2. Discrete Random Variables
Part IV: Joint PMFs

ECE 302 Fall 2009 TR 3-4:15pm
Purdue University, School of ECE
Prof. Ilya Pollak
Joint PMF $p_{X,Y}$ of $X$ and $Y$

- If $X$ and $Y$ are discrete random variables, their joint probability mass function is defined as $p_{X,Y}(x,y) = P(X=x \text{ and } Y=y)$. 
Joint PMF $p_{X,Y}$ of $X$ and $Y$

- If $X$ and $Y$ are discrete random variables, their joint probability mass function is defined as $p_{X,Y}(x,y) = P(X=x \text{ and } Y=y)$.
- The individual PMFs of $X$ and $Y$ are called the marginal PMFs and can be obtained from the joint PMF:

  \[ p_X(x) = \sum_y p_{X,Y}(x,y) \]
  \[ p_Y(y) = \sum_x p_{X,Y}(x,y) \]
Example 2.9 (Fig 2.10)

Joint PMF $p_{X,Y}(x,y)$

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Column sums: marginal PMF $p_X(x)$
Example 2.9 (Fig 2.10)

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Column sums: marginal PMF $p_X(x)$

E.g., $p_X(1) = P(X=1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$
Example 2.9 (Fig 2.10)

Column sums: marginal PMF \( p_X(x) \)

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\[ = p_{X,Y}(1,1) + p_{X,Y}(1,2) + p_{X,Y}(1,3) + p_{X,Y}(1,4) = \]
Example 2.9 (Fig 2.10)

Joint PMF $p_{X,Y}(x,y)$

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= p_{X,Y}(1,1) + p_{X,Y}(1,2) + p_{X,Y}(1,3) + p_{X,Y}(1,4) = \sum_y p_{X,Y}(1,y)$
Example 2.9 (Fig 2.10)

Column sums: marginal PMF $p_X(x)$

E.g., $p_X(1) = P(X=1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4) = 1/20 + 1/20 + 1/20 + 1/20 = 4/20$
Example 2.9 (Fig 2.10)

Joint PMF $p_{X,Y}(x,y)$

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E.g., $p_X(1) = P(X=1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$

$= p_{X,Y}(1,1) + p_{X,Y}(1,2) + p_{X,Y}(1,3) + p_{X,Y}(1,4) = \sum_y p_{X,Y}(1,y)$

$= 1/20 + 1/20 + 1/20 = 3/20$
Example 2.9 (Fig 2.10)

Joint PMF $p_{X,Y}(x,y)$

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E.g., $p_X(1) = P(X=1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$

$= p_{X,Y}(1,1) + p_{X,Y}(1,2) + p_{X,Y}(1,3) + p_{X,Y}(1,4) = \sum_{y} p_{X,Y}(1,y)$

$= 1/20 + 1/20 + 1/20 = 3/20$
Example 2.9 (Fig 2.10)

Joint PMF $p_{X,Y}(x,y)$

$$
\begin{array}{c|cccc}
  & 1 & 2 & 3 & 4 \\
\hline
1 & 1/20 & 1/20 & 1/20 & 0 \\
2 & 1/20 & 2/20 & 3/20 & 1/20 \\
3 & 1/20 & 2/20 & 3/20 & 1/20 \\
4 & 0 & 1/20 & 1/20 & 1/20 \\
\end{array}
$$

Row sums: marginal PMF $p_Y(y)$

$3/20 + 7/20 + 7/20 + 3/20 = 20/20 \times 4 = 20$ for each row.

Column sums: marginal PMF $p_X(x)$

$3/20 + 6/20 + 8/20 + 3/20 = 20/20 \times 4 = 20$ for each column.

E.g., $p_X(1) = P(X=1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$

$= p_{X,Y}(1,1) + p_{X,Y}(1,2) + p_{X,Y}(1,3) + p_{X,Y}(1,4) = \sum_y p_{X,Y}(1,y)$

$= 1/20 + 1/20 + 1/20 + 1/20 = 3/20$
Joint PMF for more than two discrete random variables

\[ p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) \]
Joint PMF for more than two discrete random variables

\[ p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) \]

The marginal PMF for each \( X_i \) can be obtained from the joint PMF by summing over all the \( x_j \)'s other than \( x_i \), for example,

\[ p_{X_1}(x_1) = \sum_{x_2} \sum_{x_3} \cdots \sum_{x_n} p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) \]
Functions of many discrete random variables

Let $Y = g(X_1, X_2, \ldots, X_n)$. Then

$$p_Y(y) = \sum_{\{(x_1, \ldots, x_n) | g(x_1, \ldots, x_n) = y\}} p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$
Functions of many discrete random variables

