Lec23-mwf

Monday, March 31, 2008 11:59 PM

Rest of proof (upper bound) - 10min

Tuesday, February 05, 2008 3:40 PM

Three constraints

① (distance)

$$\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} r_b^h \geq \lambda n T \bar{L}$$

② (bandwidth)

$$\sum_{b=1}^{\lambda n T} h(b) \leq \frac{W T n}{2}$$

③ (Interference)

$$\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} \frac{\pi \Delta^2}{16} (r_b^h)^2 \leq W \cdot T$$

They are sufficient for deriving an upper bound on $\lambda n \bar{L}$

By Cauchy - Schwartz Inequality

$$\left(\sum_n a_n^2\right)\left[\sum_n b_n^2\right] \geq \left(\sum a_n b_n\right)^2$$

$$\geq \left[\sum^{\lambda n T} \sum^{h(b)} (r_b^h)^2\right]\left[\sum^{\lambda n T} \sum^{h(b)} 1\right]$$

$$\Rightarrow \left[\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} (r_b^h)^2 \right] \left[\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} 1 \right] \geq \left(\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} r_b^h \right)^2 \quad (4)$$

$$\Rightarrow \frac{16 W T}{\pi \Delta^2} \cdot \frac{W T n}{2} \geq (\lambda n T \bar{L})^2$$

$$\Rightarrow (\lambda n \bar{L}) \leq \sqrt{\frac{8}{\pi}} \frac{1}{\Delta} W \cdot \sqrt{n}$$

(bits-meters/sec)

Note that this is a deterministic bound regardless of the placements of the source-destination pairs.

For the case when the source & destination nodes are placed uniformly inside a unit area.

$$\bar{L} = O(1)$$

$$\Rightarrow \lambda \leq O\left(\sqrt{\frac{8}{\pi}} \frac{1}{\Delta} \frac{W}{\sqrt{n}}\right)$$

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Upper bound - handout

Tuesday, February 05, 2008 3:40 PM

Three constraints

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$$\left[\sum_n a_n^2\right]\left[\sum_n b_n^2\right] \geq \left(\sum a_n b_n\right)^2$$

$$\left[\sum^{\lambda n T} \sum^{h(b)} (r^h)^2\right]\left[\sum^{\lambda n T} \sum^{h(b)} 1\right]$$

$$\Rightarrow \left(\sum_{b=1}^{\lambda n T}\sum_{h=1}^{h(b)}(r_b^h)^2\right)\left(\sum_{b=1}^{\lambda n T}\sum_{h=1}^{h(b)} 1\right)$$

$$\geq \left(\sum_{b=1}^{\lambda n T}\sum_{h=1}^{h(b)} r_b^h\right)^2 \qquad (4)$$

$\Rightarrow$

$$\Rightarrow \quad (\lambda n \bar{L}) \leq \sqrt{\frac{8}{\pi}}\frac{1}{\Delta} W\cdot\sqrt{n}$$

(bits-meters/sec)

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Tracing back the proof - 15min

Sunday, February 10, 2008 3:29 PM

Tracing back the proof, we can find the optimal value of r_b^h, $h(b)$, etc in order to attain this bound closely.

In order to reach $\lambda = \Theta\left(\frac{1}{\sqrt{n}}\right)$

Equality in (4) holds if and only if r_b^h are all equal

$$(1) \Rightarrow \quad r \cdot \sum_{b=1}^{\lambda n T} h(b) = \lambda n T \cdot \bar{L}$$

$$(2) \Rightarrow \quad \sum_{b=1}^{\lambda n T} h(b) = \frac{W T n}{2}$$

$$\Rightarrow \quad r = \frac{2 \lambda \bar{L}}{W}$$

Since we want $\lambda n \bar{L} = \sqrt{\frac{8}{\pi}} \frac{1}{\Delta} W \sqrt{n}$

$$\Rightarrow \quad r = \sqrt{\frac{32}{\pi}} \frac{1}{\Delta} \frac{1}{\sqrt{n}} = \Theta\left(\frac{1}{\sqrt{n}}\right)$$

$$h(b) = \frac{\bar{L}}{r} = \Theta(\sqrt{n})$$

Roughly speaking

- Each hop is of distance $\Theta\left(\frac{1}{\sqrt{n}}\right)$

- Need $\Theta(\sqrt{n})$ hops to reach the destination.

