happening is $P_{2,1}$. Once he gets to 1, he has probability $P_{1,0}$ to reach 0. These two stages are independent. Therefore,

$$ P_{2,0} = P_{2,1}P_{1,0} = P_{1,0}^2. $$
(iii) There are two ways to reach 0 from 1: with probability 0.5, he will go from 1 to 0 directly; with probability 0.5, he will go from 1 to 2, and, once at 2, he will have probability $P_{2,0}$ to reach 0:

$$P_{1,0} = 0.5 + 0.5P_{2,0}.$$  

(iv) Combining Parts (ii) and (iii), we have:

$$P_{1,0} = 0.5 + 0.5P^2_{1,0}\Rightarrow P^2_{1,0} - 2P_{1,0} + 1 = 0\Rightarrow P_{1,0} = 1.$$  

(v) Similarly to Part (i), if we flip the coordinate axes, the probability to get from 1 to 0 will remain the same, since the probability to make one step east is the same as the probability to make one step west. Therefore,

$$P_{-1,0} = P_{1,0} = 1.$$  

After the first step, the probability to be located at 1 is 0.5, and the probability to be located at $-1$ is 0.5, and in either case, the probability to come back home is 1:

$$P_{0,0} = 0.5P_{-1,0} + 0.5P_{1,0} = 1.$$  

Note that the probability to come home is alternatively viewed as the absorption probability for state 0 in the Gambler’s Ruin problem worked out in class (example 7.11 from [1]). In our case, both the birth and death probabilities are 1/2, and the number of states $m \to \infty$. In this scenario, it was shown in class that the probability to get to state 0 from any initial state is 1.

**Problem 2. Random walk in a plane. (Ilya Pollak.)**

When trying to compose new probability problems, Prof. Pollak likes to aimlessly wander around Purdue Mall. Each step he takes is in one of four directions: east, west, north, or south, with respective probabilities $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$. The size of each step is exactly 1 meter. Thus, if he starts out at the point (0, 0), he will be at one of the points $(-1, 0)$, $(1, 0)$, $(0, -1)$, $(0, 1)$, after his first step, etc. His wanderings are truly aimless: the direction of each step is independent of the past steps. Let $X_n$ and $Y_n$, for $n \geq 1$, be his coordinates (in meters) after $n$ steps, and let $X_0 = Y_0 = 0$.

(a) Find the probability that $X_{2012} = 302$ and $Y_{2012} = 223$. (I.e., find the probability that, after having started at the origin and making 2012 steps, he is at the point $(302, 223)$.)

(b) Find the formula, in terms of $k$ and $n$, of the probability that $X_n = k$ and $Y_n = n - k$, where $k$ and $n$ are fixed integers, with $0 \leq k < n$.

**Solution.** For Part (a), note that, after an even number of steps, $X_n + Y_n$ must be even, and therefore the probability that $X_{2012} = 302$ and $Y_{2012} = 223$ is zero. To see this, color the $XY$-plane as a chessboard, with $(0, 0)$ black, $(-1, 0)$, $(1, 0)$, $(0, -1)$, and $(0, 1)$ white, etc. Then, after 2012 steps, Prof. Pollak must be at a black point, whereas $(302, 223)$ is white.
The relevant sample space for Part (b) is the set of all \( n \)-step paths which start at the origin. Since the \( n \) steps are independent, the probability of each such path is \( 0.25^n \). [Another way of looking at it is that, since the paths are equiprobable, and since there are 4\(^n \) of them (as there are four possible directions for each of the \( n \) steps), the probability of each path is \( \frac{1}{4^n} \).]

