Nonlinear propagation of an asymmetric double exponential waveform

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A weak-shock solution is obtained for the nonlinear propagation of a waveform that initially has the form of an asymmetric double exponential. Such a wave shocks at its peak, so that shock growth and wave-amplitude attenuation occur simultaneously. Simple formulas for wave amplitude, shock amplitude, and arrival time are given as is an expression for the waveform. We also present a general technique for obtaining weak-shock solutions for the amplitude of any integrable waveform that forms a single shock.

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INTRODUCTION

In a previous paper,[1] Rogers presented a weak-shock solution for the propagation of a shock wave with an exponentially decaying tail. Such a waveform is a special case of a more general class, namely, asymmetric double exponential waveforms.

This paper treats the nonlinear propagation of an acoustic waveform that initially has the form of an asymmetric double exponential function, that is, a wave whose pressure rises exponentially with one time constant, then decays exponentially with a second time constant (see Fig. 1). Examples of such waveforms include the bubble pulses that follow the shock wave in an underwater explosion, the acoustic signal from an airgun, and an explosive shock wave that has been reflected from a water–air interface. If we consider only spherically spreading waves, except for the last example, where an early surface reflection could have a significant effect on the farfield wave amplitude, nonlinear effects will not be significant for a single airgun or explosion, no matter how large. This is due to the fact that, except for a relatively unimportant depth dependence, the amplitude and rise time of such waves scale in such a way as to leave the shock formation distance independent of the charge weight of the explosive or the pressure-volume product of the airgun. For arrays of airguns or explosives or for propagation conditions where cylindrical rather than spherical spreading pertains, nonlinear effects can cause significant changes in wave amplitude and energy.

Unlike most waveforms studied to date,[1,2] the double exponential shocks at the peak of the wave rather than at the zero-crossing. As a result, wave amplitude and energy attentuation occur simultaneously with shock growth. In this report we use weak-shock theory to obtain simple expressions for the wave amplitude and shape as a function of propagation distance. We also present a general technique for obtaining weak-shock solutions for the amplitude of any integrable waveform that forms a single shock.

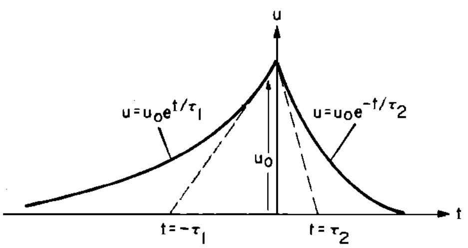

FIG. 1. Assumed particle velocity waveform at $x=0$.

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I. SOLUTION FOR SHOCK WAVE PARAMETERS

We consider a particle velocity u versus time profile, which at $x=0$ rises exponentially with time constant τ_1 for $t\leq 0$ and falls off exponentially with time constant τ_2 for $t>0$, i.e.,

$$\begin{aligned} u(0,t) &= u_0\, e^{t/\tau_1}, \quad t\leq 0\,, \\ &= u_0\, e^{-t/\tau_2}, \quad t>0\,, \end{aligned} \tag{1}$$

as shown in Fig. 1.

We now introduce the retarded time $t'=t-x/c_0$ and the characteristic length $l_0=\tau_1 c_0^2/\beta u_0$, where β is the parameter of nonlinearity of the fluid and c_0 is the sound speed. We define the dimensionless variables $\hat{u}=u/u_0$, $\hat{t}'=t'/\tau_1$, $\hat{x}=x/l_0$, $\eta=\tau_2/\tau_1$, and let $\hat{t}_s'$ denote the dimensionless retarded time corresponding to the peak of particle velocity. The parameter x is a measure of the "age" of the wave.[2] It is a linear function of range for plane waves, a logarithmic function for spherical waves, and a square root function for cylindrical waves.[3] That is, for plane waves $x=x$ and the pressure is given by $p=\rho c_0 u$. For cylindrical waves $x=r_0(r/r_0)^{1/2}$ and $p=(r_0/r)^{1/2}\rho c_0 u$. For spherical waves $x=R_0\log(R/R_0)$ and $p=(R_0/R)\rho c_0 u$, where ρ is the fluid density and r and R are cylindrical and spherical radial distances, respectively. For nonplanar waves u should be regarded as a Blokhintsev[4] invarient rather than a particle velocity.

The waveform distorts as it propagates, and at $\hat{x}=1$ a shock forms at the peak of the rising portion of the wave. For $\hat{x}\geq 1$ then, $\hat{t}_s'$ will also denote the position of the shock. Our goal is to obtain a solution for $\hat{t}_s'(x)$ as well as expressions for $\hat{u}_b(\hat{x})$, the acoustic particle velocity immediately preceding the shock, and for $\hat{u}_a(\hat{x})$, the particle velocity immediately following the shock.