Problem:

There may not be a neighbor within $\Theta(\frac{1}{\sqrt{n}})$ distance.

- The # of nodes in an area A follows a binomial distribution with $(n-1, A)$
  - expected # of nodes $= (n-1) \cdot A$

- The area of the neighbourhood should be at least $\frac{1}{n}$, so that in expectation there is at least 1 neighbor.

  Still, this is not enough because the prob. of having no neighbor is

$$(1-A)^{n-1} \simeq \left(1-\frac{1}{n}\right)^{n-1} \simeq \frac{1}{e}$$

- As there are n nodes, with high chance one of the nodes will have no neighbors in $\frac{1}{\sqrt{n}}$ radius.

— In fact, even increasing A to $A = \frac{b}{n}$ will not solve the problem.

$$(1-A)^{n-1} \approx \left(1 - \frac{b}{n}\right)^{n-1} = \frac{1}{e^b}$$

Solution:

— Increase the transmission range to $\sqrt{b \frac{\log n}{n}}$

— expected # of neighboring nodes

$$= b \cdot \log n$$

— prob. of having no neighbors in $\sqrt{b \cdot \frac{\log n}{n}}$ radius is

$$\left(1 - \frac{b \log n}{n}\right)^{n-1}$$

$$\approx \left(1 - \frac{b \log n}{n}\right)^{\frac{n}{b \log n} \cdot b \log n}$$

$$\to \left(\frac{1}{e}\right)^{b \log n} = \frac{1}{n^b}$$

— Prob. of any one node having no neighbors in $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$ radius is

$$n \cdot \left(1 - \frac{b \log n}{n}\right)^{n-1}$$

$$\approx \frac{n}{n^b}$$

With $b \geq 2$, this prob. $\to 0$ as $n \to +\infty$.

Of course the capacity will not be optimal.

— new $\log n$ factor

— attains $\lambda = \Theta\left(\sqrt{\frac{1}{n \log n}}\right)$

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— will succeed with high probability, i.e.

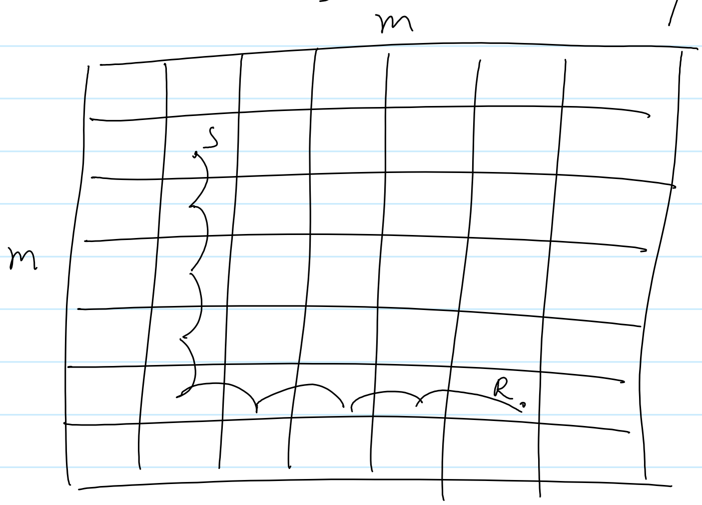
probability $\to 1$ as $n \to +\infty$

Lower (attainable) bound -10min

Tuesday, February 05, 2008 3:50 PM

Proof: (Purely centralized scheme)

Divide the unit square into a grid of $m \times m$ cells, each cell of area $\frac{1}{m^2}$

