A SUBSPACE APPROACH TO PORTFOLIO ANALYSIS

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INTRODUCTION

The aim of this paper is to introduce the subject of portfolio analysis in the subspace framework. Portfolio analysis refers to the analysis of risk and returns of a set of investments. In this paper we focus on investments in stocks (also called equities). First we will introduce factor models that will simplify the study of risks and returns of stocks.

We will discuss the single factor model that will decompose the return of a stock into a systematic component that depends on the market return and an idiosyncratic component or stock-specific return. (The term Market return refers to the return of the overall stock market).

While the single factor model has several shortcomings, it will help to motivate the subspace analysis of portfolios. Through an illustration we will show that for a set of three stocks the three dimensional space of possible portfolios can be decomposed into a two-dimensional subspace of market neutral portfolios and an orthogonal (one-dimensional) subspace of portfolios with market exposure. The returns of the portfolios in the market neutral subspace will depend only on the stock-specific returns.

To overcome the shortcomings of the single factor model, we will introduce the multi-factor model for stock returns. We will generalize the subspace decomposition of portfolios that we introduced for a single factor model. The main objective of the paper is to show that if a portfolio is optimal in a certain sense, then the decomposed portfolios will also correspond to optimal solutions under different constraints.

To analyze optimal portfolios in the subspace framework we will rely on the Markowitz mean-variance optimization framework. The mean-variance framework maximizes the expected return of the portfolio while minimizing the variance. Though the mean-variance framework has major limitations it is a valuable analysis tool. We will show that any analysis we develop using the mean-variance framework is applicable to all portfolios (irrespective of how they were formed).

We will analyze the Markowitz mean-variance optimization model (Markowitz model for short) under different constraints. Using the results we will show that any optimal portfolio (in the mean-variance framework) can be decomposed as the sum of an optimum market neutral portfolio and an optimum factor exposure portfolio.

In the later sections we will study the performance of the market neutral portfolio and the factor exposure portfolio. We will show that the performance of the market neutral component depends on the cross-sectional variation of stock returns (called dispersion) while the performance of the factor component depends on the volatility of the overall stock market.

In the last section we will look at some applications of the subspace approach. We will discuss performance attribution of general portfolios and conclude with a subspace analysis of the events surrounding the problems faced by many investment managers in August 2007.

NOTATION AND TERMINOLOGY

We will use small boldface letters like \( \mathbf{a} \) to denote vectors and large boldface letters like \( \mathbf{A} \) to denote matrices. \( \mathbf{A}^T \) will denote the transpose of \( \mathbf{A} \) and \( \mathbf{A}^{-1} \) will denote the inverse. For a vector \( \mathbf{a} \), \( ||\mathbf{a}|| \) will denote the Euclidean norm or \( L_2 \) norm. The matrix \( \mathbf{I} \) will always denote an identity matrix. The size will usually be clear from the context.

We will denote expectation of a quantity by \( \mathbb{E}(\cdot) \) and \( \text{Mean}(\cdot) \) will denote the mean of a quantity. Also \( \text{Std}(\cdot) \) will denote the standard deviation of a quantity and \( \text{Var}(\cdot) \) will denote the variance. We will use \( \text{Cov}(\cdot) \) to denote the covariance between two scalars or, if the argument is a vector, the covariances between the elements of the vector. We will use \{ \cdot \} to denote a collection of variables.

For our discussion a portfolio refers to an allocation of capital among different stocks. We express the portfolio in dollar terms (amount of cash invested in each stock) rather than the number of shares held. The gain of a portfolio refers to the change in the dollar value of a portfolio. When the allocation to a stock is negative, it implies short selling. In a short
sale we borrow the stock from the broker and sell it. The term market usually refers to a large universe of stocks representative of the economy. Typically we take it to be a major stock index like the Standard and Poor’s S&P 500 index.

While studying stock returns we will often come across the terms α (alpha) and β (beta). The term α (alpha) usually refers to that part of the expected return of a stock or portfolio that is uncorrelated to the overall market. The term β (beta) refers to the sensitivity of the stock or the portfolio to the overall market.

Market Capitalization of a company refers to the dollar value of the total number of outstanding shares of the company.

A SINGLE FACTOR MODEL FOR STOCK RETURNS

Factor models simplify risk control and can be used to forecast stock returns. The main motivation for single factor models is the Capital Asset Pricing Model (CAPM). In the single factor model the return to a stock is explained as a sum of returns to the market and stock specific returns. In this framework the return to the k-th stock in a portfolio is [1, 2]

\[ r_k = \alpha_k + \beta_k r_M + \varepsilon_k \quad k = 1, \ldots, N \]  

(1)

where \( r_M \) is the return to the market, \( \alpha_k \) is the expected stock specific return and \( \varepsilon_k \) is an error term. The term \( \beta_k \) is called the beta of the stock and reflects the sensitivity of the stock to the market movements.

Note that in (1) the return \( r_k \) is modeled as a random variable. The market return \( r_M \) is a random variable. The difference between \( r_k \) and \( \beta_k r_M \) is the stock specific return. The term \( \alpha_k \) is deterministic; it is the expected value of the stock specific return. The error term \( \varepsilon_k \) is a random variable with zero mean and variance \( \sigma^2_{\varepsilon_k} \). The term \( \sigma^2_{\varepsilon_k} \) is called the stock specific risk.

If we assume that the error term, \( \varepsilon_k \), in (1) is uncorrelated with the market return \( r_M \), then we can decompose the variance of the stock return as

\[ \text{Var}(r_k) = \beta_k^2 \text{Var}(r_M) + \sigma^2_{\varepsilon_k} \]

Thus we can calculate the variance of the stock if we know the beta, the variance of the market and the stock-specific variance. If we make the additional assumption that the stock specific errors are uncorrelated:

\[ \mathbb{E}(\varepsilon_k \varepsilon_l) = 0; \quad \forall k \neq l \]  

(2)

then we can calculate the covariance among the stock returns \( r_k \) and \( r_l \) as

\[ \text{Cov}(r_k, r_l) = \beta_k \beta_l \text{Var}(r_M); \quad \forall k \neq l \]

If we know the variance of the returns \( r_k \) and the covariance among the returns \( \{r_k, r_l\} \) then we know the covariance matrix of the returns. To perform the mean-variance optimization (which we will soon discuss in detail) we need to know the covariance matrix of the returns. In general the covariance matrix for the returns of \( N \) stocks has \( 0.5 \times N \times (N+1) \) terms. Using the single factor model the total number of terms to be estimated reduces to \( 2 \times N + 1 \): \( N \) betas, \( N \) stock specific variances and the variance of the market.

