
Throughout, define notation and work step by step. Understand the reason for each step. Anyone quibbling with your solution should be able to point to the specific step in question. Conversely, you should be able to defend each step.

1. Consider the random experiment whose procedure is to throw one six-sided die. Let the sample space be $S = \{1, ..., 6\}$, where the number indicates the side facing up. Assume that all six sides are equally likely.

(a) Let $A_5 = \text{"the outcome is a 5"}$. Find $P(A_5)$. What result in the textbook did you use?

There are six equally likely outcomes. $A_5 = \{5\}$ contains one of them. Therefore, $P(A_5) = \frac{1}{6}$.

(b) Let $A_i = \text{"the outcome is } i\text{"}$, for $i = 1, 2, ..., 6$. Find $P(A_2)$.

Analogous to Part (a), $P(A_2) = P(\{2\}) = \frac{1}{6}$.

(c) Let $E$ denote the event that the number is even. Show (carefully) that $P(E) = \frac{3}{6}$.

$$P(E) = P(A_2 \cup A_4 \cup A_6) \quad \text{by definition of } E$$
$$= P(A_2) + P(A_4) + P(A_6) \quad \text{since mutually exclusive}$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \quad \text{since equally likely outcomes}$$
$$= \frac{3}{6} \quad \text{by simplifying}$$

(d) Find $P(A_2 \cup E)$ using Result 3-1 from the textbook. State in words the event.

$$P(A_2 \cup E) = P(A_2) + P(A_4) - P(A_2 \cap E) \quad \text{by Result 3-1, p. 70, textbook}$$
$$= \frac{1}{6} + \frac{3}{6} - \frac{1}{6} \quad \text{since equally likely outcomes}$$
$$= \frac{3}{6} \quad \text{by simplifying}$$

Comment: If not asked to use Result 3-1, the probability could be obtained easier by showing that $A_2 \cup E = E$. Then $P(A_2 \cup E) = P(E)$, which is known from Part (c).
2. In each of the six blanks, provide the reason why the corresponding equality is valid. Remember that probabilities are numbers and that events are sets. Possible reasons include (1) "Axiom 1", (2) "Axiom 2", (3) "Axiom 3", (4) "events are mutually exclusive", (5) "same events", (6) "algebra", (7) "events are equally likely", and (8) "probability of the empty set is zero".

Result. For \( A \) and \( B \) are events, then \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \).

\[
P(A \cup B) = P((A \cap B^\prime) \cup (A \cap B) \cup (A^\prime \cap B)) < (5) \text{ same event } > \frac{< (4) \text{ m.e. } >}{\text{ algebra}}
\]

\[
= P(A \cap B^\prime) + P(A \cap B) + P(A^\prime \cap B) < (6) \text{ algebra } > \frac{< (3) \text{ (or maybe (4))) } >}{\text{ (3) (or maybe (4)))}}
\]

\[
= P[(A \cap B^\prime) \cup (A \cap B)] + P(A \cap B) < (5) \text{ same event } > \frac{< (6) \text{ algebra } >}{\text{ (6) algebra } >}
\]

3. (Problem 3-58.) The probability is 1% that an electrical connector that is kept dry fails during the warranty period. If the connector is ever wet, the probability of a failure during the warranty period is 5%. Suppose that 90% of all connectors are kept dry.

Consider the experiment of randomly selecting one electrical connector from a pile of 100 connectors.

(a) Define event notation for selecting a dry connector. What is the implied notation for a connector that has been wet?

Let \( D = \) "the (randomly selected) connector is kept dry".
Then \( D^\prime = \) "the (randomly selected) connector has been wet".

(b) Define event notation for selecting a connector that fails. What is the implied notation for a connector that does not fail?

Let \( F = \) "the (randomly selected) connector fails during warranty".
Then \( F^\prime = \) "the (randomly selected) connector does not fail during warranty".

(c) Write the given information in terms of your notation.

\[
P(F \mid D) = 0.01.
P(F \mid D^\prime) = 0.05.
P(D) = 0.9.
\]
(d) Use the Total Probability Rule to determine the fraction of connectors that fail during the warranty period.

\[
P(F) = P(F \mid D)P(D) + P(F \mid D')P(D') \quad \text{total probability}
\]
\[
= (0.01)(0.9) + (0.05)(1-0.9) \quad \text{substitute given information}
\]
\[
= 0.014 \quad \text{simplify}
\]

(e) Explain why the fraction of connectors that fail is the same as the probability that a randomly selected connector fails.

This relationship is fundamental, both for interpreting probabilities and later for inferential statistics.

Quick imprecise answer: "Relative frequency" is the usual interpretation of a probability when the experiment is one that can be repeated. But "relative frequency" is just a fraction.

