Simultaneous Equation Models (Book Chapter 5)

Interrelated equations with continuous dependent variables:

➢ Utilization of individual vehicles (measured in kilometers driven) in multivehicle households

➢ Interrelation between travel time from home to an activity and the duration of the activity

➢ Interrelation of average vehicle speeds by lane with the vehicle speeds in adjacent lanes.

Problem:

➢ Estimation of equation systems by the ordinary least squares (OLS) violates a key OLS assumption in that a correlation between regressors and distrubances will be present because not all independent variables are fixed in random samples (violation of A5).

Overview of the simultaneous equations problem

> ➤ Consider annual vehicle utilization equations (one for each vehicle) in two-vehicle households of the following linear form:

$$u_1 = \beta_1 Z_1 + \alpha_1 X + \lambda_1 u_2 + \varepsilon_1$$

$$u_2 = \beta_2 Z_2 + \alpha_2 X + \lambda_2 u_1 + \varepsilon_2$$

Where:

u_1 is the kilometers per year that vehicle 1 is driven,

u_2 is the kilometers per year that vehicle 2 is driven,

Z_1 and Z_2 are vectors of vehicle attributes (for vehicles 1 and 2 respectively),

X is a vector of household characteristics,

β's, α's, are vectors of estimable parameters, λ's are estimable scalars, and ε's are disturbance terms.

➢ To satisfy regression assumption A5, the value of the dependent variable (left-hand side variable) must not influence the value of an independent variable (right-hand side).

➢ This is not the case in these equations because in the first equation the independent variable u_2 varies as the dependent variable u_1 varies, and in the second equation, the independent variable u_1 varies as the dependent variable u_2 varies.

➢ Thus, u_2 and u_1 are said to be endogenous variables in Equations 5.1 and 5.2 respectively.

Reduced Form and the Identification Problem

➢ Reduced form solution: solving two equations and two unknowns to arrive at reduced forms. Substituting second equation into the first in the previous example:

$$u_1 = \beta_1 Z_1 + \alpha_1 X + \lambda_1 \left[\beta_2 Z_2 + \alpha_2 X + \lambda_2 u_1 + \varepsilon_2 \right] + \varepsilon_1$$

➢ rearranging,

$$u_1 = \frac{\beta_1}{1 - \lambda_1 \lambda_2} Z_1 + \frac{\alpha_1 + \lambda_1 \alpha_2}{1 - \lambda_1 \lambda_2} X + \frac{\lambda_1 \beta_2}{1 - \lambda_1 \lambda_2} Z_2 + \frac{\lambda_1 \varepsilon_2 + \varepsilon_1}{1 - \lambda_1 \lambda_2}$$

and similarly substituting first equation for u_1 in the second equation gives,

$$u_2 = \frac{\beta_2}{1-\lambda_2\lambda_1}Z_2 + \frac{\alpha_2+\lambda_2\alpha_1}{1-\lambda_2\lambda_1}X + \frac{\lambda_2\beta_1}{1-\lambda_2\lambda_1}Z_1 + \frac{\lambda_2\varepsilon_1+\varepsilon_2}{1-\lambda_2\lambda_1}$$

➢ Because the endogenous variables u_1 and u_2 are replaced by their exogenous determinants, the equations cand be estimated using ordinary least squares (OLS) as,

$$u_1 = a_1Z_1 + b_1X + c_1Z_2 + \xi_1 \text{ , and}$$

$$u_2 = a_2Z_2 + b_2X + c_2Z_1 + \xi_2 \text{ ,}$$

where,

$$a_1 = \frac{\beta_1}{1-\lambda_1\lambda_2}; \quad b_1 = \frac{\alpha_1+\lambda_1\alpha_2}{1-\lambda_1\lambda_2}; \quad c_1 = \frac{\lambda_1\beta_2}{1-\lambda_1\lambda_2}; \quad \xi_1 = \frac{\lambda_1\varepsilon_2+\varepsilon_1}{1-\lambda_1\lambda_2}$$

$$a_2 = \frac{\beta_2}{1 - \lambda_2 \lambda_1}; \quad b_2 = \frac{\alpha_2 + \lambda_2 \alpha_1}{1 - \lambda_2 \lambda_1}; \quad c_2 = \frac{\lambda_2 \beta_1}{1 - \lambda_2 \lambda_1}; \quad \xi_1 = \frac{\lambda_2 \varepsilon_1 + \varepsilon_2}{1 - \lambda_2 \lambda_1}.$$

➢ OLS estimation of these reduced form models (Equations 5.6 and 5.7) is called indirect least squares (ILS).

➢ **Problem:** While estimated reduced form models are readily used for forecasting purposes, if inferences are to be drawn from the model system, the underlying parameters need to be determined.

➢ Unfortunately, uncovering the underlying parameters, (the β's, α's, and λ's) in reduced form models is problematic because either too little or too much information is often available.

➢ For example, note that above equations provide two possible solutions for β_1,

$$\beta_1 = a_1\left(1 - \lambda_1\lambda_2\right) \text{ and } \beta_1 = \frac{c_2\left(1 - \lambda_2\lambda_1\right)}{\lambda_2}.$$

The Identification Problem

➢ In some instances, it may be impossible to determine the underlying parameters. In these cases, the modeling system is said to be **unidentified**.

➢ In cases where exactly one equation solves the underlying parameters, the model system is said to be **exactly identified**.

➢ When more than one equation solves the underlying parameters (as shown in Equation 5.10), the model system is said to be **over identified**.

Order Condition

➤ Determines an equation to be identified if the number of all variables excluded from an equation in an equation system is greater than or equal to the number of endogenous variables in the equation system minus one.