Note that for the single factor model, the number of parameters to be estimated grows linearly with the number of stocks. For a general covariance matrix the growth will be quadratic. If we consider the S&P 500 index (a universe of 500 stocks), the number of parameters to be estimated for a general covariance matrix is 125250 while for the single factor model it is 1001. Thus by using the single factor model we get a substantial reduction in the number of parameters to be estimated.

Market Neutral Portfolios

Consider a set of \( N \) stocks. We can stack the expression in (1) for the \( N \) stocks to obtain:

\[ \mathbf{r} = \mathbf{\alpha} + \mathbf{\beta}^T \mathbf{r}_M + \mathbf{e} \]  

(3)

where

\[ \mathbf{r} = \begin{pmatrix} r_1 \\ \vdots \\ r_N \end{pmatrix} \quad \mathbf{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \quad \mathbf{\beta} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{pmatrix} \]

(4)

The covariance matrix of \( \mathbf{e} \) is an \( N \times N \) diagonal matrix:

\[ \Sigma_e = \begin{pmatrix} \sigma^2_{\varepsilon_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma^2_{\varepsilon_N} \end{pmatrix} \]

(5)

By the assumptions of the single factor model, the covariance matrix of the returns \( \mathbf{r} \) is:

\[ \Sigma_r = \text{Cov}(\mathbf{r}) = \mathbf{\beta} \Sigma_{\mathbf{r}_M} \mathbf{\beta}^T + \Sigma_e \]  

(6)

Consider a portfolio \( \mathbf{x} \) such that:

\[ \mathbf{\beta}^T \mathbf{x} = 0 \]  

(7)

The gain of the portfolio is given by

\[ \mathbf{r}^T \mathbf{x} = \mathbf{\alpha}^T \mathbf{x} + \mathbf{\beta}^T \mathbf{r}_M + \mathbf{e}^T \mathbf{x} = \mathbf{\alpha}^T \mathbf{x} + \mathbf{e}^T \mathbf{x} \]  

(8)

By the assumptions of the single factor model, the elements of \( \mathbf{e} \) are uncorrelated with the market return \( r_M \). Hence any portfolio \( \mathbf{x} \) that satisfies the constraint \( \mathbf{\beta}^T \mathbf{x} = 0 \) will have gains that are uncorrelated to the market. Such a portfolio is called a market neutral portfolio. Also note that the gains of a market neutral portfolio depend only on the stock specific returns. Since \( \mathbf{\beta}^T \mathbf{x} = 0 \), from (6) and (7) the variance of the market neutral portfolio is

\[ \mathbf{x}^T \Sigma_r \mathbf{x} = \mathbf{x}^T \mathbf{\beta} \Sigma_{\mathbf{r}_M} \mathbf{\beta}^T \mathbf{x} + \mathbf{x}^T \Sigma_e \mathbf{x} = \mathbf{x}^T \Sigma_e \mathbf{x} \]  

(9)
orthogonal to the market neutral portfolios are called orthogonal to the subspace of market neutral portfolios. Portfolios dimensional surface. The black arrow denotes a vector orthogonal to the shaded two dimensional surface. The black arrow denotes a vector orthogonal to the shaded surface. The black arrow is orthogonal to the shaded surface. The black arrow is orthogonal to the shaded surface.

Fig. 1. The market neutral portfolios lie on the shaded two dimensional surface. The black arrow denotes a vector orthogonal to the shaded two dimensional surface. The black arrow denotes a vector orthogonal to the shaded two dimensional surface. The black arrow denotes a vector orthogonal to the shaded two dimensional surface.

The variables \( x_1, x_2 \) and \( x_3 \) are the capital allocations to the three stocks. The shaded surface is determined by \( 0.66x_1 + 1.9x_2 + 1.2x_3 = 0 \). Market neutral portfolios lie on the shaded surface. The black arrow is orthogonal to the shaded surface. Factor exposure portfolios will be parallel to the black arrow. The blue line represents a general portfolio. The red line represents the factor exposure component while the green line represents the market neutral component.

A MULTI-FACTOR MODEL FOR STOCK RETURNS

The single factor model we discussed in the previous section is motivated by the decomposition of a stock’s return into a market component and a stock-specific component. If the model holds true, then one can analyze any portfolio of stocks in terms of a market component and a stock-specific component. Before going further, it is useful to find out whether the single factor model is a sufficient approximation of reality. It turns out that the assumption of zero correlation in (2) does not usually hold. (See Chapter 1 in [3] and references therein). To overcome the deficiency we need to incorporate more than one factor in (1).

In the multi-factor framework the return to the \( k \)-th stock in a portfolio is \([1, 2, 3]\):

\[
    r_k = \alpha_k + f_k^T r_f + \varepsilon_k \quad k = 1, \ldots, N
\]

where \( f_k \) is a vector of factor exposures specific to the stock and \( r_f \) is a vector of factor returns. The vectors \( f_k \) and \( r_f \) are of size \( m \times 1 \), \( m \) being the number of factors. The factors in (10) are called systematic factors. The term \( \varepsilon_k \) is an error term. Note that in (10) the factor returns \( r_f \) are specific to the factors and they do not depend on the stock. However the factor exposures \( f_k \) vary from stock to stock. It is also assumed that the errors \( \{ \varepsilon_k \} \) are uncorrelated across stocks and uncorrelated to the factor returns \( r_f \). Note that \( f_k^T \) is of size \( 1 \times m \). Let

\[
    r = \begin{pmatrix}
        r_1 \\
        \vdots \\
        r_N
    \end{pmatrix}_{N \times 1} \quad F = \begin{pmatrix}
        f_1^T \\
        \vdots \\
        f_N^T
    \end{pmatrix}_{N \times m}
\]

\[
    e = \begin{pmatrix}
        \varepsilon_1 \\
        \vdots \\
        \varepsilon_N
    \end{pmatrix}_{N \times 1} \quad \alpha = \begin{pmatrix}
        \alpha_1 \\
        \vdots \\
        \alpha_N
    \end{pmatrix}_{N \times 1}
\]

Then we can write

\[
    r = \alpha + Fr_f + e
\]

The covariance of the returns can then be written as:

\[
    \Sigma_r = FF^T + \Sigma_s
\]

where \( \Sigma_f \) is the covariance matrix (of size \( m \times m \)) of the factor returns \( r_f \). The term \( \Sigma_s \) is a diagonal matrix with the diagonal elements \( \sigma_{\varepsilon_k}^2 = \text{Var}(\varepsilon_k) \) and it represents the stock-specific risk.