We are interested in all the paths which terminate at the point \((k, n - k)\). Suppose there are \( M \) such paths. Then the probability to end up at the point \((k, n - k)\) after \( n \) steps is

\[ M \cdot \frac{1}{4^n} \]

In order to find \( M \), notice that, on the way from \((0, 0)\) to \((k, n - k)\), Prof. Pollak has to make at least \( k \) steps east and at least \( n - k \) steps north—i.e., at least \( n \) steps altogether. Since the overall number of steps is exactly \( n \), he must make exactly \( k \) steps east and \( n - k \) steps north. So, \( M \) is the number of \( n \)-step paths which consist of \( k \) steps east and \( n - k \) steps north, which is the number of ways to choose \( k \) objects out of \( n \):

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

Answer: the probability that \( X_n = k \) and \( Y_n = n - k \) is

\[ \frac{n!}{k!(n-k)!4^n} \]

Example. If \( n = 4 \) and \( k = 2 \), there are \( \frac{4!}{2!2!} = 6 \) different 4-step paths that originate at \((0, 0)\) and terminate at \((2, 2)\):
- east, east, north, north;
- east, north, east, north;
- east, north, north, east;
- north, east, east, north;
- north, east, north, east;
- north, north, east, east.
Thus, the probability to get to \((2, 2)\) from \((0, 0)\) in four steps is \( \frac{6}{4^4} = \frac{3}{128} \).

Problem 3. 1-D random walk with reflecting barriers. (After Bertsekas-Tsitsiklis [1] Example 7.8.)

When it rains, Prof. Pollak must do all his thinking inside his office, while pacing back and forth. (He is too excited about his teaching and research to be sitting down.) Every second, he either takes one step east or one step west or stays put. The size of each step is exactly one meter. Needless to say, Prof. Pollak’s complete mental focus on his work makes his pacing a Markov chain: his each step is always independent of his past steps.

The length of the office from the window in the west to the wall in the east is \( m - 1 \) meters. The origin is located at the window, and Prof. Pollak’s coordinate \( X_n \) at time \( n \) is measured in meters. Therefore, \( X_n \) is always one of the following \( m - 1 \) integers: 0 (the window), 1, 2, . . . , \( m - 1 \) (the wall).

From each state \( i = 1, \ldots, m - 2 \), Prof. Pollak can move one step towards the wall with probability \( b \) and one step towards the window with probability \( 1 - b \). From state \( i = 0 \), he moves towards the wall with probability \( b \) and stays at \( i = 0 \) with probability \( 1 - b \). From state \( i = m - 1 \), he moves towards
the window with probability $1 - b$ and stays at $i = m - 1$ with probability $b$. The state transition diagram is shown in Fig. 2.

Let $\pi_i$ be Prof. Pollak’s steady state probability to be in state $i$. Find $\pi_i$ for $i = 0, \ldots, m - 1$. Evaluate these probabilities for $b = 0$, $b = 1/2$, and $b = 1$.

**Solution.** If $b = 0$ then Prof. Pollak always moves towards the window with probability 1. Once he reaches the window, he stays there forever. So in this case, $\pi_0 = 1$ and $\pi_i = 0$ for $i = 1, \ldots, m - 1$. Similarly, if $b = 1$ then Prof. Pollak always moves towards the wall. Once he hits the wall, he stays there. Thus $\pi_{m-1} = 1$ and $\pi_i = 0$ for $i = 0, \ldots, m - 2$.

For $0 < b < 1$, use the cumulative form of the local balance equations:

$$\pi_i = \rho^i \pi_0 \quad \text{for} \quad i = 0, \ldots, m - 1,$$

where $\rho = \frac{b}{1 - b}$.

But all the steady-state probabilities must sum to 1:

$$1 = \pi_0 + \pi_1 + \ldots + \pi_{m-1}.$$

Substituting the expression from Eq. (3) for each $\pi_i$ into this normalization equation, we have:

$$1 = \pi_0(1 + \rho + \ldots + \rho^{m-1}),$$

$$\pi_0 = \frac{1}{1 + \rho + \ldots + \rho^{m-1}}.$$

Substituting this back into Eq. (3) yields:

$$\pi_i = \frac{\rho^i}{1 + \rho + \ldots + \rho^{m-1}} \quad \text{for} \quad i = 0, \ldots, m - 1.$$

If $b = 1/2$, then $\rho = 1$, and $\pi_0 = \pi_1 = \ldots = \pi_{m-1} = 1/m$. If $b \neq 1/2$ then $\rho \neq 1$, and we can use a geometric series formula to simplify the expression for $\pi_i$:

$$\pi_i = \frac{\rho^i(1 - \rho)}{1 - \rho^m} = \frac{\rho^i - \rho^{i+1}}{1 - \rho^m}$$
Problem 4. Consider the Markov chain of Fig. 3. For all parts of this problem, the process is in state 3 immediately before the first transition.