➤ For example, in the first equation in the original equation system above, the number of elements in the vector Z_2, which is an exogenous vector excluded from the equation, must be greater than or equal to one because there are two endogenous variables in the equation system (u_1 and u_2).

Simultaneous Equation Estimation

➤ Two modeling alternatives:

single-equations estimation methods and

systems estimation methods.

➢ The distinction between the two is that systems methods consider all of the parameter restrictions (caused by over identification) in the entire equation system and account for possible contemporaneous (cross-equation) correlation of disturbance terms.

➢ Because system estimation approaches are able to utilize more information (parameter restrictions and contemporaneous correlation),

➢ they produce variance-covariance matrices that are at worst equal to, and in most cases smaller than those produced by single-equation methods (resulting in lower standard errors and higher t-statistics for estimated model parameters).

Single equation methods

1) Indirect least squares (ILS)

- ➢ Applies ordinary least squares to the reduced form models.

- ➢ Consistent but not unbiased

2) Instrumental variables (IV)

- ➢ Uses an instrument (a variable that is highly correlated with the endogenous variable it replaces, but is not correlated to the disturbance term) to estimate individual equations

- ➢ Consistent but not unbiased.

3) Two-stage least squares (2SLS)

➢ Approach finds the best instrument for endogenous variables.

　➢ **Stage 1** regresses each endogenous variable on all exogenous variables.
　➢ **Stage 2** uses regression-estimated values from stage 1 as instruments, and estimates equations with ordinary least squares.

➢ Consistent but not unbiased. Generally better small sample properties than ILS or IV.

4) Limited Information Maximum Likelihood (LIML)

➢ Uses maximum likelihood to estimate reduced form models. Can incorporate parameter restrictions in over identified equations.

➢ Consistent but not unbiased. Has same asymptotic variance-covariance matrix as 2SLS.

System equation methods

1) Three Stage Least Squares (3SLS)

➢ **Stage 1** gets 2SLS estimates of the model system.

➢ **Stage 2** uses the 2SLS estimates to compute residuals to determine cross-equation correlations.

➢ **Stage 3** uses generalized least squares (GLS) to estimate model parameters.

➢ Consistent and more efficient than single-equation estimation methods.

2) Full Information Maximum Likelihood (FIML)

➢ Similar to LIML but accounts for contemporaneous correlation of disturbances in the likelihood function.

➢ Consistent and more efficient than single-equation estimation methods. Has same asymptotic variance-covariance matrix as 3SLS.

A note on generalized least squares estimation

➢ Ordinary least squares (OLS) assumptions are that disturbance terms have equal variances and are not correlated. Generalized least squares (GLS) is used to relax these OLS assumptions. Under OLS assumptions, in matrix notation,

$$E\left(\varepsilon\varepsilon^{T}\right) = \sigma^{2}\mathbf{I}$$

where:

$E(.)$ denotes expected value,

ε is an $n \times 1$ column vector of equation disturbance terms (where n is the total number of observations in the data),

ε^T is the $1 \times n$ transpose of ε,

σ^2 is the disturbance term variance, and

$\mathbf{I}$ is the $n \times n$ identity matrix,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & 1 \end{bmatrix}.$$

When heteroskedasticity is present, $E(\varepsilon\varepsilon^T) = \Omega$, where Ω is $n \times n$ matrix,

$$\Omega = \begin{bmatrix} \sigma_1^2 & 0 & . & 0 \\ 0 & \sigma_2^2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \sigma_n^2 \end{bmatrix}.$$

For disturbance-term correlation, $E\left(\varepsilon\varepsilon^T\right) = \sigma^2\Omega$, where

$$\Omega = \begin{bmatrix} 1 & \rho & . & \rho^{N-1} \\ \rho & 1 & . & \rho^{N-2} \\ . & . & . & . \\ \rho^{N-1} & \rho^{N-2} & . & 1 \end{bmatrix}$$

Recall that in ordinary least squares, parameters are estimated from,

$$\hat{\beta} = \left(X^T X\right)^{-1} X^T Y,$$

where:

$\hat{\beta}$ is an $p \times 1$ column vector (where p is the number of parameters),

X is an $n \times p$ matrix of data,

X^T is the transpose of X, and

Y is an $n \times 1$ column vector.

➢ Using Ω, Equation 5A.5 is rewritten as,

$$\hat{\beta} = \left(X^T \Omega^{-1} X\right)^{-1} X^T \Omega^{-1} Y.$$

➢ The most difficult aspect of GLS estimation is obtaining an estimate of the Ω matrix. In 3SLS, it is estimated using the initial 2SLS parameter estimates.

3SLS application (Manering 1983)

$$u_1 = \beta_1 Z_1 + \alpha_1 X + \lambda_1 u_2 + \varepsilon_1$$

$$u_2 = \beta_2 Z_2 + \alpha_2 X + \lambda_2 u_1 + \varepsilon_2$$

Data:

- Two Vehicle Households September to November 1979

- A total of 272 households

- Monthly usage in miles

Variable	Parameter Estimate	t-statistic
Constant	1205.14	10.28
Scrappage probability (= $250/vehicle value)	-1150.63	-5.82
Cost per mile divided by income (fuel price in cents/mpg/income)	-102.23	-2.21
Age of principal driver indicator (1 if principal driver age is less than 50, 0 otherwise)	123.51	1.62

Female indicator (1 if principal drive is female, 0 otherwise)	-130.93	-1.87
Urban indicator (1 if household resides in an urabn area, 0 otherwise)	-154.44	-2.89
Use of other vehicle	-0.172	-1.18

$R^2 = 0.2395$ and 0.2611

Percent change resulting from doubling of fuel prices:

- -12.9% ignoring substutution effects.
- -11.3% accounting for within-household substution