There are many commercially available risk models. These models provide a set of factor exposures \( f_k \) for each stock and also provide an estimate of the stock specific variance \( \sigma_{\varepsilon_k}^2 \). Of course, the exact set of factors and the methodology to estimate the risk vary from model to model. Some of the commonly used commercial risk models are BARRA,
Northfield and Axioma. (For the interested reader: Northfield provides free access to all their research publications and model documentations. These are available on their website [4]. Chapter 1 in [3] includes a discussion on the BARRA risk model).

MEAN-VARIANCE OPTIMIZATION

Consider a set of $N$ stocks. Let $\mu = \mathbb{E}(r)$ be the expected value of the $N \times 1$ vector of returns $r$. Let $\Sigma_r$ be the covariance matrix of the returns. Let $x$ be a portfolio. The gain of the portfolio is given by:

$$ r_x = r^T x $$

The expected gain and variance of the portfolio are given by

$$ \mathbb{E}(r_x) = \mu^T x \quad \text{Var}(r_x) = x^T \Sigma_r x \quad (15) $$

The Markowitz mean-variance optimization problem is to find a $N \times 1$ vector of allocations $p_m$ (where the subscript $m$ denotes a Markowitz portfolio) such that $[2, 5]$

$$ p_m = \text{Arg} \max_x \mathbb{E}(r_x) - \frac{\lambda}{2} \text{Var}(r_x) \quad (16) $$

$$ = \text{Arg} \max_x \mu^T x - \frac{\lambda}{2} x^T \Sigma_r x \quad (17) $$

The first term of the objective function is the expected return of the portfolio. The second term is the variance. The scalar $\lambda$ is a positive quantity called the risk aversion factor. The higher the value of $\lambda$ the more averse we are to risk. Differentiating the objective function with respect to $x$ and setting the derivative to zero, we obtain the following solution:

$$ p_m = \frac{1}{\lambda} \Sigma_r^{-1} \mu \quad (18) $$

The variance of the portfolio is:

$$ \sigma_p^2 = p_m^T \Sigma_r p_m = \frac{1}{\lambda^2} \mu^T \Sigma_r^{-1} \mu \quad (19) $$

In practice, we fix the value of $\sigma_p$. Hence the scalar $\lambda$ is given by:

$$ \lambda = \frac{\sqrt{\mu^T \Sigma_r^{-1} \mu}}{\sigma_p} \quad (20) $$

Once we know the expected return $\mu$, the covariance $\Sigma_r$ and the required variance of the portfolio $\sigma_p^2$, we can determine $\lambda$. Plugging back in (18) the value of $\lambda$ (from (20)) we obtain the expression for $p_m$ in terms of the expected return, covariance of returns and the variance of the portfolio:

$$ p_m = \frac{\sigma_p}{\sqrt{\mu^T \Sigma_r^{-1} \mu}} \Sigma_r^{-1} \mu \quad (21) $$

If $\sigma_p$ is expressed in dollars, then the $k$-th element of $p_m$ is the dollar amount allocated to $k$-th stock. (See the next section for an illustration).

Due to various sensitivity issues the Markowitz model is seldom directly used in practice. Usually portfolios are optimized using numerical methods since closed form solutions are seldom available.

However the key feature of the Markowitz model is that for every portfolio we can find a Markowitz model implied expected return. That is, there exists an expected return $\mu_M$ that yields the same portfolio in the Markowitz framework. Hence all the analyses we develop in the Markowitz framework are applicable to any portfolio irrespective of how the portfolio was formed.

Let $p_g$ be a general portfolio. Then from (18) we can easily infer that the Markowitz implied expected return is given by:

$$ \mu_M = \lambda \Sigma_r p_g = \Sigma_r p_g \quad (22) $$

where we have set $\lambda = 1$. As we will see later our performance analysis is insensitive to scaling of the returns. The sole purpose of $\lambda$ is to set the desired risk of the portfolio. For the rest of the discussion we will assume $\mu = \mu_M$ and hence drop the subscript.

A TOY PORTFOLIO

Before we proceed further, let us look at how the concepts developed so far can be applied to a simple example. Consider a single factor model (see (3)) for a set of three stocks with the following parameters:

$$ \beta = \begin{pmatrix} 0.66 \\ 1.9 \\ 1.2 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0.0148 \\ 0.0124 \\ 0.0188 \end{pmatrix} $$

$$ \text{Var}(r_M) = (0.0143)^2 $$

$$ \Sigma_s = \begin{pmatrix} (0.0075)^2 & 0 & 0 \\ 0 & (0.0225)^2 & 0 \\ 0 & 0 & (0.0199)^2 \end{pmatrix} \quad (23) $$

From (6) we obtain the covariance of the returns as:

$$ \Sigma_r = \beta \beta^T \text{Var}(r_M) + \Sigma_s $$

$$ = \begin{pmatrix} 0.0001 & 0.0003 & 0.0002 \\ 0.0003 & 0.0012 & 0.0005 \\ 0.0002 & 0.0005 & 0.0007 \end{pmatrix} \quad (24) $$

Let us set the standard deviation of the portfolio at $10,000$. That is, we set $\sigma_p = 10000$. From (20) we get the value of $\lambda$ as:

$$ \lambda = 1.3413 \times 10^{-4} \quad (25) $$

Plugging in (21) the values of $\sigma_p, \Sigma_r$ and $\mu$, we obtain the portfolio as:

$$ p_m = \begin{pmatrix} 923216, -147728, 85604 \end{pmatrix}^T \quad (26) $$
The $k$-th element of $p_m$ denotes the dollar amount to be invested in the $k$-th stock.

**MARKOWITZ MODEL UNDER CONSTRAINTS: MARKET NEUTRAL PORTFOLIOS**

The Markowitz model can be solved under different constraints. Of particular interest is the constraint of market neutrality. Let $x$ be a portfolio of $N$ stocks. In a single factor model, the market could be interpreted as the common component among the returns of different stocks. In the case of the single factor model the gains of the market neutral portfolio are dependent only on the stock specific returns.

