Problem 1. Find the CT Fourier transforms of the following CT signals.

(a) \( x_1(t) = e^{-2|t-1|} e^{j(t-2)} \).

Solution. We can rewrite signal \( x_1(t) \) as:

\[
x_1(t) = e^{-2|t-1|} e^{j(t-2)} = e^{-2(t-1)} e^{j(t-2)} u(t-1) + e^{2(t-1)} e^{j(t-2)} u(-t+1) = e^{-2j} e^{j(t-2)} e^{-2(t-1)} u(t-1) + e^{-2j} e^{j t} e^{2(t-1)} u(-t+1).
\]  

(1)

Now, we notice that the first term in (1) is a time-shifted version of the signal \( e^{-2t} u(t) \), and is then multiplied by the complex exponential signal \( e^{jt} \). (Note that \( e^{-2j} \) is a constant independent of \( t \)). As shown in class and in Example 4.1 in the text, the Fourier transform of \( e^{-2t} u(t) \) is \( 1/(2 + j\omega) \). Using the time shifting property of the Fourier transform and then using the frequency shifting property (in that order), we get that:

\[
e^{-2j} e^{j t} e^{-2(t-1)} u(t-1) \leftrightarrow e^{-2j} \frac{e^{-j(\omega-1)}}{2 + j(\omega - 1)}.
\]

For the second term in (1), we can see that it is a time-shifted version of the signal \( e^{2t} u(-t) \), and is then multiplied by the complex exponential signal \( e^{jt} \). (again \( e^{-2j} \) is a constant independent of \( t \)). Using the time-reversal property of the Fourier transform we have that the transform of \( e^{2t} u(-t) \) is \( 1/(1 - 2j\omega) \). Using the time shifting property of the Fourier transform and then using the frequency shifting property (in that order), we get that:

\[
e^{-2j} e^{jt} e^{2(t-1)} u(-t+1) \leftrightarrow e^{-2j} \frac{e^{-j(\omega-1)}}{2 - j(\omega - 1)}.
\]

Using the linearity property of the Fourier transform we get:

\[
X_1(j\omega) = e^{-2j} \frac{e^{-j(\omega-1)}}{2 + j(\omega - 1)} + e^{-2j} \frac{e^{-j(\omega-1)}}{2 - j(\omega - 1)} = e^{-2j} e^{-j(\omega-1)} [2 - j(\omega - 1)] + e^{-2j} e^{-j(\omega-1)} [2 + j(\omega - 1)]
\]

\[
= 4e^{-j(\omega+1)} \frac{1}{4 + (\omega - 1)^2}
\]

\( \frac{\sin(2\pi t)}{\pi(t-2)} \).

(b) \( x_2(t) = \frac{\sin(2\pi t)}{\pi(t-2)} \).

Solution. As shown in class and in Example 4.5 in the text,

\[
\frac{\sin(2\pi t)}{\pi t} \leftrightarrow u(\omega + 2\pi) - u(\omega - 2\pi)
\]
Note that 
\[ x_2(t) = \frac{\sin(2\pi t)}{\pi(t-2)} = \frac{\sin(2\pi t - 4\pi)}{\pi(t-2)} = \frac{\sin(2\pi(t-2))}{\pi(t-2)}. \]

Therefore, by the time-shifting property of the Fourier transform, the Fourier transform of \( x_2(t) \) is 
\[ e^{-2j\omega}[u(\omega + 2\pi) - u(\omega - 2\pi)] \]

(c) \( x_3(t) = \text{sinc}(at + b) \) where \( a \) and \( b \) are fixed real numbers.

**Solution.** We first consider the following two special cases.

- For \( a = 0 \) and \( b \neq 0 \), \( x_3(t) = \frac{\sin(\pi b)}{\pi b} \). Then, we have that 
  \[ X_3(j\omega) = 2\pi \frac{\sin(\pi b)}{\pi b} \delta(\omega) = \frac{2}{b} \sin(\pi b) \delta(\omega). \]

- For \( a = b = 0 \), \( x_3(t) = \frac{\sin(0)}{0} = 1 \). Then, \( X_3(j\omega) = 2\pi \delta(\omega) \).

Now we consider the general case when \( a \neq 0 \). Notice that we can rewrite signal \( x_3(t) \) as: 
\[ x_3(t) = \frac{1}{a} \sin(\frac{a\pi}{t + \frac{b}{a}}). \]

Using the time shifting property of the Fourier transform we get that: 
\[ X_3(j\omega) = \frac{1}{a} [u(\omega + a\pi) - u(\omega - a\pi)] e^{j\frac{b}{a}\omega}. \]

Another way to go about this is by noticing that \( x_3(t) \) is first time-shifted and then time-scaled. Then using the time shifting property and the time scaling property of the Fourier transform we get:
\[ X_3(j\omega) = \frac{1}{|a|} \left[ u \left( \frac{\omega}{a} + \pi \right) - u \left( \frac{\omega}{a} + \pi \right) \right] e^{j\frac{b}{a}\omega}. \]

To show that this is the same as the above answer, we break the latter into two cases.

For \( a > 0 \),
\[ X_3(j\omega) = \frac{1}{a} \left[ u \left( \frac{\omega}{a} + \pi \right) - u \left( \frac{\omega}{a} + \pi \right) \right] e^{j\frac{b}{a}\omega}. \]

For \( a < 0 \),
\[ X_3(j\omega) = -\frac{1}{a} \left[ u \left( \frac{\omega}{a} + \pi \right) - u \left( \frac{\omega}{a} + \pi \right) \right] e^{j\frac{b}{a}\omega}. \]
Combining the two cases together, we can see that the answers are the same.

