1. (Montgomery and Runger, Problem 5–32 and 5–33) A marketing company performed a risk analysis for a manufacturer of synthetic fibers and concluded that new competitors present no risk 13% of the time (due mostly to the diversity of fibers manufactured), moderate risk 72% of the time (some overlapping of products), and very high risk (competitor manufactures the same products) 15% of the time. It is known that twelve international companies are planning to open new facilities for the manufacture of synthetic fibers within the next three years. Assume that the companies are independent. Let $X$, $Y$, and $Z$ denote the number of new competitors that will pose no, moderate, and very high risk for the interested company, respectively.

(a) What is the range of the joint probability distribution of $(X, Y, Z)$?

The range is the set of $(x, y, z)$ where $x$, $y$ and $z$ are non-negative integers with $x + y + z = 12$.

(b) Determine $P(X = 1, Y = 3, Z = 1)$.

$P(X = 1, Y = 3, Z = 1) = 0$ since this event is not in the range.

(c) Determine $P(Z \leq 2)$.

$Z$ is binomial with $n = 12$ and $p = 0.15$, so

$$P(Z \leq 2) = P(Z = 0) + P(Z = 1) + P(Z = 2) = \binom{12}{0}(0.15)^0(0.85)^{12-0} + \binom{12}{1}(0.15)^1(0.85)^{12-1} + \binom{12}{2}(0.15)^2(0.85)^{12-2} = 0.2923 \leftarrow$$

(d) Determine $P(Z = 1 \mid Y = 1, X = 10)$.

Since $X + Y + Z = 12$,

$$P(Z = 1 \mid Y = 1, X = 10) = 1.$$

(e) Determine $P(Z \leq 1 \mid X = 10)$.

Conditionally, $Z$ is binomial with $n = 2$ and $p = 0.15 / (0.72 + 0.15) = 0.172$, so

$$P(Z \leq 1 \mid X = 10) = 1 - P(Z = 2 \mid X = 10) = 1 - (1)(0.172)^2(0.838)^0 = 0.971 \leftarrow$$

Comment: Here $p$ is the conditional probability of very high risk given not no risk.

(f) Determine $E(Z \mid X = 10)$.

Conditionally, $Z$ is binomial with $n = 2$ and $p = 0.15 / (0.72 + 0.15) = 0.172$, so

$$E(Z \mid X = 10) = (2)(0.172) = 0.34 \leftarrow$$
(g) (From the Concise Notes) In a multinomial experiment, let \( X_i \) denote the number of trials that result in outcome \( i \) for \( i = 1, 2, \ldots, k \). (Then \( X_1 + X_2 + \cdots + X_k = n \).) The random vector \((X_1, X_2, \ldots, X_k)\) has a multinomial distribution with joint pmf
\[
P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \frac{n!}{x_1!x_2!\cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}
\]
when each \( x_i \) is a nonnegative integer and \( x_1 + x_2 + \cdots + x_k = n \); zero elsewhere.

State the correspondence between the general notation \( X_i \), \( p_i \), \( k \), and \( n \) to the marketing problem’s components.

\[
k = 3 \quad \text{and} \quad n = 12.
X_1 = X, \quad X_2 = Y, \quad \text{and} \quad X_3 = Z.
p_1 = 0.13, \quad p_2 = 0.72, \quad \text{and} \quad p_3 = 0.15.
\]

2. (Montgomery and Runger, Problems 5–58 through 5–60) Suppose that the random variables \( X \), \( Y \), and \( Z \) have the joint probability density function \( f_{XYZ}(x, y, z) = c \) over the cylinder \( x^2 + y^2 < 4 \) and \( 0 < z < 4 \).

(a) Create a context for this problem. That is, think of an experiment and random-variable definitions that match this problem.

For example, the location of contamination in a can of food.

(b) Determine the value of the constant \( c \). (Draw the picture.)