To generalize the concept of market neutrality to a multi-factor model, let us define a market neutral portfolio as a portfolio whose gains depend only on the stock specific returns. In the context of a multi-factor model, let us define a market neutral portfolio as a portfolio whose gains depend only on the stock specific returns. In a single factor model, the constraint

$$F = \text{factor model, let us define a market neutral portfolio as a portfolio whose gains depend only on the stock specific returns.}$$

If we set $F = 0$, then from (14) and (19) it follows that the covariance of the portfolio is

$$\Sigma_r = \Sigma - F^T \Sigma F$$

Thus the Markowitz optimization problem becomes:

$$p_m = \arg \max_x \mu^T x - \frac{1}{2} x^T \Sigma_r x$$

s. t. $F^T x = 0$

This problem can be solved using the Lagrange multiplier method [6]. Let $l$ be the vector of Lagrange multipliers. Then the unconstrained optimization problem becomes:

$$p_m = \arg \max_x \mu^T x - \frac{1}{2} x^T \Sigma_s x + l^T F^T x$$

Differentiating the above equation with respect to $x$ and setting the result to zero, we obtain the following equation:

$$\mu - \lambda \Sigma_s x + F l = 0$$

Solving for $x$ we get:

$$x = \frac{1}{\lambda} \Sigma_s^{-1} (\mu + Fl)$$

(32)

Using the constraint $F^T x = 0$ we can solve for the Lagrange multipliers:

$$l = - (F^T \Sigma_s^{-1} F)^{-1} F^T \Sigma_s^{-1} \mu$$

(33)

Plugging in (32) the value for $l$ we obtain the optimal market neutral portfolio as:

$$p_m = \frac{1}{\lambda} \Sigma_s^{-1} (\mu - F (F^T \Sigma_s^{-1} F)^{-1} F^T \Sigma_s^{-1} \mu)$$

(34)

Let us further simplify the expression for $p_m$. As a first step consider the Cholesky decomposition of $\Sigma_s^{-1}$:

$$\Sigma_s^{-1} = \Sigma_s^{-\frac{1}{2}} \Sigma_s^{-\frac{1}{2}}$$

(35)

where $\Sigma_s^{-\frac{1}{2}} = (\Sigma_s^{-\frac{1}{2}})^T$. Define

$$\hat{\mu} = \Sigma_s^{-\frac{1}{2}} \mu$$

$$\hat{F} = \Sigma_s^{-\frac{1}{2}} F$$

(36)

From (35) and (36) it follows that (34) can be re-written as:

$$p_m = \frac{1}{\lambda} \Sigma_s^{-\frac{1}{2}} (\hat{\mu} - \hat{F} (\hat{F}^T \hat{F})^{-1} \hat{F}^T \hat{\mu})$$

(37)

Now let us find the projection $\mu_R$ of $\hat{\mu}$ onto the range space of $\hat{F}$. The projection is given by the least squares approximation of $\hat{\mu}$ in the range space of $\hat{F}$ (see Chapter 2 in [7]). Hence it follows that:

$$\mu_R = \hat{F} (\hat{F}^T \hat{F})^{-1} \hat{F}^T \hat{\mu}$$

(38)

Therefore we can write:

$$p_m = \frac{1}{\lambda} \Sigma_s^{-\frac{1}{2}} (\mu - \mu_R) = \frac{1}{\lambda} \Sigma_s^{-\frac{1}{2}} \mu_N$$

(39)

where $\mu_N$ is the projection of $\hat{\mu}$ orthogonal to the columns of $\hat{F}$. Comparing (34) and (39) we can see that $\mu_N$ is given by:

$$\mu_N = \Sigma_s^{-\frac{1}{2}} (\mu - \hat{F} (\hat{F}^T \hat{F})^{-1} \hat{F}^T \hat{\mu})$$

(40)

where $\Sigma_s^{-\frac{1}{2}}$ is defined in (35).

The variance of the market neutral portfolio is given by:

$$\sigma_p^2 = p_m^T \Sigma_s p_m = \frac{1}{\lambda} \mu_N^T \mu_N = \frac{1}{\lambda} \|\mu_N\|^2$$

(41)

Therefore

$$\lambda = \frac{\|\mu_N\|^2}{\sigma_p^2}$$

(42)
The term $\sigma_p$ is the standard deviation or risk of the portfolio.

The results derived in this section are inspired by the derivations in [2] (see Chapter 4). However the results derived in [2] do not provide the interpretations in terms of the subspace decomposition that we have seen in (38) and (39). Note that the results we have derived do not depend on the structure of $\Sigma_s$. Even though $\Sigma_s$ is diagonal, for the derivations in this section we did not make use of that fact.

One of the important applications of the subspace approach is the numerical evaluation of $p_{mn}$. Given the matrices $F$ and $\Sigma_s$ and the expected returns $\mu$, we perform the Cholesky decomposition as in (35) and calculate $\hat{F}$ and $\hat{\mu}$ using (36). Then we calculate the QR decomposition of $\hat{F}$ as $\hat{F} = QR$ where $Q$ is an $N \times m$ matrix with orthogonal columns ($Q^T Q = I$) and $R$ is an $m \times m$ upper triangular matrix. Then we have:

$$\hat{\mu}_N = (I - QQ^T) \hat{\mu}$$

(43)

where $I$ is the $N \times N$ identity matrix. Note that the QR decomposition method avoids the computation of the inverse and hence avoids any problems due to ill-conditioning of $F$. Also since $\Sigma_s$ is diagonal the Cholesky factor is nothing but a diagonal matrix with elements equal to the square roots of the elements of $\Sigma_s$.

**Example** Consider the expected returns and the single factor model in (23). Since there is only one factor, $\mu_N$ is given by:

$$\hat{\mu}_N = \left( I - \frac{\hat{\beta} \hat{\beta}^T}{\hat{\beta}^T \hat{\beta}} \right) \hat{\mu}$$

(44)

where $I$ is the $3 \times 3$ identity matrix and

$$\mu = \Sigma_s^{-\frac{1}{2}} \mu \quad \hat{\beta} = \Sigma_s^{-\frac{1}{2}} \beta$$

(45)

Substituting for $\mu, \beta$ and $\Sigma_s$ from (23), we obtain

$$\hat{\mu}_N = (0.6546, -0.7101, 0.0388)^T$$

(46)

For $\lambda$, let us use the value in (25). Using this value of $\lambda$ and using $\mu_N$ from (46) and $\Sigma_s$ from (23) we obtain the market neutral portfolio (see (39)) as:

$$p_{mn} = (650120, -235083, 14527)^T$$

(47)

The $k$-th element of $p_{mn}$ denotes the dollar amount to be invested in the $k$-th stock.

**MARKOWITZ MODEL WITH FACTOR EXPOSURE**

So far we have considered market neutral portfolios that had no factor exposures. Now we consider portfolios with factor exposures. That is, the portfolio satisfies the constraint $F^T p = b$ where $b$ is a vector of factor exposures that is fixed. The vector $b$ might be reflective of factor exposures that are desired due to some benchmark that the portfolio has to follow. For example the manager might have a mandate to invest predominantly in stocks with large market capitalization.

