Recall: The definition of covariance is $\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$.

Recall: The variance of a sum is $\text{V}(aX + bY) = a^2\text{V}(X) + b^2\text{V}(Y) + 2ab \text{Cov}(X, Y)$.

Recall: The sample mean is $\bar{Y} = \sum_{i=1}^{n} Y_i / n$.

Recall: The sample variance is $S^2 = \frac{\sum_{i=1}^{n} (Y_i - \bar{Y})^2}{n - 1} = \frac{\sum_{i=1}^{n} Y_i^2 - n \bar{Y}^2}{n - 1}$.

Recall: The exponential pdf is $f(t) = \lambda e^{-\lambda t}$. The corresponding cdf is $F(t) = 1 - e^{-\lambda t}$ for $0 \leq t$. The corresponding mean and standard deviation are $1/\lambda$.

Recall: The second two homework assignments were based on $T = X_1 + \max\{X_2, X_3\}$, where $X_i$ is exponential with means $\mu_1$, $\mu_2$, and $\mu_3$, respectively. In HW2, $X_1$, $X_2$, and $X_3$ were independent, with standard errors estimated with micro-macro replications. In HW3, the effect of correlation between $X_2$ and $X_3$ was considered.
1. For truly random numbers, the expected value is 0.5 and variance is 1/12 and lag-1 autocorrelation is zero. In a Monte Carlo evaluation (HW2) of the 16807 random-number generator, a student reported

   Expected Value = 0.500313
   Standard error of Expected Value = 0.000001
   Variance = 0.083252
   Standard error of Variance = 0.000003
   Standard Deviation = 0.288535
   Lag-1 Correlation = −0.000012
   Standard error of Lag-1 Correlation = 0.000250

(a) (4 points) Circle the values that would change values if the experiment were rerun with a different random number seed.

(b) (6 points) Discuss the implications of the first two lines of the student’s results.

(c) (6 points) Assuming that the experiment was correctly coded and used *n* random numbers, determine the value of *n*.

(d) (6 points) There is no standard error reported for "Standard Deviation". I wonder "why not?". Circle the reason(s) that you can defend. (For partial credit for wrong answers, on the back of this page write a short defense for each of your choices.)

   (i) The standard error of the standard deviation is obvious: it is the same as the standard error of the variance.

   (ii) The standard error of the standard deviation is obvious: it is zero.

   (iii) The standard error of the standard deviation has no micro-macro estimator.
2. For a single linear control $C$, the control-variate estimator is $\hat{\theta}(\alpha) = \hat{\theta} - \alpha (C - \mu_C)$.

The optimal value of $\alpha$ is $\alpha^* = \frac{\text{Cov}(\hat{\theta}, C)}{\text{Var}(C)}$.

Consider a Monte Carlo experiment that estimates $E(T)$, where $T = X_1 + \max\{X_2, X_3\}$ and $X_1$, $X_2$, and $X_3$ are independent and exponential with means $\mu_i$ for $i = 1, 2, 3$. Assume that the experiment uses $k$ independent macro-replications of $m$ micro-replications.

(a) (6 points) Suggest a control $C$ for this application.

(b) (4 points) (circle one of the four choices) For your choice of $C$, will $\alpha^*$ be

(i) negative  (ii) zero  (iii) positive

(iv) so random that the sign of $\alpha^*$ is only known after the experiment is run.

(c) (5 points) In estimating $\alpha^*$, $\text{Var}(C)$ needs to be estimated. Suggest a point estimator of $\text{Var}(C)$.

(d) (3 points) T  F To induce a non-zero correlation between $\hat{\theta}$ and $C$, common random numbers are needed.
3. (3 points each) Based on the notation of Problem 2, circle one of "constant", "random variable", "event" or "meaningless". 

(a) $\mu_C$  constant random variable event meaningless 
(b) $\alpha^*$ constant random variable event meaningless 
(c) $\text{Cov}(\hat{\theta}, C)$ constant random variable event meaningless 

4. (3 points each) The acceptance-rejection method accepts a candidate value $x$ from $r(x)$ with probability $0 \leq f(x) / t(x) \leq 1$. 

(a) T  F  The quantity $x$ can be a vector. 
(b) T  F  The values of $x$ can be discrete or continuous. 
(c) T  F  If the values of $x$ are continuous, then the probability that the candidate value is accepted is $1 / \int_{-\infty}^{\infty} t(x) \, dx$. 

5. (3 points each) The composition method considers $f(x) = \sum_{i=1}^{k} p_i f_i(x)$. A random variate is generated from $f_i$ with probability $p_i$. 

(a) T  F  The quantity $x$ can be a vector. 
(b) T  F  The values of $x$ can be discrete or continuous. 
(c) T  F  Generation from $f_i$ must be via the inverse cdf transformation.
6. Consider the Poisson process with rate function \( \lambda(t) = 6t^2 \). Then the cumulative rate is \( \Lambda(t) = 2t^3 \). At time \( t_{i-1} \), the cdf of the time of the next arrival is \( F(t_i) = 1 - \exp(-\Lambda(t_i) - \Lambda(t_{i-1})) \).

(a) (6 points) Suppose that time is now \( t = 4.5 \) and that the next random number is \( u = 0.4 \). Determine the time of the next arrival.

(b) (6 points) For this application, do you prefer thinning or the inverse cdf transformation? Explain why.
7. NORTA (Normal to Anything) is a method for obtaining $k$-dimensional random vectors in Monte Carlo simulation experiment. In Step 1, independent standard normal random variates are converted to correlated standard normal random variates, usually via the Cholesky decomposition. In Step 2, the correlated standard normal random variates are converted to correlated $U(0,1)$ random variates, via the standard normal cdf. In Step 3, random variates with the specified properties are obtained from the correlated $U(0,1)$ random variates, via the inverse cdf transformation.

(a) (3 points) T  F  In HW3, the values of $\text{corr}(X_2, X_3)$ range from $-1$ to $1$.

(b) (3 points) T  F  In HW2, the value of $\text{corr}(X_2, X_3)$ is maximized.

(c) (3 points) T  F  In HW3, the value of $E(T)$ is a constant, and therefore does not change value as a function of $\text{corr}(X_2, X_3)$.

(d) (3 points) T  F  In NORTA, the correlations of Step 1 have the same values as the correlations of Step 3.

(e) (3 points) T  F  In NORTA’s Step 3, acceptance-rejection could be used (rather than the inverse cdf transformation) if the random numbers are synchronized using multiple random number streams in conjunction with common random numbers.

(f) (6 points) Discuss how your C code computed the standard normal cdf in Step 2.