The particle velocity $\hat{u}(\hat{x},\hat{t}')$ is given by

$$\hat{u}(\hat{x},\hat{t}')=\exp(\hat{t}'+\hat{u}\hat{x})\,, \quad \hat{t}'<\hat{t}_s'\,, \tag{2a}$$

and

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$$\hat{u}(\hat{x},\hat{t}')=\exp[-(\hat{t}'+\hat{u}\hat{x})/\eta], \quad \hat{t}'>\hat{t}_s' . \tag{2b}$$

It follows that $\hat{u}_b(\hat{x})$ and $\hat{u}_a(\hat{x})$ are related to $\hat{t}_s'$ by

$$\hat{u}_b(\hat{x}) = \exp(\hat{t}_s'+\hat{u}_b\hat{x}) , \tag{3}$$

and

$$\hat{u}_a(\hat{x}) = \exp[-(\hat{t}_s'+\hat{u}_a\hat{x})/\eta] . \tag{4}$$

Taking the logarithm and rearranging terms, we obtain

$$\hat{t}_s'(\hat{x}) = \log\hat{u}_b - \hat{u}_b\hat{x} , \tag{5}$$

and

$$\hat{t}_s'(\hat{x}) = -\eta\log\hat{u}_a - \hat{u}_a\hat{x} . \tag{6}$$

We can derive differential equations for $\hat{u}_b(\hat{x})$ and $\hat{u}_a(\hat{x})$ using Eqs. (5) and (6) in conjunction with the expression for the propagation velocity of the shock front:

$$d\hat{t}_s'/d\hat{x} = -\tfrac{1}{2}(\hat{u}_a+\hat{u}_b) . \tag{7}$$

By differentiating Eqs. (5) and (6) with respect to $\hat{x}$ and then using Eq. (7) to substitute for $d\hat{t}_s'/d\hat{x}$, we obtain

$$\tfrac{1}{2}(\hat{u}_b-\hat{u}_a) = (1/\hat{u}_b - \hat{x})(d\hat{u}_b/d\hat{x}) , \tag{8}$$

and

$$\tfrac{1}{2}(\hat{u}_b-\hat{u}_a) = (\eta/\hat{u}_a + \hat{x})(d\hat{u}_a/d\hat{x}) . \tag{9}$$

At this point it is convenient to make a change of variables. Instead of $\hat{u}_b(\hat{x})$ and $\hat{u}_a(\hat{x})$, we will work with $\tau_b(\hat{x})$ and $\tau_a(\hat{x})$, which we define, respectively, as the dimensionless time constants immediately prior to and immediately following the shock front (as shown in Fig. 2). The reason for the change of variables will soon become apparent.

From Eq. (3) we have

$$d\hat{u}_b/d\hat{t}_s' = \hat{u}_b/(1-\hat{x}\hat{u}_b) . \tag{10}$$

We define $\tau_b(\hat{x})$ such that

$$\tau_b(\hat{x}) = -[\hat{u}_b/(d\hat{u}_b/d\hat{t}_s')], \tag{11}$$

so that

$$\tau_b = \hat{u}_b\hat{x} - 1 , \tag{12}$$

or, alternately,

$$\hat{u}_b(\hat{x}) = (\tau_b+1)/\hat{x} . \tag{13}$$

Similarly we use Eq. (4) to show that

$$d\hat{u}_a/d\hat{t}_s' = -\hat{u}_a/(\eta+\hat{x}\hat{u}_a) , \tag{14}$$

and we define τ_a such that

$$\tau_a(\hat{x}) = -[\hat{u}_a/(d\hat{u}_a/d\hat{t}_s')] . \tag{15}$$

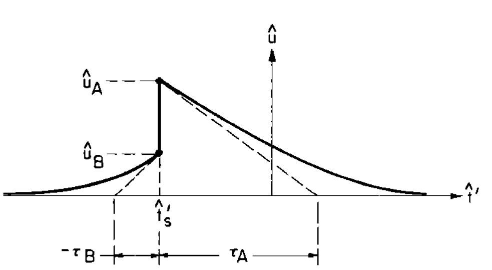

FIG. 2. Shocked particle velocity waveform ($\hat{x}>1$).

The result is that

$$\tau_a = \hat{u}_a\hat{x} + \eta , \tag{16}$$

and

$$\hat{u}_a = (\tau_a-\eta)/\hat{x} . \tag{17}$$

Expressing Eqs. (8) and (9), the differential equations for $\hat{u}_b$ and $\hat{u}_a$, in terms of τ_b and τ_a and using the substitutions indicated by Eqs. (13) and (17), we obtain

$$d\tau_b/d\hat{x} = [(\tau_b+1)/(2\tau_b\hat{x})](\tau_b+\tau_a-1-\eta) , \tag{18}$$

and

$$d\tau_a/d\hat{x} = [(\tau_a-\eta)/(2\tau_a\hat{x})](\tau_b+\tau_a+1+\eta) . \tag{19}$$

