ECE 301 Fall 2011 Division 1.
Homework 3 Solutions.

Reading: Sections 1.5, 1.6, and 2.1 in the textbook.

Problem 1. Determine whether or not each of the systems described below is memoryless and fully justify your answer. In each case, the system is specified by providing its response $y$ to an arbitrary input signal $x$.

(a) $y(t) = e^{-x(t)}$
(b) $y(t) = \frac{d^2}{dt^2} x(t)$
(c) $y[n] = x[n + 1]$

Solution.

(a) The output at time $t$ only depends on the input value at time $t$. In other words, two input signals which have the same value at $t$ will produce identical output values at time $t$. Therefore, this system is memoryless.

(b) This one is a bit tricky. It looks like this might be a memoryless system, but in fact it is not. The value of the output at time $t$ not only depends on the value of the input at time $t$, but also on the input’s behavior in the vicinity of $t$. Indeed, two inputs with identical values at $t$ are not guaranteed to produce the same output at $t$. An example of this would be $x_1(t) = t$ and $x_2(t) = t^2$. At $t = 1$, both these signals are equal to 1: $x_1(1) = x_2(1) = 1$. However, the responses of the system to these signals are, respectively, $y_1(t) = 0$ and $y_2(t) = 2$, which are different for all $t$. This example shows that the system is not memoryless.

(c) The system definition says that the $n$-th output sample depends on the $(n + 1)$-st input sample, and therefore the system is not memoryless.

Problem 2. Determine whether or not each of the systems described below is BIBO stable and fully justify your answer. In each case, the system is specified by providing its response $y$ to an arbitrary input signal $x$.

Remember that, in order to prove that a system is BIBO stable, it is necessary to show that for all bounded input signals $x$ the output signal is bounded. In order to prove that a system is not BIBO stable, it is sufficient to find one bounded input signal that produces an unbounded output signal.

(a) $y(t) = \int_0^1 x(\tau)d\tau$
(b) $y(t) = tx(t) - t$
(c) $y[n] = x[n] - x[0]$
Solution.

(a) Suppose that \( x \) is a bounded signal, i.e., that there exists a number \( L \) such that \( |x(t)| < L \) for all \( t \). Then we have:

\[
|y(t)| = \left| \int_{0}^{1} x(\tau)d\tau \right| \leq \int_{0}^{1} |x(\tau)|d\tau < \int_{0}^{1} Ld\tau = L.
\]

Thus, the output is also bounded. So, we have shown that for any bounded input signal, the output of the system is bounded. This means that the system is BIBO stable.

(b) If the input is \( x(t) = u(t) + 1 \), we get:

\[
y(t) = tu(t)
\]

This response is unbounded, since there does not exist such number \( L \) that \( tu(t) < L \) for all \( t \). Yet, the input signal is bounded since \(|u(t) + 1| < 3\) for all \( t \). Thus, we have found a bounded input such that the response of the system to it is an unbounded signal. Hence, the system is not BIBO stable.

(c) Note that \(|x[n] - x[0]| \leq |x[n]| + |x[0]|\). Therefore, for any input signal \( x \) satisfying \(|x[n]| < L\) for all \( n \), we have:

\[
|y[n]| = |x[n] - x[0]| \leq |x[n]| + |x[0]| < 2L,
\]

for all \( n \). Thus, the response of the system is a bounded signal, for any bounded input signal. Hence, the system is BIBO stable.

**Problem 3.** Determine whether or not each of the systems described below is causal and fully justify your answer. In each case, the system is specified by providing its response \( y \) to an arbitrary input signal \( x \).

(a) \( y(t) = x(t/2) \)

(b) \( y[n] = x[2 - n] \)

(c) \( y(t) = (t + 1)x(t) \)

**Solution.**

(a) For negative values of \( t \), the output depends on future values of the input. For example, \( y(-2) = x(-1) \)—i.e., the value of the output at \( t = -2 \) depends on the value of the input at \( t = -1 \). Thus, the system is not causal.

(b) For \( n < 1 \), the response \( y[n] \) depends on future values of the input signal. For example, \( y[0] = x[2] \)—i.e., the value of the output at \( n = 0 \) depends on the value of the input at \( n = 2 \). Thus, the system is not causal.

(c) The system \( y(t) = (t + 1)x(t) \) is memoryless, i.e., each output value depends only on the current value of the input. Hence, the system is causal.
Problem 4. Determine whether or not each of the systems described below is linear (according to the definition in the textbook) and fully justify your answer. In each case, the system is specified by providing its response \( y \) to an arbitrary input signal \( x \).

Remember that, in order to prove that a system is linear (according to the definition in the textbook), it is necessary to demonstrate that, for all pairs of signals \( x_1 \) and \( x_2 \), and all pairs of numbers \( a_1 \) and \( a_2 \), the response of the system to the input signal \( a_1 x_1 + a_2 x_2 \) is equal to \( a_1 y_1 + a_2 y_2 \) where \( y_1 \) and \( y_2 \) are the responses to the input signals \( x_1 \) and \( x_2 \), respectively. In order to prove that a system is nonlinear, it is sufficient to find two signals \( x_1 \) and \( x_2 \) and two numbers \( a_1 \) and \( a_2 \) such that the response to the input signal \( a_1 x_1 + a_2 x_2 \) is not equal to \( a_1 y_1 + a_2 y_2 \) where \( y_1 \) and \( y_2 \) are the responses to the input signals \( x_1 \) and \( x_2 \), respectively.

(a) \( y(t) = \int_{-\infty}^{t/2} x(\tau)d\tau \)

(b) \( y[n] = x[n^2] \)

(c) \( y(t) = x(t + 1)x(t) \)

Solution.