Another way to determine $b$ is to consider an existing portfolio. (This approach is useful for analysis). For example such a portfolio might have been obtained through (18); or we might have obtained the portfolio by solving a more complex portfolio optimization problem. As we mentioned in the discussion following (18), every portfolio will have a Markowitz implied expected return. To complete the analysis in the Markowitz framework, we also need to look at the desired factor exposures implied by the portfolio.

As a first step consider the case where we only have views on factor exposures expressed through $b$ and we have no views on the individual returns of stocks. In such a scenario, the logical way to proceed is to minimize the variance of the portfolio subject to the constraint that $F^T p = b$. Thus the problem becomes:

$$\min_{x} \ x^T \Sigma_r x \quad s.t. \ F^T x = b$$

Since the factor exposures are fixed the variance of the portfolio is

$$x^T F \Sigma_f F^T x + x^T \Sigma_s x = b^T \Sigma_f b + x^T \Sigma_s x$$

The first term does not depend on $x$ and the problem becomes

$$\min_{x} \ x^T \Sigma_s x \quad s.t. \ F^T x = b$$

(48)

The solution to the above problem can be found by using Lagrange multipliers. The solution is given by:

$$p_{fm} = \Sigma_s^{-1} F (F^T \Sigma_s^{-1} F)^{-1} b$$

(49)

where the subscript fm denotes the fact that the portfolio is a factor exposure portfolio with minimum variance.

In the special case of only one systematic factor, $F$ will be a vector and $b$ will be a scalar. In this case (48) and (49) will correspond to the Capon Estimator [8]. The problem in (48) is encountered in the same general form in robust array processing. One of the popular ways to deal with array response uncertainty is to impose a set of constraints specifying the desired array response in each direction. These are referred to as point mainbeam constraints or neighboring location constraints. See [9] for a more detailed discussion and additional references on this topic. Note that when there is an uncertainty about the desired factor exposure $b$, the problem in (48) corresponds to robust Capon beamforming (see [10] for more details).

Now let us consider the general case of a Markowitz portfolio with desired factor exposures $b$ and expected return $\mu$. As we have seen earlier, when we have a constraint on the factor exposure the variance in the Markowitz optimization problem depends only on $\Sigma_s$. Therefore the problem becomes:

$$p_{opt} = \arg \max_{x} \ \mu^T x - \frac{\lambda}{2} x^T \Sigma_s x$$
Using the techniques of Lagrange multipliers to solve the problem (see (34) and the preceding discussion), we obtain the solution as

\[
\mathbf{p}_{\text{opt}} = \frac{1}{\lambda} \Sigma_s^{-1} (\mu - F(\Sigma_s^{-1} F)^{-1} F^T \Sigma_s^{-1} \mu) \\
+ \Sigma_s^{-1} F (F^T \Sigma_s^{-1} F)^{-1} \mathbf{b}
\]

(51)

\[
= \frac{1}{\lambda} \Sigma_s^{-1} \mu_N + \Sigma_s^{-1} F (F^T \Sigma_s^{-1} F)^{-1} \mathbf{b}
\]

(52)

\[
= \mathbf{p}_{\text{mn}} + \mathbf{p}_{\text{fm}}
\]

(53)

where we have used (39) (and the results preceding it) as well as (49). Thus we see that the optimal portfolio can be written as a combination of a market neutral portfolio and a factor exposure portfolio with minimum variance. We see from (52) that once we have fixed the factor exposures the factor variance matrix \(\Sigma_f\) does not enter the picture; and the only part of the expected return \(\mu\) that matters is \(\mu_N\) that is orthogonal to the columns of \(F\).

Coming back to the Markowitz portfolio in (18) let us say that the optimal portfolio \(\mathbf{p}_m\) has factor exposures \(b_{m\text{N}}\). That is, \(F^T \mathbf{p}_m = b_{m\text{N}}\). Then we can solve the problem in (50) with \(b = b_{m}\). The solution becomes:

\[
\mathbf{p}_{\text{opt}} = \frac{1}{\lambda} \Sigma_s^{-1} (\mu - F(\Sigma_s^{-1} F)^{-1} F^T \Sigma_s^{-1} \mu) \\
+ \Sigma_s^{-1} F (F^T \Sigma_s^{-1} F)^{-1} \mathbf{b}_{m}
\]

(54)

\[
= \mathbf{p}_{\text{mn}} + \mathbf{p}_{\text{fm}}
\]

(55)

Since the optimal portfolio in (18) has the factor exposures \(b_{m}\) we do not exclude the optimal solution of (17) by imposing the constraint \(F^T \mathbf{p} = b_{m\text{N}}\). Thus the solutions in (18) and (55) should be the same. To see this equality, first consider the expansion for \(\Sigma_s^{-1}\) from (14) and use the matrix inversion lemma [7] to obtain:

\[
\Sigma_s^{-1} = (\Sigma_s + F \Sigma_f F^T)^{-1}
\]

\[
= \Sigma_s^{-1} - \Sigma_s^{-1} F (\Sigma_f^{-1} + F^T \Sigma_s^{-1} F)^{-1} F^T \Sigma_s^{-1}
\]

(56)

Now consider the expression for \(b_{m}\)

\[
b_{m} = F^T \mathbf{p}_m = \frac{1}{\lambda} F^T \Sigma_s^{-1} \mu
\]

(57)

Substitute for \(\Sigma_s^{-1}\) from (56) in (57). Use the resulting expression for \(b_{m}\) in (54). After canceling out similar terms with opposite signs we can see that the expression for \(p_{\text{opt}}\) in (55) reduces to the one in (18).

Thus we see that the Markowitz portfolio can be decomposed as a sum of a market neutral portfolio and a factor exposure component. Once the factor exposures are fixed, then only the weighted stock-specific expected return \(\mu_{N}\) and the stock specific risk \(\Sigma_s\) are needed for portfolio construction.

Of course to calculate the desired factor exposures \(b_{m}\) we need the expected factor returns and the factor covariance matrix.

Since any portfolio can be cast in the Markowitz framework (using the Markowitz implied expected return) we can conclude that any portfolio can be decomposed as a sum of a market neutral portfolio and a factor exposure component. The factor exposure component depends only on the stock-specific risks and the desired factor exposures.

The fact that any portfolio can be decomposed into a market neutral and a factor exposure component has important implications for portfolio construction and performance analysis. For example one model can be used to predict the market neutral returns and another model can be used to set the desired factor exposures.