The volume of the cylinder defined by \( f_{XYZ} \) is \( \pi r^2 h = \pi 2^2 4 = 16 \pi \).
Therefore, \( c = 1 / (16 \pi) = 0.0148 \leftarrow \)

(c) Explain why we can say that this is a trivariate uniform distribution.

All points \((x, y, z)\) that are possible are equally likely.

(d) Determine \( P(X^2 + Y^2 < 2) \). (Hint: Look at your picture.)

The volume of the cylinder of radius \( r = \sqrt{2} \) is half of the total cylinder, so \( P(X^2 + Y^2 < 2) = 1/2 \leftarrow \)

(e) Determine \( P(Z < 2) \). (Hint: Look at your picture.)

The marginal distribution of \( Z \) is uniform from zero to four. Half the cylinder is below \( Z = 2 \), so \( P(Z < 2) = 1/2 \leftarrow \)

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(f) Determine $E(Z)$.

(Hint: Look at your picture.)

The marginal distribution of $Z$ is uniform from zero to four, so $E(Z) = 2$ ←

(g) Determine the conditional distribution of $Z$ given that $X = 1$ and $Y = 1$.

$Z$ is independent of $X$ and $Y$, so the conditional distribution of $Z$ is the marginal distribution.

That is, $Z$ is uniformly distributed between one and four, regardless of $X$ and $Y$.

3. (Montgomery and Runger, Problems 5–66) A manufacturer of electroluminescent lamps knows that the amount of luminescent ink deposited on one of its products is normally distributed with a mean of 1.2 grams and a standard deviation of 0.03 grams. Any lamp with less than 1.14 grams of ink will fail to meet customer’s specifications. A random sample of 25 lamps is collected and the mass of luminescent ink of each is measured.

(a) What is the probability that at least one lamp fails to meet specifications?

Consider only the first lamp.

$P(X_1 < 1.14) = P(Z < (1.14 - 1.2)/0.03) = P(Z < -2) = 0.023$ ←

Then the probability that none succeeds is $(1 - 0.023)^{25} = 0.558$.

Therefore, the probability that at least one fails is approximately $1 - 0.558 = 0.442$ ←

(b) Why is the joint probability distribution of the twenty five lamps not needed to answer Part (a)?

The lamps’ ink amounts are assumed to be independent.

(c) Let $X_i$ denote the ink weight from lamp $i$, $i = 1, 2, \ldots, 25$. Write the 25-dimensional pdf of $(X_1, X_2, \ldots, X_{25})$.

$f_{X_1 \ldots X_{25}}(x_1, x_2, \ldots, x_{25}) = f_X(x_1) \cdots f_X(x_{25})$, where

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2}$$

is the normal pdf with $\mu = 1.2$ and $\sigma = 0.03$. 

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4. Consider the experiment of choosing a random student from the class who has taken both Exam 1 and Exam 2. Let $X_i$ denote the student’s score of Exam $i$, for $i = 1, 2$. (We viewed the scatter plot in class Friday.)

(a) Visually, where is the bivariate mean $(\mu_1, \mu_2)$.

At about $(\mu_1, \mu_2) = (70, 70)$ ←

(b) In words, state the meaning of $E(X_2 \mid X_1 = 90)$.

The expected Exam 2 score of a randomly chosen student who scored 90 on Exam 1.

(c) In words, state the meaning of $\text{std}(X_2 \mid X_1 = 90)$.

The standard deviation of the Exam 2 score of a randomly chosen student who scored 90 on Exam 1.

(d) Draw the line $E(X_2 \mid X_1 = x)$. State the line algebraically.

(Assume that it is linear. Your answer will be approximate.)

The line

$$E(X_2 \mid X_1 = x) = 10 + 0.9x_1$$

looks about right.

Comment: One might expect $E(X_2 \mid X_1 = x_1) = x_1$.

Notice, though, that the first exam had 110 points and the second exam had 98 points, which yields a coefficient of $98 / 110 \approx 0.89$.

This line suggests that (one the average) students who had a strong first exam did less well than students who had a weak exam one.

Comment: Notice that $x_1$ is a dummy variable.

I could have used, for example, $x$. 