**Problem 2.** The response of a continuous-time LTI system $S$ to the input signal $x(t) = e^{-2t}u(t)$ is $y(t) = e^{-t}u(t) - e^{-3t}u(t)$. Find the frequency response $H(j\omega)$ and the impulse response $h(t)$ for this system.

**Solution.** The Fourier transform of $x(t) = e^{-2t}u(t)$ is

$$X(j\omega) = \frac{1}{2 + j\omega}.$$  

The Fourier transform of the output signal $y(t)$ is

$$Y(j\omega) = \frac{1}{1 + j\omega} - \frac{1}{3 + j\omega}.$$  

Since system $S$ is LTI, we can use the convolution property of the Fourier transform to get its frequency response:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 + j\omega} - \frac{1}{3 + j\omega}.$$  

Now, in order to find the impulse response of system $S$, we need to find the partial fraction expansion of $H(j\omega)$. We set up the following equation:

$$\frac{4 + 2j\omega}{(1 + j\omega)(3 + j\omega)} = \frac{A}{1 + j\omega} + \frac{B}{3 + j\omega},$$  

and then we solve for $A$ and $B$. To do that, we multiply both sides by $(1 + j\omega)(3 + j\omega)$ to get:

$$4 + 2j\omega = 3A + B + (A + B)j\omega.$$  

Comparing both sides, we get that $A + B = 2$ and $3A + B = 4$. Thus, $A = B = 1$. So now we have that

$$H(j\omega) = \frac{1}{1 + j\omega} + \frac{1}{3 + j\omega}.$$  

Taking the inverse Fourier transform of the above equation, we get:

$$h(t) = e^{-t}u(t) + e^{-3t}u(t).$$  

**Problem 3.** Find the inverse CT Fourier transforms of the following spectra:
(a) \( X_1(j\omega) = \pi e^{-2|\omega|} \).

(b) \( X_2(j\omega) = u(\omega - 10) - u(\omega - 20) \) where \( u \) is the unit step.

**Solution.**

(a) Using the inverse Fourier transform:

\[
x_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi e^{-2|\omega|} \cdot e^{j\omega t} d\omega
\]

\[
= \frac{1}{2} \int_{0}^{0} e^{2\omega} e^{j\omega t} d\omega + \frac{1}{2} \int_{0}^{\infty} e^{-2\omega} e^{j\omega t} d\omega
\]

\[
= \frac{1}{2} \int_{0}^{0} e^{(2+jt)\omega} d\omega + \frac{1}{2} \int_{0}^{\infty} e^{(jt-2)\omega} d\omega
\]

\[
= \frac{2}{2(2+jt)} - \frac{1}{2(jt-2)}
\]

\[
= \frac{2}{4+t^2}.
\]

(b) We can write \( X_2(j\omega) \) as:

\[
X_2(j\omega) = u((\omega - 15) + 5) - u((\omega - 15) - 5).
\]

Taking the inverse Fourier transform we get:

\[
x_2(t) = \frac{\sin(5t)}{\pi t} e^{j15t}.
\]

**Problem 4.** Let

\[
x(t) = \sin(100\pi t) + 2\sin(200\pi t)
\]

\[
g(t) = x(t) \sin(500\pi t)
\]

\[
r(t) = g(t) \sin(500\pi t)
\]

\[
H(j\omega) = \begin{cases} 
1, & \text{for } |\omega| \leq 500\pi \\
0, & \text{for } |\omega| > 500\pi
\end{cases}
\]

If the signal \( r \) is the input to the low-pass filter whose frequency response is \( H \), find the signal obtained at the output of the low-pass filter.

**Solution.** Using the multiplication (modulation) property of the Fourier transform as illustrated in...
Example 4.21 in the text, we get:

\[
G(j\omega) = \frac{1}{2j} X(j(\omega - 500\pi)) - \frac{1}{2j} X(j(\omega + 500\pi)),
\]

\[
R(j\omega) = \frac{1}{2j} G(j(\omega - 500\pi)) - \frac{1}{2j} G(j(\omega + 500\pi))
\]

\[
= -\frac{1}{4} X(j(\omega - 1000\pi)) + \frac{1}{4} X(j\omega) + \frac{1}{4} X(j\omega) - \frac{1}{4} X(j(\omega + 1000\pi))
\]

\[
= -\frac{1}{4} X(j(\omega - 1000\pi)) + \frac{1}{2} X(j\omega) - \frac{1}{4} X(j(\omega + 1000\pi)).
\]

Since \( X \) is bandlimited to \(|\omega| \leq 200\pi\), and since the ideal low-pass filter \( H \) has cutoff frequency \( \omega = 500\pi \) and passband gain of 1, here is what will happen when the signal \( r \) is put through this filter:

1. Both \(-\frac{1}{4} X(j(\omega - 1000\pi))\) which is a scaled replica of \( X(j\omega) \) centered at frequency \( 1000\pi \), and \(-\frac{1}{4} X(j(\omega + 1000\pi))\) which is a scaled replica of \( X(j\omega) \) centered at frequency \(-1000\pi\), will be completely suppressed.

2. The piece \( \frac{1}{2} X(j\omega) \) will be multiplied by the filter’s passband gain, to obtain the output spectrum \( \frac{1}{2} X(j\omega) \).

Therefore, the output of the low-pass filter will be \( y(t) = \frac{1}{2} x(t) = \frac{1}{2} \sin(100\pi t) + \sin(200\pi t) \).