Factor returns are usually trend following; that is, there are distinct trends in the factor returns that persist for a while. The primary reason for the persistence of trends is that factor returns are often driven by general economic conditions that tend to persist for a while. Desired factor exposures are often set based on the expected macroeconomic climate [11]; we do not need to specify the expected returns of individual stocks, \(\mu\). A method is proposed in [12] where by relying on the confidence of expected factor returns, different factor exposures are set.

Market neutral returns on the other hand are usually mean reverting. That is, we believe that the excess returns of the stocks will return to their mean value. A mean reversion based method for predicting (and trading) stock specific returns is considered in [13].

♦ Example ♦ Consider the optimal portfolio in (26) and the single factor model in (23). The exposure of the optimal portfolio to \(\beta\) is \(b_{m\beta} = \beta F^T \mathbf{p}_m = 432220\). From (49) the minimum variance portfolio, \(\mathbf{p}_{\text{fm}}\), that satisfies the constraint \(\beta^T \mathbf{p}_m = b_{m\beta}\) is given by:

\[
\mathbf{p}_{\text{fm}} = \frac{\Sigma_s^{-1} \beta}{\beta^T \Sigma_s^{-1} \beta} b_{m\beta}
\]

(58)

Substituting the values for \(\beta\) and \(\Sigma_s\) from (23) and using \(b_{m\beta} = 432220\) in the above equation, we get:

\[
\mathbf{p}_{\text{fm}} = (273096, 87355, 71077)^T
\]

(59)

The \(k\)-th element of \(\mathbf{p}_{\text{fm}}\) is the dollar amount to be invested in the \(k\)-th stock. From (26) and (47) we have:

\[
\mathbf{p}_m = (923216, -147728, 85604)^T
\]

(60)

\[
\mathbf{p}_{\text{mn}} = (650120, -235083, 14527)^T
\]

(61)

From the above equations and (59), we can see that the optimal portfolio in (26) can be decomposed as:

\[
\mathbf{p}_m = \mathbf{p}_{\text{mn}} + \mathbf{p}_{\text{fm}}
\]

(62)
PERFORMANCE ANALYSIS OF MARKET NEUTRAL PORTFOLIOS

Let $r(t)$ be the $N \times 1$ vector of stock returns on a given day $t$. Then using (13), $r(t)$ can be decomposed as

$$r(t) = r_s(t) + r_R(t)$$  \hspace{1cm} (63)

where $r_R(t)$ is in the range space of $F$ and $r_s(t)$, the vector of stock specific returns, is orthogonal to the columns of $F$. Comparing (63) with (13) we get:

$$r_R(t) = Fr_f(t) \quad r_s(t) = \alpha + e(t)$$  \hspace{1cm} (64)

where $\alpha$ is the expected value of the stock specific returns $r_s(t)$.

The market neutral portfolio $p_m$ satisfies the constraint $F^T p_m = 0$. Therefore it follows from (63) that the gain of the market neutral portfolio on any given day, $G_t$, is given by

$$G_t = r^T(t) p_m = r^T_s(t) p_m + r^T_R(t) p_m = r^T_s(t) p_m + 0$$  \hspace{1cm} (65)

since $r_R(t)$ is a linear combination of the columns of $F$ and $p_m$ is orthogonal to all the columns of $F$. Expanding the expression for $G_t$

$$G_t = r^T_s(t) p_m = \frac{1}{\lambda} r^T_s(t) \Sigma^{-\frac{1}{2}} r_s(t) \mu_N \quad \text{(from (39))}$$  \hspace{1cm} (66)

$$= (\Sigma^{-\frac{1}{2}} r_s(t))^T \mu_N \frac{\sigma_p}{||\mu_N||} \quad \text{(from (42))}$$  \hspace{1cm} (67)

$$= \frac{(\Sigma^{-\frac{1}{2}} r_s(t))^T \mu_N}{||\mu_N||} \frac{\sigma_p}{||\Sigma^{-\frac{1}{2}} r_s(t)||} \quad \text{(from (64))}$$  \hspace{1cm} (68)

$$= \cos(\theta(t)) \sigma_p ||\Sigma^{-\frac{1}{2}} r_s(t)||$$  \hspace{1cm} (69)

where $\cos(\theta(t))$ is the cosine of the angle between the vectors $\Sigma^{-\frac{1}{2}} r_s(t)$ and $\mu_N$. Usually the cross-sectional sample correlation between the returns and the expected returns is called the Information Coefficient [1]. However, as we saw in the above derivation, for our particular analysis the performance really depends on the cosine of the angle, $\cos(\theta(t))$, between the weighted stock-specific returns $\Sigma^{-\frac{1}{2}} r_s(t)$ and the weighted expected stock-specific returns $\mu_N$. We call $\cos(\theta(t))$ the modified information coefficient. By a slight abuse of usual notation we will denote the modified information coefficient by $IC(t)$.

Example Consider the set of three stocks with the single factor model in (23). Let the returns on a given day be $r(t) = (0.03, 0.01, -0.01)^T$. The returns orthogonal to the factors are the stock-specific returns. In the case of a single factor model, the stock specific returns, $r_s(t)$, are orthogonal to $\beta$. The part of $r(t)$ orthogonal to $\beta$ is:

$$r_s(t) = (0.0268, 0.001, -0.0159)^T$$  \hspace{1cm} (70)

Using the value of $\Sigma_s$ from (23) we get:

$$\Sigma^{-\frac{1}{2}} r_s(t) = (3.570, 0.035, -0.796)^T$$  \hspace{1cm} (71)

From (46) we have

$$\hat{\mu}_N = (0.6546, -0.7101, 0.0388)^T$$  \hspace{1cm} (72)

The cosine of the angle, $\cos(\theta(t))$, between the vectors in (71) and (72) is $IC(t)$. In this example we have $IC(t) = 0.6452$.