We impose initial conditions at $\hat{x}=1$, the propagation distance at which shock formation begins. The only mechanism for attenuation we are considering consists of points on the trailing edge of the wave catching up with the shock front; hence, the peak of particle velocity must equal unity prior to shock formation and

$$\hat{u}_b = \hat{u}_a = 1, \quad \text{at } \hat{x}=1 . \tag{20a}$$

From Eqs. (12) and (16) we see that this implies that

$$\tau_a = 1+\eta , \tag{20b}$$

and

$$\tau_b = 0, \quad \text{at } \hat{x}=1 .$$

We digress for a moment to consider the behavior of these functions in the limit where $\hat{x}$ approaches infinity. The asymptotic shape for our waveform as $\hat{x}$ becomes very large is a simple sawtooth,[3] so that τ_a becomes proportional to $\sqrt{\hat{x}}$ and $\hat{u}_a$ becomes proportional to $1/\sqrt{\hat{x}}$. It is also clear that $\hat{u}_b$ must approach zero quicker than $\hat{u}_a$, so that we can write the asymptotic limit of Eq. (8) as follows:

$$-C/\sqrt{\hat{x}} = (1/\hat{u}_b)(d\hat{u}_b/d\hat{x}) , \tag{21}$$

where C is some constant. We have also used the fact that $1/\hat{u}_b$ must dominate $\hat{x}$ as $\hat{x}$ approaches infinity. Otherwise Eq. (8) tells us that $\hat{u}_b$ would increase with $\hat{x}$. The solution of Eq. (21) is

$$u_b \sim \exp(-2C\sqrt{\hat{x}}), \quad \text{as } \hat{x}\to\infty . \tag{22}$$

From Eq. (12) then,

$$\lim_{\hat{x}\to\infty}(\tau_b) = -1 , \tag{23}$$

which essentially states that in the limit as $\hat{x}\to\infty$, the waveform to the left of the shock is undistorted.

Only a few straightforward manipulations are required to express τ_a in terms of τ_b. We multiply Eq. (19) by τ_a and Eq. (18) by τ_b and then subtract, giving

$$\tau_a(d\tau_a/d\hat{x}) - \tau_b(d\tau_b/d\hat{x}) = (1/2\hat{x})(\tau_a^2-\tau_b^2+1-\eta^2) , \tag{24}$$

or

$$(d/d\hat{x})(\tau_a^2-\tau_b^2) = (1/\hat{x})(\tau_a^2-\tau_b^2+1-\eta^2) . \tag{25}$$

The solution to this equation is

$$\tau_a = (A\hat{x}+\tau_b^2+\eta^2-1)^{1/2} . \tag{26}$$

From Eq. (20b) we find that the constant A is given by

$$A = 2(\eta + 1)\,, \tag{27}$$

so that

$$\tau_a = [2(\eta+1)\hat{x} + \tau_b^2 + \eta^2 - 1]^{1/2}\,. \tag{28}$$

We note here that as an alternative to the procedure of Eqs. (24) through (28), we could easily find a relation between u_a and u_b using Landau's law of equal areas.[5] This more general method is demonstrated in the Appendix.

Now that we know τ_a in terms of τ_b, we can use Eqs. (5) and (6) to find τ_b as an implicit function of $\hat{x}$. We equate the right-hand sides of these equations and make the appropriate substitutions for $\hat{u}_a$ and $\hat{u}_b$ with the result that

$$-\eta \log[(\tau_a - \eta)/\hat{x}] - \tau_a + \eta = \log[(\tau_b + 1)/\hat{x}] - \tau_b - 1\,. \tag{29}$$

Taking the exponential function of Eq. (29) and making a few manipulations gives the following:

$$f e^{\tau_b} = \tau_b + 1\,, \tag{30a}$$

where

$$f = x^{(\eta+1)}(\tau_a - \eta)^{-\eta} e^{(\eta+1-\tau_a)}\,, \tag{30b}$$

with τ_a given by Eq. (28).

For $0 \leqslant \hat{x} \leqslant 1$ (that is, until shock formation occurs), we can express τ_b and τ_a as explicit functions of $\hat{x}$:

$$\tau_b(\hat{x}) = \hat{x} - 1\,, \tag{31}$$

and

$$\tau_a(\hat{x}) = \hat{x} + \eta\,. \tag{32}$$

For $\hat{x} > 1$, this result is no longer valid and we can no longer obtain τ_b as an explicit function of $\hat{x}$. However, we can use Newton's rule to obtain the following iterative solution to Eq. (30):

$$(\tau_b)_{n+1} = (\tau_b)_n - \frac{f_n e^{(\tau_b)_n} - [(\tau_b)_n + 1]}{f_n e^{(\tau_b)_n}\{1 - (\tau_b)_n/[(\tau_a)_n - \eta]\} - 1}\,. \tag{33}$$

The only problem remaining is to find a good starting point for our iterative process. Our first approximation for τ_b should approach -1 as $\hat{x}$ becomes large and should have the right behavior for $\hat{x} \to 1$. Such a function can be found by solving Eq. (18) under the assumption that $\hat{x} = 1$. First, we obtain approximate expressions for τ_a and τ_b. Eq. (19) yields

$$d\tau_a/d\hat{x} = 1\,,$$

and

$$d^2\tau_a/d\hat{x}^2 = -3/[4(1+\eta)], \quad \text{for } \hat{x} = 1\,. \tag{34}$$