(a) Find the variance for $J$, the number of transitions up to and including the transition on which the process leaves state 3.

Solution. Each transition originating in state 3 can be thought of a Bernoulli trial with probability of success equal to $1 - \frac{4}{10} = \frac{6}{10}$, where “success” means leaving state 3. Since transitions are independent, random variable $J$ counts the number of independent Bernoulli trials until the first success. Therefore, $J$ is a geometric random variable with parameter $p = \frac{6}{10}$. Its variance is

$$\text{var}(J) = \frac{1 - p}{p^2} = \frac{10}{9}.$$  

(b) Find the expectation for $K$, the number of transitions up to and including the transition on which the process enters state 4 for the first time.

Solution. If the process enters state 2 or state 7, which happens with a non-zero probability, it will never enter state 4. In this case, $K$ is undefined. Since the values of $K$ for part of the sample space are not defined, it is not a legitimate random variable, and therefore its expectation is undefined. Alternatively, we could say that if state 4 is never reached then $K$ is infinite. In this case, the expectation of $K$ would be infinite.

(c) Find $\pi_i$ for $i = 1, 2, \ldots, 7$, the probability that the process is in state $i$ after $10^{10}$ transitions or explain why these probabilities cannot be found.

Solution. The Markov chain has three different recurrent classes. The first recurrent class consists of states 1 and 2; the second recurrent class consists of state 7; and the third recurrent class consists of states 4, 5, and 6. The probability of getting absorbed into the first recurrent class starting from the transient state 3 is

$$\frac{1/10}{1/10 + 2/10 + 3/10} = \frac{1}{6}.$$
which is the conditional probability of transitioning from state 3 to the first recurrent class given that there is a change of state. Similarly, the probability of getting absorbed into the second and third recurrent classes are $\frac{2}{6}$ and $\frac{2}{6}$, respectively.

Now, we solve the balance equations within each recurrent class, which give us the steady-state probabilities conditioned on getting absorbed from state 3 into that recurrent class. The unconditional steady-state probabilities are found by weighting the conditional steady-state probabilities by the probability of absorption to the recurrent classes. The first recurrent class is a birth-death process. We must therefore have: $p_1 = \frac{1}{2}p_2$, and in addition $p_1 + p_2 = 1$, i.e. $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$, where $p_1$ and $p_2$ are conditional steady-state probabilities. For the second recurrent class, $p_7 = 1$.

The third recurrent class is also a birth-death process, and so we have:

\[
p_4 = 2p_5 \\
p_5 = 2p_6 \\
p_4 + p_5 + p_6 = 1,
\]

and thus $p_4 = \frac{1}{4}$, $p_5 = \frac{2}{7}$, $p_6 = \frac{1}{7}$.

Using these conditional probabilities, the unconditional steady-state probabilities are found as follows:

\[
\begin{align*}
\pi_1 &= \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18} \\
\pi_2 &= \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9} \\
\pi_3 &= 0 \text{ (transient state)} \\
\pi_4 &= \frac{4}{7} \cdot \frac{2}{6} = \frac{4}{21} \\
\pi_5 &= \frac{2}{7} \cdot \frac{2}{6} = \frac{2}{21} \\
\pi_6 &= \frac{1}{7} \cdot \frac{2}{6} = \frac{1}{21} \\
\pi_7 &= 1 \cdot \frac{3}{6} = \frac{1}{2}
\end{align*}
\]

(d) Given that the process never enters state 4, find the $\pi_i$'s as defined in Part (c) or explain why they cannot be found.

**Solution.** The given conditional event, that the process never enters state 4, changes the probabilities of being absorbed into the three recurrent classes. The conditional probability of getting absorbed into the first recurrent class is now $\frac{1}{4}$, into the second recurrent class is $\frac{3}{4}$, and into the third recurrent class is 0. Hence the steady-state probabilities are now given by:

\[
\begin{align*}
\pi_1 &= \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12} \\
\pi_2 &= \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6} \\
\pi_3 = \pi_4 = \pi_5 = \pi_6 &= 0 \\
\pi_7 &= 1 \cdot \frac{3}{4} = \frac{3}{4}
\end{align*}
\]
Consider the three-state Markov chain shown in Fig. 4. Determine the three-step transition probabilities $r_{11}(3)$, $r_{12}(3)$, and $r_{13}(3)$.