Let $Y = g(X_1, X_2, \ldots, X_n)$. Then

$$p_Y(y) = \sum_{\{(x_1, \ldots, x_n) | g(x_1, \ldots, x_n) = y\}} p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$

$$E[g(X_1, X_2, \ldots, X_n)] = \sum_{x_1, \ldots, x_n} g(x_1, \ldots, x_n) p_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$$
Functions of many discrete random variables

Let $Y = g(X_1, X_2, \ldots, X_n)$. Then

$$p_Y(y) = \sum_{\{(x_1, \ldots, x_n) : g(x_1, \ldots, x_n) = y\}} p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$

$$E[g(X_1, X_2, \ldots, X_n)] = \sum_{x_1, \ldots, x_n} g(x_1, \ldots, x_n)p_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$$

If $g(X_1, X_2, \ldots, X_n) = a_0 + a_1X_1 + a_2X_2 + \ldots + a_nX_n$, then

$$E[g(X_1, X_2, \ldots, X_n)] = a_0 + a_1E[X_1] + a_2E[X_2] + \ldots + a_nE[X_n]$$
Example 2.9 (continued)

Joint PMF $p_{X,Y}(x,y)$

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$Z = X + 2Y$

(a) Find $p_z(k)$.
(b) Find $E[Z]$. 
Example 2.9 (continued)

Joint PMF $p_{X,Y}(x,y)$

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$Z = X + 2Y$ is between 3 and 12, with probability 1

(a) Find $p_Z(k)$.
(b) Find $E[Z]$. 
Example 2.9 (continued)

Z = X+2Y is between 3 and 12, with probability 1

Joint PMF $p_{X,Y}(x,y)$

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Z = X+2Y

(a) Find $p_Z(k)$.
(b) Find $E[Z]$.

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Example 2.9 (continued)

$Z = X + 2Y$ is between 3 and 12, with probability 1

$p_z(3) = P(X + 2Y = 3) = P(X = 1$ and $Y = 1) = 1/20$

(a) Find $p_z(k)$.
(b) Find $E[Z]$. 

Joint PMF $p_{X,Y}(x,y)$

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$Z = X + 2Y$ is between 3 and 12, with probability 1

$p_z(3) = P(X + 2Y = 3) = P(X=1 \text{ and } Y=1) = 1/20$

(a) Find $p_z(k)$.
(b) Find $E[Z]$. 

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Joint PMF $p_{X,Y}(x,y)$

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 2 & 1/20 & 2/20 & 3/20 & 1/20 \\
 1 & 1/20 & 1/20 & 1/20 & 0 \\
\end{array}
\]

$Z = X + 2Y$ is between 3 and 12, with probability 1

$p_z(3) = P(X + 2Y = 3) = P(X=1 \text{ and } Y=1) = 1/20$

$p_z(4) = P(X + 2Y = 4) = P(X=2 \text{ and } Y=1) = 1/20$

(a) Find $p_z(k)$.
(b) Find $E[Z]$.  

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Joint PMF $p_{X,Y}(x,y)$

$Z = X + 2Y$ is between 3 and 12, with probability 1

$p_z(3) = P(X + 2Y = 3) = P(X=1 \text{ and } Y=1) = 1/20$

$p_z(4) = P(X + 2Y = 4) = P(X=2 \text{ and } Y=1) = 1/20$

(a) Find $p_z(k)$.

(b) Find $E[Z]$. 

<table>
<thead>
<tr>
<th>$k$</th>
<th>$p_z(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
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</tr>
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<tr>
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</tbody>
</table>
Example 2.9 (continued)

Z = X + 2Y is between 3 and 12, with probability 1

\[ p_z(3) = P(X + 2Y = 3) = P(X=1 \text{ and } Y=1) = 1/20 \]

\[ p_z(4) = P(X + 2Y = 4) = P(X=2 \text{ and } Y=1) = 1/20 \]

\[ p_z(5) = P(X + 2Y = 5) = P(X=1 \text{ and } Y=2) + P(X=3 \text{ and } Y=1) = 2/20 \]
Example 2.9 (continued)

Z = X + 2Y is between 3 and 12, with probability 1

\[ p_Z(3) = P(X+2Y = 3) = P(X=1 \text{ and } Y=1) = 1/20 \]
\[ p_Z(4) = P(X+2Y = 4) = P(X=2 \text{ and } Y=1) = 1/20 \]
\[ p_Z(5) = P(X+2Y = 5) = P(X=1 \text{ and } Y=2) + P(X=3 \text{ and } Y=1) = 2/20 \]
Example 2.9 (continued)

