One page of notes, front and back. Closed book. 50 minutes.

Score ___ < 100 /100 > ___
1. True or false. (2 points if correct, 1 point if left blank.)
   (If you wish, write an explanation of your thinking.)
   (a) T  F ← In the inverse cdf transformation to generate a random variate from the
distribution $F_X$, if $P(X = x)$ has positive probability, then there is exactly one random
number $u$ that yields the value $X = x$.
   (b) T ← F If $U$ has a uniform distribution on the interval $[0, 1]$ and if $X$ has an
exponential distribution with mean 2, then the fractional part of $U + X$ has a uniform
distribution on the interval $[0, 1]$.
   (c) T  F ← Micro-macro replications can be an effective method for reducing the
variance of the point estimator $\hat{\theta}$.
   (d) T  F ← For independent sampling, obtaining one more meaningful digit in the
point estimator requires increasing the sample size by a factor of 10.
   (e) T ← F The performance measure to be estimated is a property of the
distribution of the output data.
   (g) T ← F When common random numbers are used to compare two systems, the
two input models remain unchanged (compared to independent sampling).
   (h) T ← F If random numbers are truly random in three dimensions, they are also
truly random in two dimensions.
   (i) T  F ← If random numbers are truly random in three dimensions, they are also
truly random in four dimensions.
   (j) T ← F Floyd’s algorithm can be used even if the functional form of the
random-number generator is unknown.

2. Consider the sentences

"When using common random numbers to compare two systems, say $A$ and
$B$, we obtain point estimators $\hat{\theta}_A$ and $\hat{\theta}_B$. Let $\rho = \text{corr}(\hat{\theta}_A, \hat{\theta}_B)$. Positive
values of $\rho$ are good in the sense that $V(\hat{\theta}_A - \hat{\theta}_B)$ is smaller than when the
two point estimators are independent.

Suppose that you rerun a simulation experiment, keeping everything the same except the
random-number seeds. Using the above three sentences for context, circle "constant",
"random", and "undefined" for each part below.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Category 1</th>
<th>Category 2</th>
<th>Category 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}_A$</td>
<td>constant</td>
<td>random</td>
<td>undefined</td>
</tr>
<tr>
<td>$\rho$</td>
<td>constant</td>
<td>random</td>
<td>undefined</td>
</tr>
<tr>
<td>$(\hat{\theta}_A)^2$</td>
<td>constant</td>
<td>random</td>
<td>undefined</td>
</tr>
<tr>
<td>$V(\hat{\theta}_A)$</td>
<td>constant</td>
<td>random</td>
<td>undefined</td>
</tr>
<tr>
<td>$P(A - B)$</td>
<td>constant</td>
<td>random</td>
<td>undefined</td>
</tr>
<tr>
<td>$\text{corr}(A - B)$</td>
<td>constant</td>
<td>random</td>
<td>undefined</td>
</tr>
</tbody>
</table>
3. Monte Carlo simulation can be used to evaluate an integral by interpreting it as an expected value

\[ E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx, \]

where \( f_X \) is the density function of \( X \).

In class, we considered the integral

\[ \int_{0}^{\infty} \int_{0}^{\infty} \min\{x_1, x_2\} f_{X_1}(x_1) f_{X_2}(x_2) \, dx_1 \, dx_2, \]

where \( f_{X_i} \) is the density function of the time until failure of component \( i \).

(a) In this integral, what is the input model?

________________________________________________

\[ f_X(x) = f_{X_1}(x_1) f_{X_2}(x_2) \]

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(b) In this integral, what is the logic model?

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\[ g(x) = \min\{x_1, x_2\} \]

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(c) Write pseudo-code to estimate the value of the integral using Monte Carlo simulation. (You do not need to specify how to generate random variates; just say what you want to generate.)

________________________________________________

\[ gsum = 0 \]

loop for \( i=1 \ldots n \)

\[ \text{generate } x_1 \text{ from } f_{X_1} \]

\[ \text{generate } x_2 \text{ from } f_{X_2} \]

\[ \text{set } y = \min\{x_1, x_2\} \]

\[ gsum = gsum + y \]

endloop

\[ \text{average} = gsum / n \]

________________________________________________

(d) In your pseudo-code, circle your point estimator.

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Circle \( \text{average} \) and/or \( gsum / n \).
4. Consider using micro/macro replications to estimate the standard error of $Y$, the sample average of $n = 100,000$ independent random variables $Y_i$. Let $Y_j$ denote the $j$th macro-replication point estimator, $j = 1, 2, ..., k$, and assume that $k = 100$. Let $Y$ denote the average of the $Y_j$'s.

(a) What is the performance measure?

$E(Y) \leftarrow$

Comment:
The purpose of the Monte Carlo simulation experiment always is to estimate the performance measure, $\theta$, which is a property of the model. The purpose of micro/macro replications always is to estimate $\text{ste}(\theta)$, the standard error of the point estimator, which is a property of the experiment.

(b) What is the relationship between $Y$ and $\bar{Y}$?

$Y = \bar{Y} \leftarrow$

Comment:
That is, both are algebraically equal. Therefore, their distributions are equal, and in turn their distribution properties are equal. For example, $E(Y) = E(\bar{Y})$.

(c) Suppose that $Y_i$ are binary, one if an event $A$ occurs and zero otherwise. Explain an alternative to micro/macro replications for estimating $\text{ste}(\bar{Y})$.

$$\text{ste}(\bar{Y}) = \left[ \frac{S_Y^2}{n} \right]^{1/2} = \left[ \frac{\bar{Y}(1 - \bar{Y})}{n - 1} \right]^{1/2}$$

Comment:
Here, $\bar{Y} = \rho$, the fraction of successes. Both answers are algebraically equal; both are correct; the second is easier to implement.

(d) If $Y = 123.456789$ and $\text{ste}(Y) = 0.0111111111$, which digits of $Y$ are meaningful?

something close to 123.46 $\leftarrow$
5. Suppose that the random variable $X$ has density function $f_X(x) = (x - 1)/50$ for $1 \leq x \leq 11$ and zero elsewhere. The cdf is then $F_X(x) = (x - 1)^2/100$ over the interesting values of $x$. If we use the inverse transformation to generate random variates from this distribution, for every random number $u$ in $[0, 1]$, what is the corresponding random variate $x$?

For every $u$ in $[0, 1]$, set $u = F_X(x) = (x - 1)^2/100$.
Solve for $x$:
$$x = 1 + 10\sqrt{u} \leftarrow.$$ 

6. Suppose that a particular linear congruential random-number generator $X_{i+1} = (aX_i + c) \pmod{m}$ produces the values $u_1, u_2,$ ....

(a) If $u_{32} = u_1$, then what do you know about $u_{34}$?

$$u_{34} = u_3 \leftarrow$$
(Notice that this does not tell us the value of $m$.)

(b) If $a = 1$ and $c = 3$, how many Marsaglia hyperplanes will occur for $k = 2$ dimensions?

In $k = 2$ dimensions, a hyperplane is a line.
Most 2-dimensional points, $(i, i+3)$ are almost on the main diagonal.
Therefore, I accept the answer 1 $\leftarrow$

Three points $(m-3, 0)$, $(m-2, 1)$, and $m-1, 2$ lie in in corner on a second line.
Therefore, I think that the better answer is 2 $\leftarrow$.

(c) If we discuss 3-dimensional uniformity, what is it that is 3 dimensional?

The $m$ three-tuples of the form $(u_i, u_{i+1}, u_{i+2}) \leftarrow$