**Solution.** Recall that the entry in the $i$-th row and $j$-th column of the state transition matrix is the one-step transition probability from state $i$ to state $j$. Therefore, for the Markov chain in Fig. 4, the state-transition matrix is

$$P = \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}.$$  

By the Chapman-Kolmogorov equations, the three-step transition matrix, $R(3)$, is equal to $P^3$. Computing $P^3$ (by hand or with Matlab) yields:

$$R(3) = P^3 = \begin{pmatrix} 0.419 & 0.294 & 0.287 \\ 0.333 & 0.458 & 0.209 \\ 0.372 & 0.372 & 0.256 \end{pmatrix}.$$  

The transition probabilities $r_{11}(3)$, $r_{12}(3)$, and $r_{13}(3)$ are the entries in the first row of $R(3)$:

$$r_{11}(3) = 0.419,$$
$$r_{12}(3) = 0.294,$$
$$r_{13}(3) = 0.287.$$
Problem 6. (After Drake [2], Problem 5.05. Solutions by Ilya Pollak and Bin Ni.)

Consider the Markov process pictured in Fig. 5. Given that this process is in state 0 just before the first transition, determine the probability that:

(a) The process enters state 2 for the first time as a result of the $K$-th transition.

Solution. By inspecting the transition graph, we can see that it takes at least 2 steps to reach state 2 from state 0. If $K \geq 2$, the only way from state 0 to state 2 with exactly $K$ transitions is:

$$0 \rightarrow 3 \rightarrow 3 \rightarrow \cdots \rightarrow 3 \rightarrow 2$$

The probability for this to happen is:

$$P_K = p_{03}p_{33}^{K-2}p_{32} = (1/3)(1/4)^{K-2}(1/4) = (1/3)(1/4)^{K-1}, \quad K = 2, 3, \ldots$$

(b) The process enters state 4 for the first time as a result of the $K$-th transition.

Solution. By inspecting the transition graph, we can see that it takes at least 2 steps to reach state 4 from state 0. If $K \geq 2$, the only way from state 0 to state 2 with exactly $K$ transitions is:

$$0 \rightarrow 3 \rightarrow 3 \rightarrow \cdots \rightarrow 3 \rightarrow 4$$

The probability of this is:

$$P_K = p_{03}p_{33}^{K-2}p_{34} = (1/3)(1/4)^{K-2}(1/2) = (1/6)(1/4)^{K-2}, \quad K = 2, 3, \ldots$$

(c) The process never enters state 4.

Solution. If the process goes directly from 0 to 1 or to 5, it will never enter state 4. The probability of this is $p_{01} + p_{05} = 2/3$. If the first transition is from zero to three, then it will never get to 4 under one of the following two scenarios: (i) if it stays at 3 forever—the probability of this is zero; (ii) if it will eventually go to 2. Since $p_{32} = 0.5p_{34}$, the conditional probability...
that the process will eventually go to 2 is twice as small as that it will go to four. Overall, the probability that the process will go from 0 to 3 and then eventually to 2 is therefore

\[
\frac{1}{3} \cdot \frac{1}{1/2 + 1/4} = \frac{1}{9}.
\]

The total probability that the process never enters state 4 is then \(1/9 + p_{01} + p_{05} = 7/9\).

(d) The process never enters state 3.

**Solution.** From zero, the process can only go to 3, 1, or 5. In the two latter cases, it will never reach 3. Therefore, the probability to never enter 3 is 2/3.

(e) The process enters state 2, and leaves state 2 on the transition immediately after it entered state 2.

**Solution.** In order to eventually get to state 2, the first transition must be from 0 to 3. As shown in the solution to Part (c), the probability that the process goes from 0 to 3 and then eventually to 2 is 1/9. The probability that the next transition takes it out of 2 is \(p_{21} = 1/2\), and therefore the overall probability to enter state 2 and then immediately leave it is \(1/9 \cdot 1/2 = 1/18\).