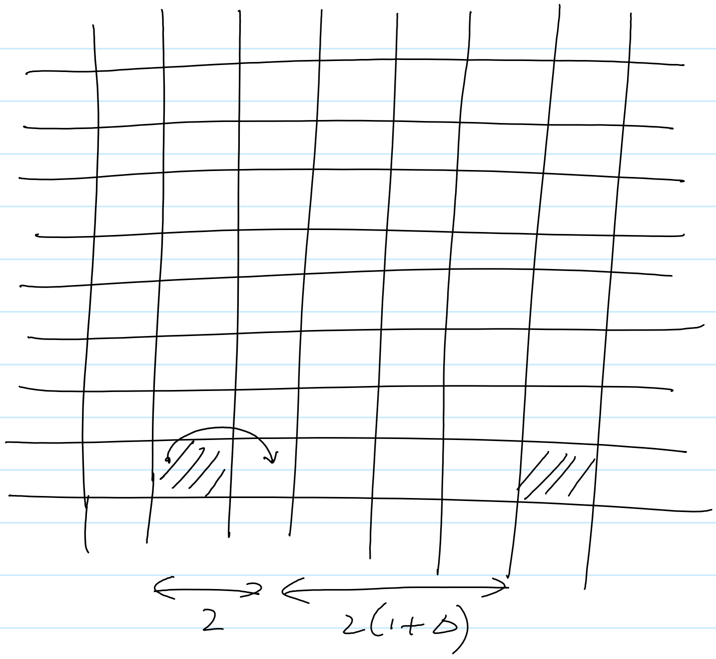


We only use transmission across neighboring cells, and packets are routed along one direction first, then along another direction.

For this to work:

1. each cell must have at least one node
2. no cells can be overly congested for relay traffic.
3. the schedules must satisfy the protocol model

Claim: Each cell can be activated at least once every $(4+2\Delta)^2$ time-slots



The protocol model is satisfied.

Hence, assuming that all cells can be activated simultaneously only lead to a constant factor of difference.

Now, let the cell size be

$$\left\lfloor \frac{2\log n}{n} \right\rfloor^{\frac{1}{2}} \times \left\lfloor \frac{2\log n}{n} \right\rfloor^{\frac{1}{2}}$$

i.e., $m = \sqrt{\frac{n}{2\log n}}$

(When n is large, the rounding factor can be ignored).

Claim: all cells have at least one node with prob $\to 0$ as $n \to +\infty$

Sketch: given a cell, the prob. that it is empty is

$$\left(1 - \frac{2\log n}{n}\right)^n \approx e^{-2\log n} = \frac{1}{n^2}$$

Any cell is empty with prob.

$$m^2 \cdot \frac{1}{n^2} \to 0 \quad \text{as } n \to +\infty$$

Hence, all cells can forward traffic with high probability.

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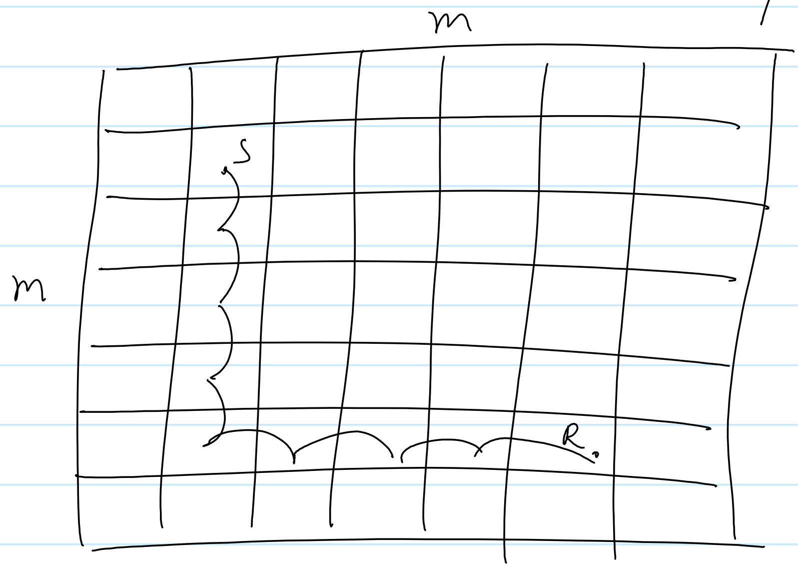
Attainable bound - handout

Tuesday, February 05, 2008 3:50 PM

Divide the unit square into a grid of $m \times m$ cells, each cell of area $\frac{1}{m^2}$



We only use transmission across neighboring cells, and packets are routed along one direction first, then along another direction.