(a) Suppose we have two signals \( x_1(t) \) and \( x_2(t) \). Let their corresponding outputs be \( y_1(t) \) and \( y_2(t) \), respectively. We get:

\[
y_1(t) = \int_{-\infty}^{t/2} x_1(\tau)d\tau,
\]

and

\[
y_2(t) = \int_{-\infty}^{t/2} x_2(\tau)d\tau.
\]

Now, let \( x(t) = a_1 x_1(t) + a_2 x_2(t) \), where \( a_1 \) and \( a_2 \) are any numbers. Then, the response to \( x(t) \) is:

\[
y(t) = \int_{-\infty}^{t/2} [a_1 x_1(\tau) + a_2 x_2(\tau)]d\tau
\]

\[
= a_1 \int_{-\infty}^{t/2} x_1(\tau)d\tau + a_2 \int_{-\infty}^{t/2} x_2(\tau)d\tau
\]

\[
= a_1 y_1(t) + a_2 y_2(t).
\]

Hence, the system is linear.

(b) Suppose we have two signals \( x_1[n] \) and \( x_2[n] \). Let their corresponding outputs be \( y_1[n] \) and \( y_2[n] \), respectively. We get:

\[
y_1[n] = x_1[n^2],
\]

and

\[
y_2[n] = x_2[n^2].
\]
Now, let $x[n] = a_1 x_1[n] + a_2 x_2[n]$, where $a_1$ and $a_2$ are any numbers. Then, the response to $x[n]$ is:

$$y[n] = a_1 x_1[n^2] + a_2 x_2[n^2] = a_1 y_1[n] + a_2 y_2[n].$$

Hence, the system is linear.

(c) Let $x_1(t) = t$ and $x_2(t) = 1$ for all $t$. We get:

$$y_1(t) = t^2 + t,$$

and

$$y_2(t) = 1.$$

Now, let $x(t) = x_1(t) + x_2(t) = t + 1$. Then,

$$y(t) = (t + 2)(t + 1) = t^2 + 3t + 2 \neq y_1(t) + y_2(t).$$

Hence, the system is not linear.

**Problem 5.** Determine whether or not each of the systems described below is time-invariant and fully justify your answer. In each case, the system is specified by providing its response $y$ to an arbitrary input signal $x$.

Remember that, in order to prove that a CT system is time-invariant, it is necessary to demonstrate, for all signals $x$ and all real numbers $T$, that the response of the system to the input signal $x_T$ is $y_T$, where $x_T(t) = x(t - T)$, $y_T(t) = y(t - T)$, and $y$ is the response of the system to the input signal $x$.

In order to prove that a CT system is time-varying (i.e., not time-invariant), it is sufficient to find one signal $x$ and one number $T$ that violate this condition. The situation with DT systems is similar, except the shifts must be integer numbers.

(a) $y[n] = x[n^2]$

(b) $y(t) = x(2t)$

(c) $y[n] = x[n - 1]$

**Solution.**

(a) Let $x[n] = \delta[n]$ and $x_1[n] = \delta[n - 2]$. The responses to these two input signals are $y[n] = \delta[n]$ and $y_1[n] = 0$. Since these signals are not shifted versions of each other, this system is a time-varying system.

(b) Let $x(t) = t$ and $x_1(t) = t - 1$. The responses to these two input signals are, respectively, $y(t) = 2t$ and $y_1(t) = 2t - 1$. Note that $y(t - 1) = 2t - 2 \neq y_1(t)$ for any $t$. Therefore, the system is time-varying.
(c) Let $x_1[n]$ be an input signal to the system, then $y_1[n] = x_1[n - 1]$. Let $x_2[n] = x_1[n - n_0]$, we get $y_2[n] = x_1[n - n_0 - 1] = y_1[n - n_0]$. Hence, the system is time-invariant.

**Problem 6.** Consider the following six CT signals:

- $x_1(t) = u(t) - u(t - 1)$
- $x_2(t) = -u(t - 2) + u(t - 3)$
- $x_3(t) = x_1(t) - x_2(t)$
- $y_1(t) = \begin{cases} 0, & t \leq 0 \\ t, & 0 \leq t \leq 1 \\ 2 - t, & 1 \leq t \leq 2 \\ 0, & t \geq 2 \end{cases}$
- $y_2(t) = \begin{cases} 0, & t \leq 2 \\ 2 - t, & 2 \leq t \leq 3 \\ t - 4, & 3 \leq t \leq 4 \\ 0, & t \geq 4 \end{cases}$
- $y_3(t) = y_1(t) + y_2(t)$

(a) Sketch each of the six signals and carefully label the axes.

(b) Consider a system whose responses to the input signals $x_1$, $x_2$, and $x_3$ are $y_1$, $y_2$, and $y_3$, respectively. Nothing else is known about this system. Based on the information provided, could this system be

(i) memoryless?
(ii) causal?
(iii) linear?
(iv) time-invariant?

**Fully justify your answers.**

**Solution.**

(a) The plots of the six signals are in Fig. 1.

(b) (i) The system cannot be memoryless. Even though $x_2(1.5) = x_3(1.5) = 0$, we have that $y_2(1.5) = 0$ is different from $y_3(1.5) = 0.5$.

(ii) The system could be causal. The given input-output pairs do not violate causality.

(iii) The system cannot be linear. Note that $x_3 = x_1 - x_2$. Therefore, if the system were linear, the response to $x_3$ would be $y_1 - y_2$. However, the response is $y_3 = y_1 + y_2$. Hence, the system is nonlinear.

(iv) None among the given input signals is a shifted version of another given input signal. Therefore, the given information does not violate time invariance. Hence, the system could be time-invariant.
Figure 1: The graphs of the six signals for Problem 6.