Note that the term $||\Sigma^{-\frac{1}{2}} r_s(t)||$ can be written as:

$$||\Sigma^{-\frac{1}{2}} r_s(t)|| = \sqrt{N} \frac{1}{\sqrt{N}} ||\Sigma^{-\frac{1}{2}} r_s(t)||$$  \hspace{1cm} (73)

The factor $(1/\sqrt{N}) \cdot ||\Sigma^{-\frac{1}{2}} r_s(t)||$ is called dispersion. That is:

$$\text{Dispersion}(t) = \frac{1}{\sqrt{N}} ||\Sigma^{-\frac{1}{2}} r_s(t)||$$  \hspace{1cm} (74)

When the cross-sectional sample mean of $\Sigma^{-\frac{1}{2}} r_s(t)$ is approximately zero, dispersion is equal to the cross-sectional sample standard deviation; that is, the sample standard deviation computed by averaging over the $N$ stocks on a given day $t$ [2]. Using (74) in (69) we can write:

$$G_t = IC(t) \times \text{Portfolio Risk} \times \sqrt{\Sigma^{-\frac{1}{2}} r_s(t)}$$  \hspace{1cm} (75)

The portfolio risk is usually set at a fixed value. The number of stocks, $N$, is also a constant. Usually we use a model to predict the expected returns. In (75) the expected returns enter through $IC(t)$. Hence by modeling the expected returns we are indirectly modeling $IC(t)$. That leaves us with the dispersion term which we discuss now.

Note that from (64) and (14) it follows that $\Sigma^{-\frac{1}{2}} r_s(t)$ is a $N$ dimensional vector with a covariance matrix equal to the $N \times N$ Identity matrix; however this covariance is across time and not cross-sectional. Consider the case when cross-sectional sample mean of $\Sigma^{-\frac{1}{2}} r_s(t)$ is approximately zero. Then dispersion will be the cross-sectional standard deviation of $\Sigma^{-\frac{1}{2}} r_s(t)$. If the process $\{\Sigma^{-\frac{1}{2}} r_s(t)\}$ is stationary and the principle of cross-sectional ergodicity holds, that is expectations across time are equal to expectations across samples, then

$$\text{Dispersion} = \text{Std} \left( \Sigma^{-\frac{1}{2}} r_s(t) \right) = 1$$  \hspace{1cm} (76)

where Std denotes the cross-sectional standard deviation. In practice it turns out that dispersion is not constant. This is primarily because of the means of the process $\Sigma^{-\frac{1}{2}} r_s(t)$. They are time varying and also vary from one component to another. Also there exists no theoretical justification for using any particular probability distribution to model the values of dispersion.

An empirical study of dispersion was done on all the US stocks that are traded on New York or NASDAQ stock exchanges. All stocks that were priced above $5$, had a market
capitalization of at least $1\ billion and had a 21-day trailing average volume of at least 250,000 shares per day were considered. The time period between January 1, 1998 and June 30, 2010 was used. On average there were 2000 stocks that passed the selection criteria. The daily log returns from yesterday’s closing price to today’s closing price were used. (The log returns are calculated as the difference in logarithms of prices). Using the Northfield fundamental US equity risk model (to obtain $F$ and $\Sigma_s$) the daily dispersion was calculated. (Details of the risk model and the estimation method are available on the Northfield website under model documentation [4]). To avoid any data issues and outliers, dispersion values greater than 3 were rejected. Out of 3142 data points 31 had values above 3 and hence were omitted from the study. A plot of the 21-day trailing average of daily dispersion across time is shown in Fig. 2. The time period of 21 days was chosen for the average since, on average, a month has 21 trading days. The probability density function (pdf) of the daily dispersion is shown in Fig. 3. The empirical pdf was estimated by the histogram method. The normal and lognormal fits to the empirical pdf are also shown. From the figure we can see that lognormal distribution provides a better approximation of the actual data.

![Fig. 2. Trailing 21-day average of dispersion of daily returns. The red line shows the constant value of 1. For daily returns, the daily log returns from yesterday’s closing price to today’s closing price were used.](image)

The modified information coefficient, $IC(t)$, on the other hand is a sum of a large number of random variables that are independent for the most part. Therefore it follows approximately a normal distribution. (Note that $IC(t)$ should always lie between +1 and -1 and hence it can never be truly normally distributed).

The modified information coefficient was calculated for a proprietary model for the time period between January 1, 2003 and June 30, 2010. The 21-day trailing average of $IC(t)$ is shown in Fig. 4. The pdf of the daily values of $IC(t)$ is shown in Fig. 5.

**Fig. 3.** Empirical modeling of the probability density function of dispersion. The blue bars represent the density estimate of the dispersion by the histogram method. The red line is a lognormal density fit and the green line is a normal density fit.

**Fig. 4.** Trailing 21-day average of the modified information coefficient $IC(t)$.

**EFFECTS OF TIME VARYING DISPERSION**

If dispersion was a constant then from (75) it follows that for a given risk of the portfolio ($\sigma_P = \text{constant}$) the gains of the portfolio will depend only on the Information Coefficient. One of the metrics used to measure the consistency of the returns of a portfolio is the Information Ratio [1, 2]. The information ratio is usually measured for active gains. Active gains are the excess gains with respect to some benchmark like the market gain or some other factor gain. Since we are considering a market neutral portfolio, the gains in (75) are not influenced by the market or factor gains and hence, in this
and the dispersion on day $t$ given by (see Chapter 4 in [2]):

$$IC = \frac{\text{Mean}(G_t)}{\text{Std}(G_t)}$$

When dispersion is a constant the Information Ratio will be equal to the ratio of the mean of $IC(t)$ to the standard deviation of $IC(t)$. However when dispersion varies with time, the Information Ratio depends on the correlation between dispersion and $IC(t)$. Let $\rho$ denote the correlation between $IC(t)$ and the dispersion on day $t$. Then the Information Ratio is given by (see Chapter 4 in [2]):

$$\text{Information Ratio} = \frac{\text{Mean}(IC(t))}{\text{Std}(IC(t))} + \rho \frac{\text{Std}(\text{Dispersion})}{\text{Mean}(\text{Dispersion})}$$

If $IC(t)$ is positively correlated with dispersion ($\rho > 0$) it implies that we have higher values of $IC(t)$ when the dispersion is higher and the Information ratio will increase. So far there have been no detailed studies on the interaction between $IC(t)$ and dispersion.

**ANALYZING FACTOR RETURNS**

When we consider the performance analysis of factor gains of a portfolio, we would like to separate the active gains from the passive gains. Let $b^k(t)$ be the exposure of the portfolio to the $k$-th factor on day $t$. Over a period of time, $T$, the average factor exposure is $(1/T) \sum_{t=1}^{T} b^k(t)$. Let $r^k(t)$ be the return to factor $k$ on day $t$. Also, let $\langle r^k(T) \rangle$ be the average daily return for the factor $k$ over a period of time $T$. That is,

$$\langle r^k(T) \rangle = \frac{1}{T} \sum_{t=1}^{T} r^k(t)$$

Then the passive daily gain of the portfolio for factor $k$ can be defined as

$$\frac{1}{T} \sum_{t=1}^{T} b^k(t) \cdot \langle r^k(T) \rangle$$

The passive gain can be seen as the average daily gain generated if the investor passively held a factor exposure equal to the average of $\{b^k(t)\}$. To achieve this we do not need any portfolio management or prediction model. So the passive gain can be considered as a benchmark against which we measure the performance of our models. The factor gain of the portfolio over and above the benchmark is the active gain.