We express τ_a as the first few terms of its Taylor series:

$$\tau_a = (1+\eta) + (\hat{x} - 1) - \{3/[8(1+\eta)]\}(\hat{x} - 1)^2\,. \tag{35}$$

Likewise, Eq. (18) and repeated applications of l'Hōpital's rule yield

$$d\tau_b/d\hat{x} = -\tfrac{1}{2}\,,$$

and

$$d^2\tau_b/d\hat{x}^2 = \tfrac{3}{16}[(2\eta+3)/(\eta+1)], \quad \text{for } \hat{x} = 1\,, \tag{36}$$

so that an approximate expression for τ_b is given by

$$\tau_b = -\tfrac{1}{2}(\hat{x} - 1) + \tfrac{3}{32}[(2\eta+3)/(\eta+1)](\hat{x} - 1)^2\,. \tag{37}$$

We now use Eqs. (35) and (37) to approximate $(\tau_b + \tau_a - 1 - \eta)/\tau_b$ to first order terms in $(\hat{x} = 1)$. We obtain

$$(\tau_b + \tau_a - 1 - \eta)/\tau_b = -1 - \tfrac{3}{8}[(2\eta+1)/(\eta+1)](\hat{x} - 1)\,. \tag{38}$$

This can be replaced by

$$(\tau_b + \tau_a - 1 - \eta)/\tau_b = -1 + \tfrac{3}{4}[(2\eta+1)/(\eta+1)]\tau_b\,, \tag{39}$$

which is also correct to the first order in $(x - 1)$. We now define a new constant

$$\gamma = \tfrac{3}{4}(2\eta+1)/(\eta+1)\,. \tag{40}$$

Note that γ is a weak function of η lying between $\frac{3}{4}$ and $\frac{3}{2}$ as η varies from zero to infinity. Plugging Eqs. (39) and (40) into Eq. (18), we arrive at

$$d\tau_b/[(\tau_b + 1)(\gamma\tau_b - 1)] = d\hat{x}/2\hat{x}\,. \tag{41}$$

The solution to Eq. (41), subject to the condition that $\tau_b = 0$ at $\hat{x} = 1$, is

$$\tau_b = (1 - \hat{x}^{(\gamma+1)/2})/(\gamma + \hat{x}^{(\gamma+1)/2})\,. \tag{42}$$

The approximate solution given by Eq. (42) has the proper behavior for $\hat{x} = 1$ (up to second order terms) and does approach -1 as $\hat{x}$ approaches infinity. Thus we use Eq. (42) as the starting point for the iterative solution given by Eq. (33).

We might be interested in an even simpler starting point for our iterative solution. In that case, we need only notice that when $\eta = \frac{1}{2}$, then Eq. (42) reduces to

$$\tau_b = (1 - \hat{x})/(1 + \hat{x}), \quad \eta = \tfrac{1}{2}\,. \tag{43}$$

Eq. (43) has the correct first order behavior in the vicinity of $\hat{x} = 1$ for any η and still approaches -1 as $\hat{x} \to \infty$. With Eq. (43) as a starting point, the first iteration yields a solution accurate to within 0.6% for all η. With Eq. (42) as the starting point, the first iteration is accurate to within 0.2%. Repeated iterations will, of course, yield greater accuracy.

Knowing $\tau_b(\hat{x})$, we can now obtain the values of τ_a, $\hat{u}_b$, and $\hat{u}_a$ through the use of Eqs. (28), (13), and (17), respectively. It is preferable to use Eq. (6) rather than Eq. (5) to find $\hat{t}_s'$.

The three quantities which are of most practical interest are the peak amplitude $\hat{u}_a$, the shock amplitude $\hat{u}_s = \hat{u}_a - \hat{u}_b$, and the arrival time $\hat{t}_s'$. These quantities are relatively weak functions of τ_b, so one can obtain them by substituting τ_b directly from Eqs. (42) or (43) into the above equations *without going through the iteration process* at all. Sufficiently accurate results can, in fact, be obtained by using Eq. (43) which is independent of η rather than Eq. (42) for τ_b. The principal results of this report can thus be expressed as

$$\hat{u}_a = 1, \quad \hat{x} \leqslant 1\,,$$

$$\hat{u}_a = (\{(1+\hat{x})^2[\eta^2 - 1 + 2(\eta+1)\hat{x}] + (1-\hat{x})^2\}^{1/2} - \eta(\hat{x}+1))/\hat{x}(\hat{x}+1), \quad \hat{x} \geqslant 1\,, \tag{44}$$

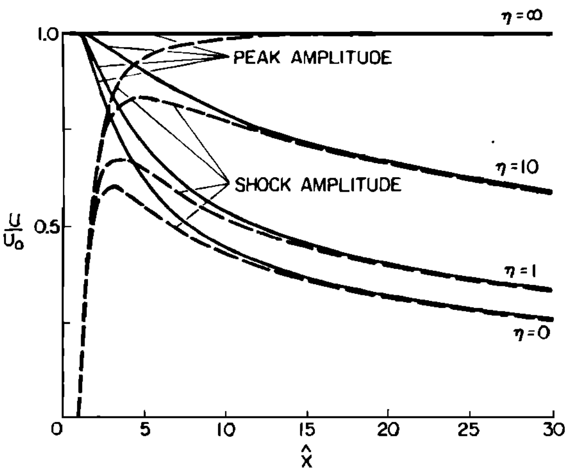

FIG. 3. Peak amplitude (solid lines) and shock-discontinuity amplitude (dashed line) as a function of "age" $\hat{x}$ for $\eta = 0$, 1, 10, and infinity.

