3. General Random Variables

Part VIII: Sums of Independent Random Variables

ECE 302 Spring 2012
Purdue University, School of ECE
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Sum of two independent discrete r.v.’s

- $X$ and $Y$ are independent integer-valued r.v.’s
- PMF of $X$ is $p_X$, PMF of $Y$ is $p_Y$
- Let $Z = X + Y$
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$$p_z(z) = P(X + Y = z)$$
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p_Z(z) = P(X + Y = z) = \sum_{\{(x,y)\mid x+y=z\}} P(X = x, Y = y) = \sum_x P(X = x, Y = z-x)
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= \sum_{\{(x,y)\mid x+y=z\}} P(X = x, Y = y) \\
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= \sum_x p_X(x) p_Y(z - x)
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$$= \sum_x P(X = x, Y = z - x)$$

$$= \sum_x p_X(x)p_Y(z - x)$$

$$\equiv p_X * p_Y(z)$$

This is the **discrete convolution** of $p_X$ and $p_Y$
Sum of two independent continuous r.v.’s

- \( X, Y \) independent continuous r.v.’s with respective PDFs \( f_x, f_y \)
- Let \( Z = X + Y \)
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- $X, Y$ independent continuous r.v.'s with respective PDFs $f_X, f_Y$
- Let $Z = X + Y$
- To find PDF of $Z$, first find the joint PDF of $X$ and $Z$, then integrate
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$$F_{Z|X}(z \mid x) = P(Z \leq z \mid X = x)$$
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$$F_{Z|X}(z \mid x) = P(Z \leq z \mid X = x) = P(X + Y \leq z \mid X = x) = P(x + Y \leq z \mid X = x) = P(Y \leq z - x \mid X = x)$$
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$$= P(Y \leq z - x) \quad \text{using the independence of } X \text{ and } Y$$
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$$= F_Y(z - x)$$
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$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Z}(x,z) dx$$
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$$= \int_{-\infty}^{\infty} f_X(x)f_Y(z - x)dx = f_X * f_Y(z)$$

This is the continuous convolution of $f_X$ and $f_Y$
Example 4.10: sum of independent uniform r.v.’s
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If $X$ and $Y$ are independent, and $Z = X + Y$, then $f_Z = f_X \ast f_Y$
Example 4.10: sum of independent uniform r.v.’s

If $X$ and $Y$ are independent, and $Z = X + Y$, then $f_Z = f_X \ast f_Y$
Sum of independent normal r.v.’s is normal

Let $X_i$ be a normal r.v. with mean $\mu_i$ and variance $\sigma_i^2$, for $i = 1, 2, \ldots, n$, 
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i.e., let $f_{X_i}(x) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}$ for $i = 1, 2, \ldots, n$
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Suppose $X_1, X_2, \ldots, X_n$ are independent.
Sum of independent normal r.v.’s is normal

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Suppose $X_1, X_2, \ldots, X_n$ are independent.
Let $a_0, a_1, \ldots, a_n$ be real numbers, and let

$Y = a_0 + a_1 X_1 + \ldots + a_n X_n$
Sum of independent normal r.v.’s is normal

Let $X_i$ be a normal r.v. with mean $\mu_i$ and variance $\sigma_i^2$, for $i = 1, 2, \ldots, n$,
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Let $a_0, a_1, \ldots, a_n$ be real numbers, and let
$Y = a_0 + a_1 X_1 + \ldots + a_n X_n$
Then $Y$ is a normal r.v.
Sum of independent normal r.v.’s is normal

Let $X_i$ be a normal r.v. with mean $\mu_i$ and variance $\sigma_i^2$, for $i = 1, 2, \ldots, n$, i.e., let $f_{X_i}(x) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}$ for $i = 1, 2, \ldots, n$

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$Y = a_0 + a_1 X_1 + \ldots + a_n X_n$

Then $Y$ is a normal r.v. with mean

$\mu_Y = a_0 + a_1 \mu_1 + \ldots + a_n \mu_n$
Sum of independent normal r.v.’s is normal

Let $X_i$ be a normal r.v. with mean $\mu_i$ and variance $\sigma_i^2$, for $i = 1, 2, \ldots, n$,
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Suppose $X_1, X_2, \ldots, X_n$ are independent.
Let $a_0, a_1, \ldots, a_n$ be real numbers, and let
$Y = a_0 + a_1 X_1 + \ldots + a_n X_n$
Then $Y$ is a normal r.v. with mean
$\mu_Y = a_0 + a_1 \mu_1 + \ldots + a_n \mu_n$
and variance
$\sigma_Y^2 = a_1^2 \sigma_1^2 + \ldots + a_n^2 \sigma_n^2$