Closed book. Some handwritten notes. No calculator. 50 minutes.

Score ___________________________
1. True or false. (three points each. if false, state why.)

(a)  T ← F  The expected number of iterations for the acceptance/rejection method depends only upon the choice of the majorizing function \( t(x) = c \cdot r(x) \).

(b)  T ← F  Composition is applicable for generating either continuous or discrete random variates.

(c)  T  F ← A Poisson process’s rate function \( \lambda(t) \) must be non-negative and continuous, but can be non-differentiable at some times \( t \).

Does not need to be continuous.

(d)  T ← F  Always, the purpose of "control variates" is to reduce the variance of the point estimator.

(e)  T  F ← The SimLib routine "timing" is internal to the SimLib logic, never called by user-written code.

The routine "timing" is called in the user-written main program.

(f)  T  F ← The SimLib routine "sampst" is internal to the SimLib logic, never called by user-written code.

The routine "sampst" is called in user-written event routines.

(g)  T ← F  For a time-based random variable (such as number in the system, \( N \)), \( \text{Var}(N) \) is a well-defined performance measure.

2. (two points each) Suppose you rerun a simulation experiment, keeping everything the same except the random-number seeds. The point estimator is \( \hat{\theta} \), used to estimate the performance measure \( \theta \). Which are constant and which are random?

(a)  \( \theta \)<constant> 

(b)  \( E(\hat{\theta}) \)<constant> 

(c)  \( \text{MSE}(\hat{\theta}, \theta) \)<constant> 

3. (ten points) Suppose the rate function of a nonhomogeneous process is \( \lambda(t) = 6 + t \). If we are currently at time \( t_i = 13.3 \) minutes, what is the inverse transformation to generate the next time \( t_{i+1} \)? (You don’t need to solve for a numerical value.)

Given that the current time \( t_i \), the conditional cdf of the next time \( t_{i+1} \) is

\[
F_{T_{i+1}|T_i = t_i}(t_{i+1}) = 1 - \exp\left[-\int_{t_i}^{t_{i+1}} \lambda(t) \, dt\right].
\]

The inverse cdf is obtained by setting a \( U(0, 1) \) random number \( u \) to

\[
F_{T_{i+1}|T_i = t_i}(t_{i+1}) = 1 - \exp\left[-\lambda(t_i+1) \int_{t_i}^{t_{i+1}} \left(\frac{t^2}{2}\right) \, dt\right]
\]

In this case, \( t_i = 13.3 \).

Let’s solve using the more-general rate function \( \lambda(t) = t_0 + t \). The cdf is

\[
F_{T_{i+1}|T_i = t_i}(t_{i+1}) = 1 - \exp\left[-\int_{t_i}^{t_{i+1}} (t_0 + t) \, dt\right]
\]

\[
= 1 - \exp\left[-(t_0 t + \frac{t^2}{2})\bigg|_{t_i}^{t_{i+1}}\right]
\]

\[
= 1 - \exp\left[-\left[t_0 (t_{i+1} - t_i) + \frac{(t_{i+1}^2 - t_i^2)}{2}\right]\right]
\]

\[
= 1 - \exp\left[-\left[\left(t_{i+1} + t_0\right)^2 - (t_i + t_0)^2\right] - \frac{(t_{i+1} + t_0)^2 - (t_i + t_0)^2}{2}\right].
\]

Solving yields \( t_{i+1} = \left[(t_0 + t_i)^2 - 2 \ln(1-u)\right]^{1/2} - t_0 \).

(Notice that \( u = 0 \) yields \( t_{i+1} = t_i \), an easy check for correctness.)

4. (ten points) Discuss the relationship between a model’s event graph and a SimLib implementation.

Each node of the event graph is an event routine in SimLib.
Each arc of the event graph describes which events scheduled other events, and if and when.
Each node in the event graph that has no incoming arcs must be manually scheduled in the user’s main program.
5. Consider using NORTA (normal-to-anything) to generate a 3-dimensional random-vector observation \((x_1, x_2, x_3)\) having \(U(0, 1)\) marginal distributions and correlations \(\rho_{12} = \rho_{13} = \rho_{23} = 0.2\).

Recall: NORTA generates a multivariate standard-normal random vector \((z_1, z_2, z_3)\), calculates \(u_i = \Phi(z_i)\) for \(i = 1, 2, 3\), and calculates \(x_i = F_i^{-1}(u_i)\) for \(i = 1, 2, 3\).

(a) (six points) What do you know about the distribution of the random variable \(U_3\)?

\[ U(0, 1) \leftarrow \]

(b) (six points) What do you know about \(F_3^{-1}\)?

It is the inverse cdf of \(X_3\), which is \(U(0, 1)\). Therefore,
\[ x_3 = F_3^{-1}(u) = u \quad \text{for} \quad 0 < u < 1, \]
zero if \(u \leq 0\), and one if \(1 \leq u\).

(c) (six points) What do you know about the covariance matrix of \((Z_1, Z_2, Z_3)\)?

The covariance matrix is \(3 \times 3\) and symmetric.
All three random variables are standard normal, so the three diagonal elements are one and the three covariances are correlations.
Because of the symmetry among \(X_1, X_2,\) and \(X_3\),
\[ \text{corr}(Z_1, Z_2) = \text{corr}(Z_1, Z_3) = \text{corr}(Z_2, Z_3). \]
Because the three given correlations are \(\rho_{12} = \rho_{13} = \rho_{23} = 0.2\),
\[ \text{corr}(Z_i, Z_j) \approx 0.2. \]

(d) (six points) Sketch the function \(\Phi\).

\(\Phi\) is the standard-normal cdf. Sketch two perpendicular axes.
Label the horizontal axis with \(z\); scale it from about \(-3\) to \(+3\).
Label the vertical axis with \(\Phi(z)\); scale it from zero to one.
Sketch \(\Phi\) as a continuous s-shaped function rising from visual zero at \(z = -3\) to \(1/2\) at \(z = 0\) to visual one at \(z = +3\).
6. Consider the following piece of a SERVO output report. Here the model is of a single-server queueing system. The random variable \textit{ibusy} is the indicator variable, one when the server is busy and zero when idle. The purpose of the simulation is to estimate \( p \), the steady-state probability that the server is busy.

Random Variable... \textit{ibusy}

<table>
<thead>
<tr>
<th>Distribution Property</th>
<th>Point Estimate</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.796</td>
<td>(0.0154647)</td>
</tr>
<tr>
<td>Std. Deviation</td>
<td>0.496645</td>
<td>(0.0107279)</td>
</tr>
<tr>
<td>Variance</td>
<td>0.246661</td>
<td>(0.008485)</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.231791</td>
<td>(0.116534)</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.5659</td>
<td>(0.377775)</td>
</tr>
</tbody>
</table>

(a) (six points) What is the best guess of \( p \) ?

\[ \hat{p} = 0.796 \]

(b) (six points) In our general view of simulation, \( U \) denotes random numbers, \( X \) denotes input random variables, \( Y \) denotes output random variables, and \( \theta \) denotes point estimators of performance measures \( \theta \). Which of these is \textit{ibusy}?

\[ Y, \text{ output} \]

(c) (six points) Which of the five point estimates in the report are time based (as opposed to observation based)?

All five are based on \textit{ibusy}, so all five are time based.

(d) (six points) If the experiment’s sample size is increased to infinity, which of the ten numbers in the report will go to zero?

The five standard errors (in parentheses).