$$\hat{u}_s = 0, \quad \hat{x} \leq 1,$$

$$\hat{u}_s = \hat{u}_a - \hat{u}_b = (\{(1+\hat{x})^2[\eta^2 - 1 + 2(\eta+1)\hat{x}] + (1-\hat{x})^2\}^{1/2} - (\eta\hat{x} + \eta + 2))/\hat{x}(\hat{x}+1), \quad \hat{x} > 1, \tag{45}$$

and

$$\hat{t}_s' = -\eta \log \hat{u}_a - \hat{u}_a \hat{x}. \tag{46}$$

The peak amplitude, shock amplitude, and the advance in the arrival time $\hat{t}_s'$ are plotted as a function of $\hat{x}$ for $\eta = 0$, 1, 10, and ∞ (shown in Figs. 3 and 4). Note that $\eta = \infty$ corresponds to a waveform that at $\hat{x} = 0$ rises exponentially to a value of unity, then remains there for all time. Such a wave never attenuates at all. Once the shock is completely formed, it is equivalent to a simple shock. Of course, one must continually put energy in at the source to support such a wave, and it must be one dimensional, i.e., in a pipe. The case $\eta = 0$ corresponds to a waveform that at $\hat{x} = 0$ rises exponentially to unity, then discontinuously falls to zero. The discontinuity is not a shock and disappears as soon as $\hat{x} > 0$. Though it may not be immediately apparent, $\eta = 0$ also handles the case of the surface-reflected explosive shock wave, i.e., a wave for which the pressure discontinuously falls to some negative value, then rises exponentially to zero.

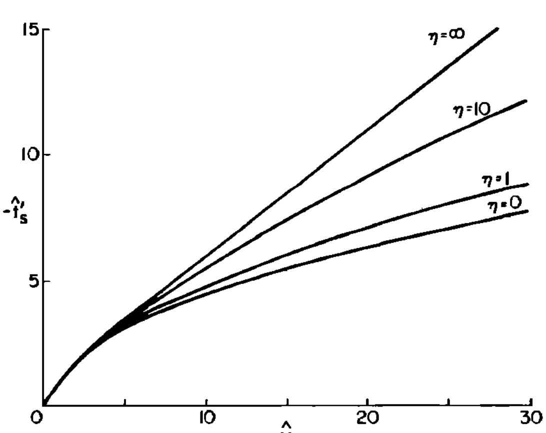

FIG. 4. Advance in arrival time as a function of "age" $\hat{x}$ for $\eta = 0$, 1, 10, and infinity.

Figures 3 and 4, which are calculated directly from Eqs. (44), (45), and (46), show the results obtained by essentially holding τ_1 constant and changing η by varying τ_2. This keeps $\hat{x}$ (which is inversely proportional to τ_1) the same in all cases. If we define a new dimensionless "age" parameter $\hat{x}'$ in terms of τ_2 such that $\hat{x}' = \hat{x}/\eta$, this new age parameter will then be equivalent to the age parameter in Ref. 1. Expressed in terms of $\hat{x}'$, $\eta \to \infty$ corresponds to a shock wave with an exponential tail and Eq. (44) reduces to the result given in Ref. 1.

II. SOLUTION FOR THE WAVEFORM SHAPE

We now have a method for determining the position of the shock front and the particle velocity immediately before and after its arrival, but as yet we do not have a solution for the shape of the velocity waveform. An exact expression cannot be obtained, but we can easily derive an excellent approximation.

For $\hat{t}' \leq \hat{t}_s'$, the acoustic particle velocity satisfies Eq. (2). Taking the logarithm of both sides, we find that

$$\log \hat{u} = \hat{t}' + \hat{u}\hat{x}. \tag{47}$$

We now introduce two new dimensionless variables $\hat{g}$ and $\hat{z}$ such that

$$\hat{u} = \hat{g}\, e^{\hat{t}'}, \tag{48}$$

and

$$\hat{z} = \hat{x}\, e^{\hat{t}'}. \tag{49}$$

Eq. (47) is then equivalent to

$$\log \hat{g} - \hat{g}\hat{z} = 0. \tag{50}$$

Using Newton's rule, we obtain the following iterative solution:

$$\hat{g}_{n+1} = (1 - \log \hat{g}_n)/[(1/\hat{g}_n) - \hat{z}]. \tag{51}$$

Before choosing a trial solution, let us consider some requirements that this trial solution must satisfy. First, the minimum value of $\hat{z}$ is zero, corresponding to