$Z = X + 2Y$ is between 3 and 12, with probability 1

- $p_Z(3) = P(X + 2Y = 3) = P(X=1 \text{ and } Y=1) = 1/20$
- $p_Z(4) = P(X + 2Y = 4) = P(X=2 \text{ and } Y=1) = 1/20$
- $p_Z(5) = P(X + 2Y = 5) = P(X=1 \text{ and } Y=2) + P(X=3 \text{ and } Y=1) = 2/20$, etc

### Joint PMF $p_{X,Y}(x,y)$

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### Table for $p_Z(k)$

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<td>1/20</td>
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<tr>
<td>12</td>
<td>1/20</td>
</tr>
</tbody>
</table>
Example 2.9 (continued)

Z = X + 2Y
(a) Find \( p_Z(k) \).
(b) Find \( E[Z] \).

\[ E[Z] = 3 \cdot (1/20) + 4 \cdot (1/20) + 5 \cdot (2/20) + 6 \cdot (2/20) + 7 \cdot (4/20) + 8 \cdot (3/20) + 9 \cdot (3/20) + 10 \cdot (2/20) + 11 \cdot (1/20) + 12 \cdot (1/20) = 7.55 \]
Example 2.9 (continued)

Joint PMF $p_{X,Y}(x,y)$

Z = X + 2Y
(a) Find $p_z(k)$.
(b) Find $E[Z]$.

Alternatively, $E[Z] = E[X] + 2E[Y]$
Example 2.9 (continued)

Joint PMF \( p_{X,Y}(x,y) \)

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<td>(1/20)</td>
<td>(1/20)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

\[ Z = X + 2Y \]

(a) Find \( p_Z(k) \).

(b) Find \( E[Z] \).

Alternatively, \( E[Z] = E[X] + 2E[Y] \)

\[
E[X] = 1 \cdot (3/20) + 2 \cdot (6/20) + 3 \cdot (8/20) + 4 \cdot (3/20) = 51/20
\]
Example 2.9 (continued)

Joint PMF $p_{X,Y}(x,y)$

$Z = X + 2Y$

(a) Find $p_z(k)$.

(b) Find $E[Z]$.  

$\begin{array}{c|c|c|c|c|}
 y & 0 & 1/20 & 1/20 & 1/20 \\
4 & 1/20 & 2/20 & 3/20 & 1/20 \\
3 & 1/20 & 2/20 & 3/20 & 1/20 \\
2 & 1/20 & 1/20 & 1/20 & 0 \\
1 & 1/20 & 1/20 & 3/20 & 3/20 \\
\end{array}$

$Z = X + 2Y$

Alternatively, $E[Z] = E[X] + 2E[Y]$

$E[X] = 1 \cdot (3/20) + 2 \cdot (6/20) + 3 \cdot (8/20) + 4 \cdot (3/20) = 51/20$

$E[Y] = 1 \cdot (3/20) + 2 \cdot (7/20) + 3 \cdot (7/20) + 4 \cdot (3/20) = 50/20$

$k$ | $p_z(k)$
--- | ---
3  | 1/20 
4  | 1/20 
5  | 2/20 
6  | 2/20 
7  | 4/20 
8  | 3/20 
9  | 3/20 
10 | 2/20 
11 | 1/20 
12 | 1/20
Example 2.9 (continued)

Joint PMF $p_{X,Y}(x,y)$

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<td></td>
</tr>
</tbody>
</table>

Z = X + 2Y
(a) Find $p_z(k)$.
(b) Find $E[Z]$.

$E[X] = 1 \cdot (3/20) + 2 \cdot (6/20) + 3 \cdot (8/20) + 4 \cdot (3/20) = 51/20$

$E[Y] = 1 \cdot (3/20) + 2 \cdot (7/20) + 3 \cdot (7/20) + 4 \cdot (3/20) = 50/20$

Alternatively, $E[Z] = E[X] + 2E[Y] = 51/20 + 2 \cdot (50/20) = 7.55$

<table>
<thead>
<tr>
<th>k</th>
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<tbody>
<tr>
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<td>1/20</td>
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<td>12</td>
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Example 2.10: mean of a binomial random variable

- $N$ students in class
- Each has probability $p$ of getting an A
- All grades are independent
- $X =$ number of students that get an A
- What is $E[X]$?
Example 2.10: mean of a binomial random variable

Let $X_i = \begin{cases} 1, & \text{if the } i\text{-th student gets an A} \\ 0, & \text{otherwise} \end{cases}$
Example 2.10: mean of a binomial random variable

Let \( X_i = \begin{cases} 1, & \text{if the } i\text{-th student gets an A} \\ 0, & \text{otherwise} \end{cases} \)

\( X_1, X_2, \ldots, X_N \) are Bernoulli random variables with \( E[X_i] = p \).
Example 2.10: mean of a binomial random variable

Let $X_i = \begin{cases} 1, & \text{if the } i\text{-th student gets an A} \\ 0, & \text{otherwise} \end{cases}$

$X_1, X_2, \ldots, X_N$ are Bernoulli random variables with $E[X_i] = p$.