Better precise answer: Let the sample space \( S \) be a large, but finite, set of connectors; let \( n \) denote the number of connectors in \( S \). The experiment is to select one connector, with each connector being equally likely. \( F \) is then the set of connectors that fail. Let \( k \) denote the number of connectors in \( F \). Let \( F_i = "i\text{th failed component is selected, } i = 1, 2, \ldots, k. \) Then

\[
P(F) = P(F_1 \cup F_2 \cup \cdots \cup F_k) \quad \text{same event}
\]
\[
= P(F_1) + P(F_2) + \cdots + P(F_k) \quad \text{mutually exclusive}
\]
\[
= (1/n) + (1/n) + \cdots +(1/n) \quad \text{equally likely components}
\]
\[
= k/n \quad \text{simplifying } k \text{ terms}
\]
\[
= \text{fraction of components that fail} \quad \text{definition of } k \text{ and } n
\]

4. (Problem 3-54.) Suppose that \( P(A \mid B) = 1 \).

(a) Use the definition of conditional probability to show that the result is true in the special case when \( A = B \).

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{def. of cond. prob.}
\]
\[
= \frac{P(B \cap B)}{P(B)} \quad \text{because } A = B
\]
\[
= \frac{P(B)}{P(B)} \quad \text{same event}
\]
\[
= 1 \quad \text{algebra}
\]
(b) Make up an example for which \( A \neq B \) and the result is true. (For example, when tossing a die as in Problem 1, \( A = \) "the number is odd" and \( B = \) "the number is 3".)

Consider the experiment of randomly choosing a student from this class. Let \( B \) denote the event that the student gets a "B" grade. Let \( A \) denote the event that the student passes the course. Then \( P(A \mid B) = 1 \).

(c) Draw a Venn diagram to explain the relationship between \( A \) and \( B \) that implies \( P(A \mid B) = 1 \).

In general, the result is true whenever \( B \) is a subset of \( A \). The \( B \) occurring is sufficient to guarantee that \( A \) occurs.

Therefore, the Venn diagram should show \( B \) completely inside \( A \), which is completely inside the sample space \( S \).

5. (Problem 3-51.) A lot contains 15 castings from a local supplier and 25 casting from a supplier in the next state. Two castings are selected at random, without replacement, from the lot of 40 castings. Let \( L_1 \) denote the event that the first casting selected is from the local supplier; let \( L_2 \) denote the event that the second casting selected is from the local supplier.

(a) Determine \( P(L_1) \).

\[
P(L_1) = \frac{15}{15 + 25} = 0.375 \text{ because all are equally likely.}
\]

(b) Determine \( P(L_2 \mid L_1) \).

\[
P(L_2 \mid L_1) = \frac{25}{14 + 25} = 0.641
\]

(c) Determine \( P(L_1 \cap L_2) \).

\[
P(L_1 \cap L_2) = P(L_1)P(L_2 \mid L_1) = \left(\frac{15}{40}\right)\left(\frac{25}{39}\right) = 0.24
\]

(d) Rework (a-c) assuming that sampling is with replacement.

\[
P(L_1) = \frac{15}{15 + 25} = 0.375 \text{ because all are equally likely.}
P(L_2 \mid L_1) = \frac{25}{15 + 25} = 0.625
\]
\[
P(L_1 \cap L_2) = P(L_1)P(L_2 \mid L_1) = \left(\frac{15}{40}\right)\left(\frac{25}{40}\right) = 0.234
\]
Comment: Ask yourself why replacement reduces \( P(L_2 \mid L_1) \).
6. Spreadsheet. (Email a copy to "ie230@ecn.purdue.edu").

Consider flipping a fair coin. Let $T$ denote the coin landing tails up. Because the coin is fair, we know that $P(T) = 0.5$. We want to perform a Monte Carlo simulation in the spreadsheet. The simulation will flip the coin 100 times and tally the relative frequency that $T$ occurs.

(a) At the top of the sheet, place your name, class, homework number, and problem number. Leave two or three blank rows. Suppose that we are now in Row 7.

(b) In Cell A9, label this column "Trial". Beginning in Row 11, in Column A enter the integers 1 through 100.

(c) In Cell B7, enter the constant 0.5. To its left, label this cell "$P(T)$".

(d) In Column B, beginning in Row 11, generate 100 Uniform(0,1) pseudo-random numbers. (You may use the "=rand()" function.) Label this column "U(0,1)" in Row 9.

(e) In Column C, enter a "1" in every row for which the corresponding random number is less than the cell B7, currently containing 0.5. Otherwise, enter a "0". (You may use the "if" function.) Label this column "Outcome" in Row 9.

(f) In Cell C7, in the same row as used for Part (c), enter the relative frequency of ones that occurred in Column C. (You can use the "average" function.) In Cell D7 label this cell "Relative Frequency".

(g) Compare the relative frequency with the probability next to it. Now hit the "F9" key to recalculate the spreadsheet (that is, to reflip the 100 coins). Again compare these two numbers. Recalculate a few more times. Does the probability $P(T)$ seem to be the long-run relative frequency of obtaining a head? If not, debug your spreadsheet.

(h) Copy your work to a second sheet. Change the experiment in the second sheet to flip an unfair coin by changing the probability to $P(T) = 0.7$. Hit F9 enough to convince yourself that the logic is correct.

A solution is in the posted MSExcel spreadsheet.