$$\hat{g} = 1, \quad \text{at } \hat{z} = 0. \tag{52a}$$

Second, at $\hat{z} = e^{-1}$, the waveform is shocked so that $\hat{z} = e^{-1}$ is the maximum value of $\hat{z}$ in which we are interested. This corresponds to

$$\hat{g} = e, \quad \text{at } \hat{z} = e^{-1}. \tag{52b}$$

Finally,

$$d\hat{g}/d\hat{z} = \infty, \quad \text{at } \hat{z} = e^{-1}, \tag{52c}$$

which simply states that the shock occurs at $\hat{z} = e^{-1}$. A function which satisfies the condition of Eq. (52) is

$$\hat{g}_0 = e - \alpha(1/e - \hat{z})^{1/3}, \tag{53a}$$

where

$$\alpha = e^{1/3}(e-1) \approx 2.398. \tag{53b}$$

The first iteration yields

$$\hat{g}_1 = \frac{1 - \log[e - \alpha(1/e - \hat{z})^{1/3}]}{1/[e - \alpha(1/e - \hat{z})^{1/3}] - \hat{z}}, \tag{54}$$

where $\hat{g}_1$ differs from the exact value of $\hat{g}$ by less than 0.6% and is exactly correct for $\hat{z} = 1/e$. We take $\hat{g}_1$ as our solution for $\hat{t}' \le \hat{t}'_s$.

We now duplicate this procedure for $\hat{t}' > \hat{t}'_s$. Taking the logarithm of Eq. (2b), we obtain

$$\log\hat{u} = -(\hat{t} + \hat{u}\hat{x})/\eta\,. \tag{55}$$

We introduce two more dimensionless variables $\hat{f}$ and $\hat{y}$, where

$$\hat{u} = \hat{f}\, e^{-\hat{t}1/\eta}, \tag{56}$$

and

$$\hat{y} = (1/\eta)\hat{x}\, e^{-t^{1/\eta}}\,. \tag{57}$$

Equation (55) is reduced to the following:

$$\log\hat{f} = -\hat{f}\hat{y}. \tag{58}$$

Rogers[1] previously obtained the following iterative solution for this equation:

$$\hat{f}_3 = \frac{1 + \log\{(2\hat{y}+1)/[1+\log(\hat{y}+1)]\}}{\hat{y} + (2\hat{y}+1)/[1+\log(\hat{y}+1)]}\,. \tag{59}$$

The error in $\hat{f}_3$ is less than 1% for $\hat{y} \le 10^6$. Our approximate solution for the waveform shape then is given by Eq. (48):

$$\begin{aligned} &\hat{u} = \hat{g}_1 e^{\hat{t}'}, \quad \hat{t}' \le \hat{t}'_s, \\ &\hat{u} = \hat{f}_3 e^{-\hat{t}'/\eta}, \quad \hat{t}' \ge \hat{t}'_s\,. \end{aligned} \tag{60}$$

Figures 5, 6, and 7 show $\hat{u}$ as a function of dimensionless retarded time for several values of $\hat{x}$ and η. As anticipated, $\hat{u}_a$, $\hat{u}_b$, and $\hat{t}'_s$ all decrease with $\hat{x}$. If we compare the three figures, we see that as η increases, the shock front attentuates less rapidly and moves more rapidly in the negative $\hat{t}'$ direction. Furthermore, if we keep η constant and vary $\hat{x}$, we see that the various waveforms coincide when $\hat{u}$ is very small. This too is expected, when $\hat{u}$ is small, very little distortion takes place, and the shape of the original waveform is preserved.

The results in Figs. 5 and 7 were checked against,

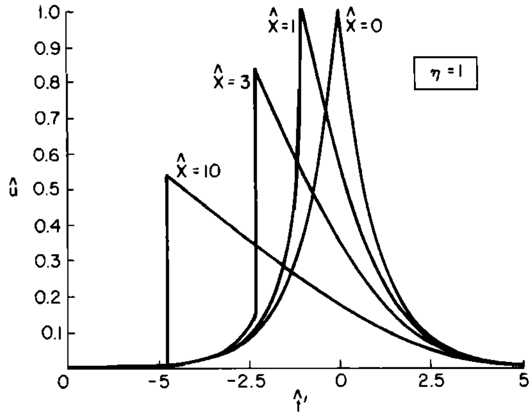

FIG. 6. Particle velocity as a function of retarded time for $\eta = 1$ and for $\hat{x} = 0$, 1, 3, and 10.

and are in excellent agreement with, a computer alogrithm that models the original waveform as a set of discrete points and that allows each point to propagate individually according to the laws of weak-shock theory.

APPENDIX

In this section we use Landau's law of equal areas to obtain a relation between $\hat{u}_a$ and $\hat{u}_b$ that is equivalent to that of Eq. (28). Landau's law[5] states that the area under any closed waveform propagating according to the laws of weak-shock theory remains constant. (By closed we mean that the initial and final points of the waveform must lie on the $\hat{t}'$-axis, i.e., $\hat{u} = 0$.)