$X_1 + X_2 + \ldots + X_N = X$ is the number of successes in $N$ independent Bernoulli trials.
Example 2.10: mean of a binomial random variable

Let \( X_i = \begin{cases} 1, & \text{if the } i\text{-th student gets an A} \\ 0, & \text{otherwise} \end{cases} \)

\( X_1, X_2, \ldots, X_N \) are Bernoulli random variables with \( E[X_i] = p \).

\( X_1 + X_2 + \ldots + X_N = X \) is the number of successes in \( N \) independent Bernoulli trials.

Therefore, \( X \) is a binomial random variable, with the following PMF:

\[
p_X(k) = \binom{N}{k} p^k (1 - p)^{N-k}
\]
Example 2.10: mean of a binomial random variable

Let \( X_i = \begin{cases} 1, & \text{if the } i\text{-th student gets an A} \\ 0, & \text{otherwise} \end{cases} \)

\( X_1, X_2, \ldots, X_N \) are Bernoulli random variables with \( E[X_i] = p. \)

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Hard way of evaluating \( E[X] \):

\[
E[X] = \sum_{k=0}^{N} k \binom{N}{k} p^k (1 - p)^{N-k}
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$p_X(k) = \binom{N}{k} p^k (1-p)^{N-k}$

Hard way of evaluating $E[X]$:

$E[X] = \sum_{k=0}^{N} k \binom{N}{k} p^k (1-p)^{N-k}$

Easy way:

$E[X] = E[X_1] + E[X_2] + \ldots + E[X_N] = Np$
Example 2.10: mean of a binomial random variable

Let \( X_i = \begin{cases} 1, & \text{if the } i\text{-th student gets an A} \\ 0, & \text{otherwise} \end{cases} \)

\( X_1, X_2, \ldots, X_N \) are Bernoulli random variables with \( E[X_i] = p \).

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Hard way of evaluating \( E[X] \):

\[
E[X] = \sum_{k=0}^{N} k \binom{N}{k} p^k (1 - p)^{N-k}
\]

Easy way:

\[
E[X] = E[X_1] + E[X_2] + \ldots + E[X_N] = Np
\]

E.g., if \( N = 60 \) and \( p = \frac{1}{5} \), then \( E[X] = 12 \).
Example 2.11

• n people throw their hats in a box
• each picks up one hat at random
• $X = \text{number of people who get their own hat back}$
• What is $E[X]$?
Example 2.11

Let \( X_i = \begin{cases} 1, & \text{if the } i \text{-th person gets his own hat} \\ 0, & \text{otherwise} \end{cases} \)
Example 2.11

Let $X_i = \begin{cases} 
1, & \text{if the } i\text{-th person gets his own hat} \\
0, & \text{otherwise}
\end{cases}$

$p_{X_i}(k) = \begin{cases} 
1/n, & \text{if } k = 1 \\
1 - 1/n, & \text{if } k = 0 \\
0, & \text{otherwise}
\end{cases}$
Example 2.11

Let \( X_i = \begin{cases} 1, & \text{if the } i\text{-th person gets his own hat} \\ 0, & \text{otherwise} \end{cases} \)

\[ p_{X_i}(k) = \begin{cases} 1/n, & \text{if } k = 1 \\ 1 - 1/n, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases} \]

\[ E[X_i] = 1 \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right) = \frac{1}{n} \]
Example 2.11

Let \( X_i = \begin{cases} 1, & \text{if the } i\text{-th person gets his own hat} \\ 0, & \text{otherwise} \end{cases} \)

\[
p_{X_i}(k) = \begin{cases} \frac{1}{n}, & \text{if } k = 1 \\ 1 - \frac{1}{n}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}
\]

\[
E[X_i] = 1 \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right) = \frac{1}{n}
\]

\[
X = X_1 + X_2 + \ldots + X_n
\]
Example 2.11

Let $X_i = \begin{cases} 1, & \text{if the } i\text{-th person gets his own hat} \\ 0, & \text{otherwise} \end{cases}$