The average active gain of the portfolio for exposure to factor $k$ is:

$$G^k_a = \frac{1}{T} \sum_{t=1}^{T} b^k(t) r^k(t) - \frac{1}{T} \sum_{t=1}^{T} b^k(t) \cdot \langle r^k(T) \rangle$$

$$= \frac{1}{T} \sum_{t=1}^{T} b^k(t) (r^k(t) - \langle r^k(T) \rangle)$$

$$= \cos(\theta_k) \frac{1}{\sqrt{T}} \left[ \sum_{t=1}^{T} (b^k(t))^2 \right]^{\frac{1}{2}}$$

$$\cdot \frac{1}{\sqrt{T}} \left[ \sum_{t=1}^{T} (r^k(t) - \langle r^k(T) \rangle)^2 \right]^{\frac{1}{2}}$$

$$= \langle IC^k_f \rangle \frac{1}{\sqrt{T}} \left[ \sum_{t=1}^{T} (b^k(t))^2 \right]^{\frac{1}{2}} \text{Std} \{r^k(t)\}$$

where $\cos(\theta_k)$ is the angle between the vectors $[b^k(1) \ldots b^k(T)]^T$ and $[r^k(1) - \langle r^k(T) \rangle \ldots r^k(T) - \langle r^k(T) \rangle]^T$. The modified information coefficient is given by $\langle IC^k_f \rangle = \cos(\theta_k)$ where we have used the notation $\langle \cdot \rangle$ to denote the fact that the modified information coefficient is calculated across time (as opposed to across the stocks in (75)). The standard deviation of $\{r^k(t)\}$ is also calculated across time. This quantity is a generalization of what is usually called volatility in the markets. It is a time varying quantity. When factor $k$ corresponds to the stocks’ exposure to the overall market (often called $\beta$) then the standard deviation is the market volatility. (The volatilities of the other factors are usually highly correlated with the market volatility).

From (75) we see that the performance of the market neutral portion of a portfolio depends on the cross-sectional dispersion. From (80) it follows that the performance of the factor exposures (as measured by the active gain) depends on the temporal volatility of factor returns.
APPLICATIONS OF SUBSPACE ANALYSIS

In this section we consider some practical applications of the ideas we have developed so far. The first one we consider is performance attribution. From the sub-space decomposition of optimal portfolios in (53), we know that any portfolio can be decomposed into a market neutral portfolio and a factor exposure portfolio. When analyzing the performance of a general portfolio, we should analyze the performance of the market neutral portfolio and the factor exposure portfolio separately. This split will help us identify whether the performance of the portfolio is being influenced by stock-specific returns (through the market neutral portfolio) or by factor returns (through the factor exposure portfolio). By calculating the factor exposures of the portfolio we can further determine the performance due to active gains (see (80)).

In practice we will not know the expected returns of the stocks, \( \mu \), ex ante. We will use a model to predict \( \mu \). By analyzing the market neutral and factor exposure portfolios we can see how good the models are at predicting stock-specific returns and how good they are at timing the factor exposures. Also note that the performance of the market neutral portfolio depends on dispersion while the performance of the factor exposure portfolio depends on market and factor volatilities. During times of volatile performance it helps to look at the root cause of volatility, whether it is factor volatility or dispersion, and adjust the respective exposures accordingly.

Another related application is performance analysis of market-neutral portfolios. When considering actual performance of portfolios one often wonders whether the performance is driven by the model or market conditions. From (75) we can break down the performance in terms of \( IC(t) \) (that we model indirectly by modeling the expected returns \( \mu \)) and the dispersion. In a recent study [14] the authors looked at the predictability of dispersion. (What we call dispersion in (75) is called cross-sectional dispersion of Information Ratio in [14]). The study found little evidence of predictability of dispersion. Thus dispersion is something that influences the performance; but we can neither predict nor control dispersion. In Fig. 6 we have shown the effects of time varying dispersion. We calculated the \( IC \) from a proprietary market-neutral model. Then we calculated the gains assuming that the dispersion remained constant at its mean value over the period. We also calculated the gains with actual dispersion.

We see that since 2009 the actual gains have been lower compared to the earlier periods. However if dispersion had remained constant this would not have occurred. Thus we conclude that the relative under-performance is not due to model failure but rather due to market conditions that are beyond our control.

**Fig. 6.** Illustration of the effect of time-varying dispersion on Portfolio Performance. The blue line shows the cumulative Dollar gains per unit risk if dispersion remained at a constant value (equal to the mean value of actual dispersion). The red line shows the performance with actual dispersion.

**AUGUST 2007: A SUBSPACE ANALYSIS**

In August 2007, the quantitative investing strategies (or quants for short) in the equity world went through some rough weather [15]. Many papers, articles and books have looked at what went wrong during that period. One of the commonly agreed upon conclusions is that some quants started unwinding their portfolio and the effect of the unwinding was felt all over the space. The losses we saw during that period were (at that time) unprecedented. The unprecedented losses prompted some to question whether the models were wrong and if quant investing would survive.

From a subspace analysis perspective, there were two issues. For one, the factor volatilities increased substantially except for the volatility of the market factor. From (80) this implied that people who had factor exposures were seeing some large swings in their gains. Also the dispersion during the first few days in August reached new highs. From (75) this would have induced some large swings in the performance of market neutral portfolios.

At Santa Fe Partners we run a market neutral portfolio. During the first few days of August we saw some wild swings in our portfolio and that prompted us to do a subspace analysis. We first calculated the risk and the implied expected gain of the portfolio. The risk was within limits. What we did notice was that the dispersion had really increased and the increase was rapid. We calculated our \( IC \) and figured that the \( IC \) was negative but was well within the historical ranges. So the conclusion was that the sole driver of the unprecedented losses was the unprecedented dispersion. There was nothing wrong with the models (from the \( IC \) perspective) or the risk.

Historically the information coefficient had always been
mean reverting to the long term (positive) value and neither the speed of mean reversion nor the level were correlated to dispersion. Hence we decided to hold on to our positions and trade as usual. Of course the IC turned positive by the end of the second week (August 10th, to be precise) but dispersion was still very high. In a single day we recouped all our losses and more. We ended up with August 2007 as one of the best months on record.

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