Consider the waveform given by

$$\begin{aligned} &\hat{u} = e^{\hat{t}'}, \quad \hat{t}' < 0\,, \\ &\hat{u} = e^{-\hat{t}'/\eta}, \quad \hat{t}' > 0\,, \end{aligned} \tag{A1}$$

for $\hat{x} = 0$, as shown in Fig. 8(a). The area under this curve is

$$\begin{aligned} A_0 &= \int_{-\infty}^{0} e^{\hat{t}'}\, d\hat{t}' + \int_{0}^{\infty} e^{-\hat{t}'/\eta}\, d\hat{t}' \\ &= 1 + \eta\,. \end{aligned} \tag{A2}$$

Now we let the wave propagate a distance $\hat{x}$. The result is shown in Fig. 8(b). Not only must the total area

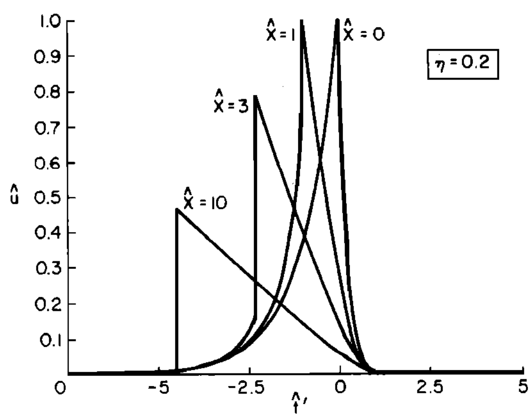

FIG. 5. Particle velocity as a function of retarded time for $\eta = 0.2$ and for $\hat{x} = 0$, 1, 3, and 10.

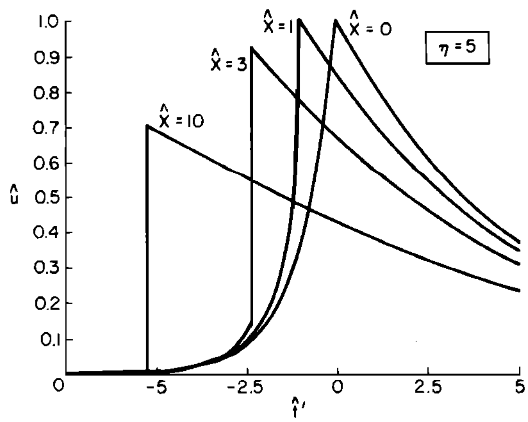

FIG. 7. Particle velocity as a function of retarded time for $\eta = 5$ and for $\hat{x} = 0$, 1, 3, and 10.

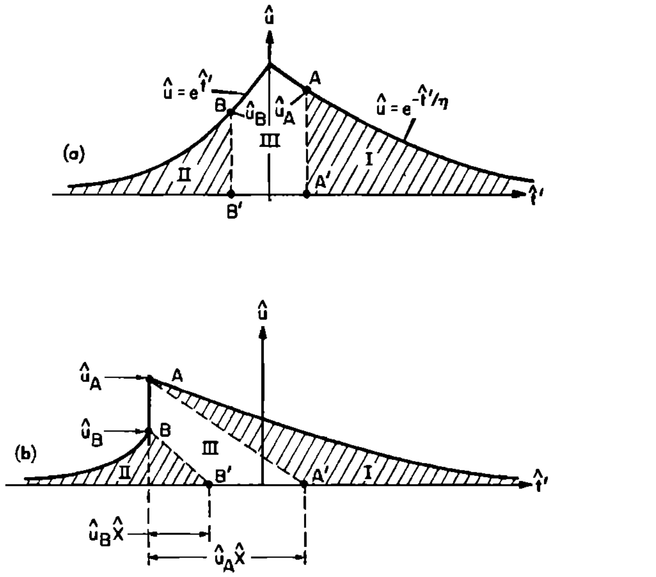

FIG. 8. (a) Initial waveform ($\hat{x}=0$). The total area under the curve, as well as the area under each of Regions I, II, and III, remains constant. (b) Distorted waveform ($\hat{x}>1$). The total area under the curve, as well as the area under each of regions I, II, and III, remains constant.

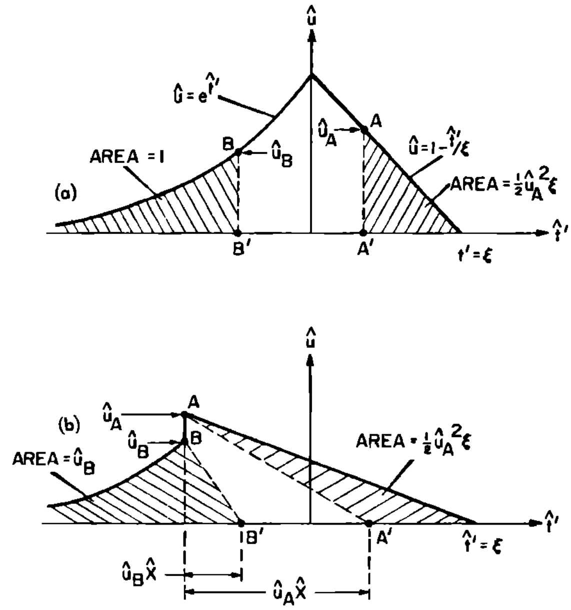

FIG. 9. (a) Initial waveform ($\hat{x}=0$). (b) Distorted waveform ($\hat{x}>1$).