$p_{X_i}(k) = \begin{cases} \frac{1}{n}, & \text{if } k = 1 \\ 1 - \frac{1}{n}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}$

$E[X_i] = 1 \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right) = \frac{1}{n}$

$X = X_1 + X_2 + \ldots + X_n$

$E[X] = E[X_1] + E[X_2] + \ldots + E[X_n] = n \cdot \frac{1}{n} = 1$
Conditioning a random variable on an event

• Conditional PMF of a random variable $X$, conditioned on an event $A$ with $P(A)>0$:

$$p_{X|A}(x) = P(X = x \mid A) = \frac{P(\{X = x\} \cap A)}{P(A)}$$
Conditioning a random variable on an event

• Conditional PMF of a random variable $X$, conditioned on an event $A$ with $\mathbf{P}(A)>0$:

$$p_{X|A}(x) = \mathbf{P}(X = x | A) = \frac{\mathbf{P}(\{X = x\} \cap A)}{\mathbf{P}(A)}$$

Note:

$$\mathbf{P}(A) = \sum_x \mathbf{P}(\{X = x\} \cap A), \text{ and therefore}$$

$$\sum_x p_{X|A}(x) = 1$$
Conditioning a random variable on another random variable

- If $X$ and $Y$ are random variables, conditional PMF $p_{X|Y}$ of $X$ given $Y$ is defined, for all $y>0$, as:

$$p_{X|Y}(x \mid y) = P(X = x \mid Y = y) = \frac{P(\{X = x\} \cap \{Y = y\})}{P(\{Y = y\})} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$
Conditional PMF: examples

Joint PMF $p_{X,Y}(x,y)$

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$p_{X|Y}(x|4)$

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**Conditional PMF: examples**

The Joint PMF $p_{X,Y}(x,y)$ is shown in the table below:

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The conditional PMFs $p_{X|Y}(x|y)$ are:

- For $y=4$:
  
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<tr>
<td>$p_{X</td>
<td>Y}(x</td>
<td>4)$</td>
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- For $y=3$:
  
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<th>4</th>
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<td>Y}(x</td>
<td>3)$</td>
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</tbody>
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**Joint PMF $p_{X,Y}(x,y)$**

<table>
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**Conditional PMF $p_{X|Y}(x|y)$**

- For $y=4$:
  
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<td>$p_{X</td>
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<td>4)$</td>
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</table>

- For $y=3$:
  
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Conditional PMF: examples

Joint PMF $p_{X,Y}(x,y)$

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<tbody>
<tr>
<td>$p_{X</td>
<td>Y}(x</td>
<td>4)$</td>
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<tr>
<td>$p_{X</td>
<td>Y}(x</td>
<td>3)$</td>
<td>1/7</td>
<td>2/7</td>
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</table>

| y   | $p_{Y|X}(y|3)$ |
|-----|----------------|
| 1   | 1/8            |
| 2   | 3/8            |
| 3   | 3/8            |
| 4   | 1/8            |
Example: memoryless property of a geometric random variable

- \( X = \) the number of independent coin flips until first H, with \( P(H) = p \).

\[
p_x(k) = \begin{cases} 
(1 - p)^{k-1}p, & \text{if } k = 1, 2, \ldots \\
0, & \text{otherwise}
\end{cases}
\]
Example: memoryless property of a geometric random variable

- $X =$ the number of independent coin flips until first H, with $P(H) = p$.
  
  $$p_x(k) = \begin{cases} (1 - p)^{k-1} p, & \text{if } k = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- Given that the first trial is a T, what is the conditional PMF of $X - 1$?
Example: memoryless property of a geometric random variable

- $X = \text{the number of independent coin flips until first H, with } P(H) = p.$
  
  $$p_X(k) = \begin{cases} (1 - p)^{k-1} p, & \text{if } k = 1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

- Given that the first trial is a T, what is the conditional PMF of $X - 1$?
  
  $Y = X - 1$
  
  $A = \{X \geq 2\}$
  
  $p_{Y|A}(m) = ?$
Example: memoryless property of a geometric random variable

- \( X = \) the number of independent coin flips until first \( H \), with \( P(H) = p \).

\[
p_X(k) = \begin{cases} 
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0, & \text{otherwise}
\end{cases}
\]

- Given that the first trial is a \( T \), what is the conditional PMF of \( X - 1 \)?

\[
Y = X - 1 \\
A = \{ X \geq 2 \} \\
\mathbb{P}_{Y|A}(m) = ?
\]

- Intuition: since flips are independent, the outcome of the first flip does not influence the future.
Example: memoryless property of a geometric random variable

- $X$ = the number of independent coin flips until first H, with $P(H) = p$.
  \[ p_X(k) = \begin{cases} 
(1 - p)^{k-1} p, & \text{if } k = 1, 2, \ldots \\
0, & \text{otherwise} 
\end{cases} \]

- Given that the first trial is a T, what is the conditional PMF of $X - 1$?
  