(f) The process enters state 3, and leaves state 3 on the transition immediately after it entered state 3.

**Solution.** This is the probability to enter state 3, which is 1/3, times the probability to immediately leave state 3, which is \(1/4+1/2 = 3/4\). Answer: \(1/3 \cdot 3/4 = 1/4\).

(g) The process enters state 1 for the first time as a result of the third transition.

**Solution.** The only path that takes the process from state 0 to state 1 in exactly 3 steps is:

\[0 \rightarrow 3 \rightarrow 2 \rightarrow 1\]

The probability of this is:

\[p_{03}p_{32}p_{21} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}.
\]

(h) The process enters state 5 for the first time as a result of the third transition.

**Solution.** The only path that takes the process from state 0 to state 5 in exactly 3 steps is:

\[0 \rightarrow 3 \rightarrow 4 \rightarrow 5\]

The probability of this is:

\[p_{03}p_{34}p_{45} = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.
\]

(i) The process is in state 3 as a result of the \(N\)-th transition.

**Solution.** The only way for this to happen is:

\[\underbrace{0 \rightarrow 3 \rightarrow 3 \rightarrow \cdots \rightarrow 3}_{N \text{ transitions}}\]

The probability of this sequence is:

\[p_{03}p_{33}^{N-1} = (1/3)(1/4)^{N-1}.
\]
Problem 7. (Bertsekas-Tsitsiklis [1] Example 7.6.)
An absent-minded professor commutes from home to office and back. She has two umbrellas. If it rains and an umbrella is available, she takes it. If it’s not raining, she always forgets an umbrella. It rains with probability \( p \) \((0 < p < 1)\) each time she commutes, independently of other times. What is her steady-state probability to get wet?

Solution. Let state \( i \) be the number of umbrellas at her current location. Then \( i = 0 \) or \( i = 1 \) or \( i = 2 \). Each commute corresponds to a state transition. If \( i = 0 \) then the other location she commutes to necessarily has two umbrellas. Therefore, starting from \( i = 0 \) she goes to \( i = 2 \) with probability 1: \( p_{02} = 1 \). If she starts at a location with two umbrellas, then one of two events can happen:

- With probability \( p \), it rains, and she takes an umbrella with her. When she arrives at the other location, this location now has an umbrella—namely, then one she brings with her.
- With probability \( 1 - p \), it doesn’t rain, and she forgets an umbrella. She then arrives at the other location which has zero umbrellas.

Therefore, \( p_{21} = p \) and \( p_{20} = 1 - p \). Finally, if she starts at a location with one umbrella, also one of two events can happen:

- With probability \( p \), it rains, and she takes the umbrella with her. When she arrives at the other location, this location now has both umbrellas.
- With probability \( 1 - p \), it doesn’t rain, and she forgets an umbrella. She then arrives at the other location which has one umbrella.

Therefore, \( p_{12} = p \) and \( p_{11} = 1 - p \). The corresponding state transition diagram is shown in Fig. 6.

This Markov chain has a single recurrent class and all three states are aperiodic. Therefore, steady-state probabilities exist. We obtain them by solving the balance equations in conjunction with the normalization equation:

\[
\begin{align*}
\pi_0 &= (1 - p)\pi_2 \\
\pi_2p &= \pi_1p \\
\pi_0 + \pi_1 + \pi_2 &= 1
\end{align*}
\]

Substituting \( \pi_0 \) from the first equation into the third and \( \pi_1 = \pi_2 \) from the second equation into the third, we get from the third equation:

\[
(1 - p)\pi_2 + \pi_2 + \pi_2 = 1.
\]
Therefore,

\[
\pi_1 = \pi_2 = \frac{1}{3 - p}, \\
\pi_0 = \frac{1 - p}{3 - p}
\]

The steady state probability to get wet is

\[
P(\text{wet}) = \pi_0 \cdot P(\text{rain}) = \pi_0 p = \frac{(1 - p)p}{3 - p}.
\]

References