under the curve remain constant, but according to the law of equal areas, the area in each of regions I, II, and III must also remain unchanged. Therefore,

$$A_{\mathrm{I}}=\eta\hat{u}_a\,,$$

and (A3)

$$A_{\mathrm{II}}=\hat{u}_b\,,$$

where A_{I} denotes the area in region I, etc. The area A_{III} is found geometrically from Fig. 8(b); it is simply the difference of two triangles:

$$A_{\mathrm{III}}=\tfrac{1}{2}(\hat{u}_a\hat{x})\hat{u}_a-\tfrac{1}{2}(\hat{u}_b\hat{x})\hat{u}_b\,. \tag{A4}$$

Equating the total areas, we have

$$\eta\hat{u}_a+\hat{u}_b+\tfrac{1}{2}\hat{u}_a^2\hat{x}-\tfrac{1}{2}\hat{u}_b^2\hat{x}=1+\eta\,. \tag{A5}$$

Now we make the substitutions indicated in Eqs. (13) and (17) and simplify to obtain

$$\tau_a^2=2(\eta+1)\hat{x}+\tau_b^2+\eta^2-1\,, \tag{A6}$$

which is the same as Eq. (28).

The method outlined above is quite powerful and can be applied to many other problems. Consider, for example, a waveform that at $\hat{x}=0$ rises exponentially for $t<0$ and falls off linearly for $t>0$, that is,

$$\hat{u}=e^{\hat{t}'},\quad \hat{t}'<0\,,$$
$$\hat{u}=1-\hat{t}'/\xi,\quad \hat{t}'>0\,, \tag{A7}$$

as shown in Fig. 9(a), where ξ is dimensionless. Note that the initial area under the curve is

$$A_0=1+\xi/2\,. \tag{A8}$$

Figure 9(b) shows the distorted waveform at a propagation distance $\hat{x}$. The total area is now given by

$$A=\hat{u}_a^2\xi/2+\hat{u}_a^2\hat{x}/2+\hat{u}_b-\hat{u}_b^2\hat{x}/2\,. \tag{A9}$$

Equating Eqs. (A8) and (A9), we obtain

$$\hat{u}_a=\{[2/(\xi+\hat{x})](-\hat{u}_b+\hat{u}_b^2\hat{x}/2+1+\xi/2)\}^{1/2}\,. \tag{A10}$$

We can now derive a second relation between $\hat{u}_a$ and $\hat{u}_b$. First we write down the shape of the waveform for arbitrary $\hat{x}$ and ξ:

$$\hat{u}=e^{\hat{t}'+\hat{u}\hat{x}},\quad \hat{t}'\leq\hat{t}_s'\,, \tag{A11a}$$

$$\hat{u}=1-(\hat{t}'+\hat{u}\hat{x})/\xi,\quad t'\geq t_s'\,. \tag{A11b}$$

Now we make the following substitutions. In Eq. (A11a) we replace $\hat{u}$ with $\hat{u}_b$ and $\hat{t}'$ with $\hat{t}_s'$; similarly we replace $\hat{u}$ with $\hat{u}_a$ and $\hat{t}'$ with $\hat{t}_s'$ in Eq. (A11b). We solve each equation for $\hat{t}_s'$ and equate the resulting expressions, giving

$$\hat{u}_a(\xi+\hat{x})=\hat{u}_b+\hat{x}\hat{u}_b+\xi\,. \tag{A12}$$

By combining this with Eq. (A7) and introducing the time constant τ_b as given in Eq. (12), we find that

$$\hat{x}G\,e^{\tau_b+1}-(\tau_b+1)=0\,, \tag{A13a}$$

where

$$G=e^{\xi-s}\,, \tag{A13b}$$

with

$$s=\{2(\hat{x}+\xi)[(1/2\hat{x})(\tau_b^2-1)+1+\xi/2]\}^{1/2}\,. \tag{A13c}$$

This solved iteratively using Newton's rule yields

$$(\tau_b)_{n+1}=(\tau_b)_n-\{\hat{x}G\,e^{(\tau_b)_n+1}-[(\tau_b)_n+1]\}\times\left[\hat{x}G\,e^{(\tau_b)_n+1}\left(1-\frac{(\xi+\hat{x})(\tau_b)_n}{s\hat{x}}\right)-1\right]^{-1}\,. \tag{A14}$$

A good trial solution to Eq. (A14) can be found by writing the differential equation for τ_b and using the method of Eqs. (34) through (42) to solve it approximately. One obtains the same approximate solution as before [that is, Eqs. (40) and (42)] provided only that η is replaced by ξ in Eq. (40). This is no real surprise since the linear decay at $\hat{x}=1$ and $\hat{t}'=\hat{t}_s'$ looks identical to the exponential decay.

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