  $Y = X - 1$
  
  $A = \{X \geq 2\}$
  
  $p_{Y|A}(m) = ?$

- Intuition: since flips are independent, the outcome of the first flip does not influence the future. Therefore, the remaining flips after the first one should have the same conditional distribution as $p_X$. In other words, the answer must be $p_{Y|A} = p_X$. 
Example: memoryless property of a geometric random variable

• $X$ = the number of independent coin flips until first H, with $P(H) = p$.

$$p_X(k) = \begin{cases} (1 - p)^{k-1} p, & \text{if } k = 1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

• Given that the first trial is a T, what is the conditional PMF of $X - 1$?

$Y = X - 1; \quad A = \{X \geq 2\}$

$$p_{Y|A}(m) = \frac{P(Y = m, X \geq 2)}{P(X \geq 2)}$$
Example: memoryless property of a geometric random variable

• \( X \) = the number of independent coin flips until first H, with \( P(H) = p \).

\[
p_X(k) = \begin{cases} 
(1 - p)^{k-1} p, & \text{if } k = 1, 2, \ldots \\
0, & \text{otherwise}
\end{cases}
\]

• Given that the first trial is a T, what is the conditional PMF of \( X - 1 \)?

\[ Y = X - 1; \quad A = \{ X \geq 2 \} \]

\[
p_{Y|A}(m) = \frac{P(Y = m, X \geq 2)}{P(X \geq 2)} = \frac{P(X = m + 1, X \geq 2)}{P(X \geq 2)}
\]
Example: memoryless property of a geometric random variable

- \( X \) = the number of independent coin flips until first H, with \( \Pr(H) = p \).
  \[
  p_X(k) = \begin{cases} 
  (1-p)^{k-1}p, & \text{if } k = 1, 2, \ldots \\
  0, & \text{otherwise}
  \end{cases}
  \]

- Given that the first trial is a T, what is the conditional PMF of \( X - 1 \)?

\[
Y = X - 1; \quad A = \{X \geq 2\}
\]

\[
p_{Y|A}(m) = \frac{\Pr(Y = m, X \geq 2)}{\Pr(X \geq 2)} = \frac{\Pr(X = m + 1, X \geq 2)}{\Pr(X \geq 2)} = \begin{cases} 
  \frac{\Pr(X = m + 1)}{\Pr(X \geq 2)}, & \text{if } m = 1, 2, \ldots \\
  0, & \text{otherwise}
  \end{cases}
\]
Example: memoryless property of a geometric random variable

• $X$ = the number of independent coin flips until first H, with $P(H) = p$.

\[ p_X(k) = \begin{cases} (1 - p)^{k-1} p, & \text{if } k = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases} \]

• Given that the first trial is a T, what is the conditional PMF of $X - 1$?

$Y = X - 1; \quad A = \{X \geq 2\}$

\[ p_{Y|A}(m) = \frac{P(Y = m, X \geq 2)}{P(X \geq 2)} = \frac{P(X = m + 1, X \geq 2)}{P(X \geq 2)} = \begin{cases} \frac{P(X = m + 1)}{P(X \geq 2)}, & \text{if } m = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases} \]

\[ = \begin{cases} \frac{(1 - p)^m p}{1 - p}, & \text{if } m = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases} \]
Example: memoryless property of a geometric random variable

- \( X \) = the number of independent coin flips until first \( H \), with \( P(H) = p \).
  
  \[
  p_X(k) = \begin{cases} 
  (1 - p)^{k-1}p, & \text{if } k = 1, 2, \ldots \\
  0, & \text{otherwise}
  \end{cases}
  \]

- Given that the first trial is a \( T \), what is the conditional PMF of \( X - 1 \)?

  \( Y = X - 1; \quad A = \{ X \geq 2 \} \)

  \[
  p_{Y|A}(m) = \frac{P(Y = m, X \geq 2)}{P(X \geq 2)} = \frac{P(X = m + 1, X \geq 2)}{P(X \geq 2)} = \begin{cases} 
  \frac{P(X = m + 1)}{P(X \geq 2)}, & \text{if } m = 1, 2, \ldots \\
  0, & \text{otherwise}
  \end{cases}
  \]

  \[
  = \begin{cases} 
  (1 - p)^m p, & \text{if } m = 1, 2, \ldots \\
  0, & \text{otherwise}
  \end{cases} = p_X(m)
  \]
Example: memoryless property of a geometric random variable

• $X$ = the number of independent coin flips until first $H$, with $P(H) = p$.

$\begin{align*}
p_X(k) &= \begin{cases} 
(1 - p)^{k-1} p, & \text{if } k = 1,2,\ldots \\
0, & \text{otherwise}
\end{cases}
\end{align*}$

• Given that the first $N$ trials are all $T$’s, find conditional PMF of $X - N$.

$Y_N = X - N; \quad A_N = \{X \geq N + 1\}$
Example: memoryless property of a geometric random variable

- $X$ = the number of independent coin flips until first H, with $P(H) = p$.
  
  $$p_X(k) = \begin{cases} (1 - p)^{k-1} p, & \text{if } k = 1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

- Given that the first $N$ trials are all T’s, find conditional PMF of $X - N$.

  $$Y_N = X - N; \quad A_N = \{X \geq N + 1\}$$

  $$p_{Y_N|A_N}(m) = \begin{cases} (1 - p)^{m-1} p, & \text{if } m = 1,2,\ldots \\ 0, & \text{otherwise} \end{cases} = p_X(m)$$
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$. 

Joint PMF $p_{R,S}(r,s)$

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</tbody>
</table>
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.

$p_S(s) = \begin{cases} 
12/45, & \text{if } s = 1 \\
?, & \text{if } s = 2 \\
?, & \text{if } s = 3 \\
0, & \text{otherwise}
\end{cases}$
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_{S}(s)$ and $p_{S|A}(s)$.

\[ p_{S}(s) = \begin{cases} 
12/45, & \text{if } s = 1 \\
18/45, & \text{if } s = 2 \\
?, & \text{if } s = 3 \\
0, & \text{otherwise}
\end{cases} \]
Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.

$p_S(s) = \begin{cases} 
\frac{12}{45}, & \text{if } s = 1 \\
\frac{18}{45}, & \text{if } s = 2 \\
\frac{15}{45}, & \text{if } s = 3 \\
0, & \text{otherwise}
\end{cases}$
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.

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12/45, & \text{if } s = 1 \\
18/45, & \text{if } s = 2 \\
15/45, & \text{if } s = 3 \\
0, & \text{otherwise}
\end{cases}$$

$$P(A) = 12/45 + 18/45 = 30/45$$
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.

\[ p_S(s) = \begin{cases} 
12/45, & \text{if } s = 1 \\
18/45, & \text{if } s = 2 \\
15/45, & \text{if } s = 3 \\
0, & \text{otherwise} 
\end{cases} \]

\[ p_{S|A}(s) = \begin{cases} 
\frac{12/45}{30/45} = \frac{12}{30}, & \text{if } s = 1 \\
?, & \text{if } s = 2 \\
0, & \text{otherwise} 
\end{cases} \]
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.

$\begin{align*}
p_S(s) &= \begin{cases} 
12/45, & \text{if } s = 1 \\
18/45, & \text{if } s = 2 \\
15/45, & \text{if } s = 3 \\
0, & \text{otherwise}
\end{cases} \\
p_{S|A}(s) &= \begin{cases} 
12/45, & \text{if } s = 1 \\
18/45, & \text{if } s = 2 \\
0, & \text{otherwise}
\end{cases}
\end{align*}$

$P(A) = 12/45 + 18/45 = 30/45$
Example

Let \( A = \{S \neq 3\} \), \( Y = R - S \), \( X = R + S \)

(a) Find \( p_S(s) \) and \( p_{S|A}(s) \).

(b) Find \( p_{R,Y}(r,y) \).

Joint PMF \( p_{R,S}(r,s) \):

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Joint PMF \( p_{R,Y}(r,y) \):

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Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.
(b) Find $p_{R,Y}(r,y)$.

Joint PMF $p_{R,S}(r,s)$

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Joint PMF $p_{R,Y}(r,y)$

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Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

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Joint PMF $p_{R,S}(r,s)$

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Joint PMF $p_{R,Y}(r,y)$

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Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.
(b) Find $p_{R,Y}(r,y)$.
(c) Find $p_{X|A}(x)$.

Joint PMF $p_{R,S}(r,s)$

\[
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 6/45 & 9/45 & 0 \\
2 & 6/45 & 9/45 & 3/45 \\
1 & 4/45 & 6/45 & 2/45 \\
\end{array}
\]

Event $A$

$\mathbb{P}(A) = 30/45$
Example

Let \( A = \{S \neq 3\}, \ Y = R - S, \ X = R + S \)

(a) Find \( p_S(s) \) and \( p_{S|A}(s) \).
(b) Find \( p_{R,Y}(r,y) \).
(c) Find \( p_{X|A}(x) \).

### Joint PMF \( p_{R,S}(r,s) \)

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Event \( A \)

\[ \mathbb{P}(A) = \frac{30}{45} \]
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.
(b) Find $p_{R,Y}(r,y)$.
(c) Find $p_{X|A}(x)$.

Event $A$
$P(A) = \frac{30}{45}$
Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$.

(a) Find $p_S(s)$ and $p_{S|A}(s)$.
(b) Find $p_{R,Y}(r,y)$.
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### Joint PMF $p_{R,S}(r,s)$

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Event $A$  
$P(A) = 30/45$
Example

Let \( A = \{S \neq 3\} \), \( Y = R - S \), \( X = R + S \)

(a) Find \( p_S(s) \) and \( p_{S|A}(s) \).
(b) Find \( p_{R,Y}(r,y) \).
(c) Find \( p_{X|A}(x) \).

Event \( A \)
\[
P(A) = \frac{30}{45}
\]
Example

Let \( A = \{S \neq 3\} \), \( Y = R - S \), \( X = R + S \).

(a) Find \( p_S(s) \) and \( p_{S|A}(s) \).

(b) Find \( p_{R,Y}(r,y) \).

(c) Find \( p_{X|A}(x) \).

\[
\begin{align*}
p_{X|A}(x) &= \begin{cases} 
\frac{4}{45}, & \text{if } x = 2 \\
\frac{30}{45}, & \text{if } x = 3 \\
?, & \text{if } x = 4 \\
?, & \text{if } x = 5 \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.
(b) Find $p_{R,Y}(r,y)$.
(c) Find $p_{X|A}(x)$.

Event $A$

\[ P(A) = \frac{30}{45} \]

Joint PMF $p_{R,S}(r,s)$

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\[
p_{X|A}(x) = \begin{cases} 
  \frac{4}{45} = \frac{4}{30}, & \text{if } x = 2 \\
  \frac{30}{45} = \frac{30}{30}, & \text{if } x = 3 \\
  \frac{6/45 + 6/45}{45} = \frac{12}{30}, & \text{if } x = 4 \\
  ?, & \text{if } x = 5 \\
  0, & \text{otherwise} 
\end{cases}
\]
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.
(b) Find $p_{R,Y}(r,y)$.
(c) Find $p_{X|A}(x)$.

**Joint PMF** $p_{R,S}(r,s)$

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Event $A$

$\mathbb{P}(A) = 30/45$

$x = 2$

$x = 3$

$x = 4$

$x = 5$

$p_{X|A}(x) = \begin{cases} 
\frac{4/45}{30/45} = \frac{4}{30}, & \text{if } x = 2 \\
\frac{30/45}{6/45 + 6/45} = \frac{30}{12} = \frac{30}{3}, & \text{if } x = 3 \\
\frac{2/45 + 9/45}{30/45} = \frac{11}{30}, & \text{if } x = 4 \\
?, & \text{if } x = 5 \\
0, & \text{otherwise}
\end{cases}$
Example

Let $A = \{S \neq 3\}$, $Y = R - S$, $X = R + S$

(a) Find $p_S(s)$ and $p_{S|A}(s)$.
(b) Find $p_{R,Y}(r,y)$.
(c) Find $p_{X|A}(x)$.

Joint PMF $p_{R,S}(r,s)$

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<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4/45</td>
<td>6/45</td>
<td>2/45</td>
</tr>
<tr>
<td>3</td>
<td>6/45</td>
<td>9/45</td>
<td>0</td>
</tr>
</tbody>
</table>

Event $A$

$P(A) = \frac{30}{45}$

$p_{X|A}(x) = \begin{cases} 
\frac{4}{30} = \frac{2}{15}, & \text{if } x = 2 \\
\frac{30}{45} = \frac{2}{3}, & \text{if } x = 2 \\
\frac{6/45 + 6/45}{30/45} = \frac{12}{30}, & \text{if } x = 3 \\
\frac{2/45 + 9/45}{30/45} = \frac{11}{30}, & \text{if } x = 4 \\
\frac{3/45}{30/45} = \frac{1}{10}, & \text{if } x = 5 \\
0, & \text{otherwise}
\end{cases}$