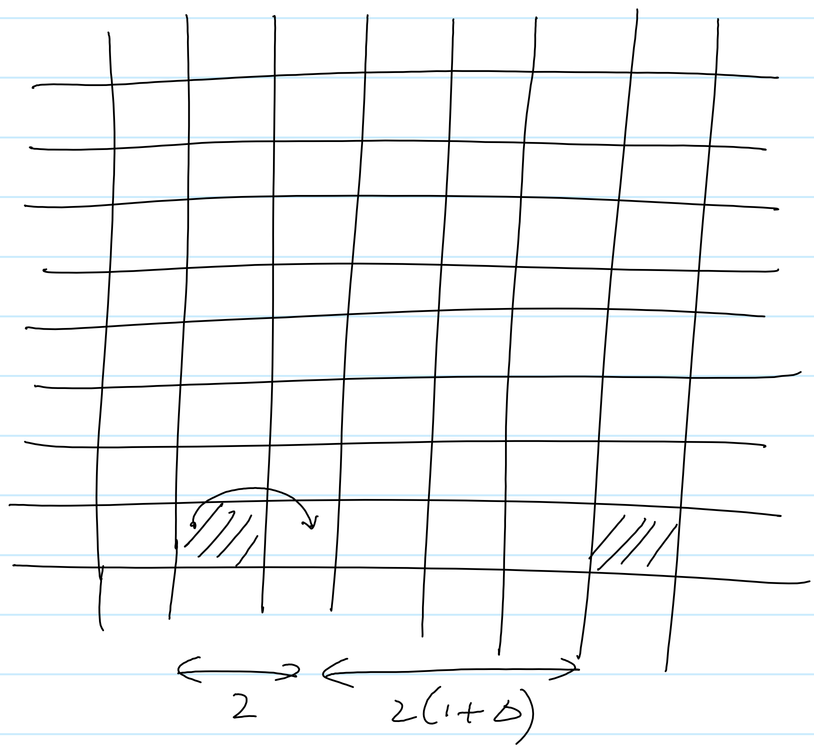
For this to work:

1.

2.

3.

2. Claim: Each cell can be activated at least once every $(4+2\Delta)^2$ time-slots



The protocol model is satisfied.

Hence, assuming that all cells can be activated simultaneously only lead to a constant factor of difference.

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① Now, let the cell size be

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i.e., $\quad m = \sqrt{\frac{n}{2\log n}}$

(When n is large, the rounding factor can be ignored).

Claim: all cells have at least one node with prob $\to 0$ as $n \to +\infty$

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$$\left(1 - \frac{2\log n}{n}\right)^n \approx e^{-2\log n} = \frac{1}{n^2}$$

Any cell is empty with prob.

$$m^2 \cdot \frac{1}{n^2} \to 0 \quad \text{as } n \to +\infty$$

Hence, all cells can forward traffic with high probability. ✓

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(3) On average, each cell have $\frac{n}{m} = \sqrt{2n\log n}$ connections passing through it in each direction.

Sketch: Fix a particular cell. The prob. that a connection goes through the cell $= \frac{1}{m}$

Let X be the # of connections going through the cell.

$$E[X] = \frac{n}{m} = \sqrt{2n\log n}$$

However, some cell may have to support more that $E(X)$ connections

Claim: With high probability, all cell has at most $\frac{2n}{m} = \sqrt{8n\log n}$

cell has at most $\frac{2n}{m} = \sqrt{8n\lg n}$ connections passing through it in both directions.

Routing - 10min

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We have shown:
- each cell is of size $O\left(\sqrt{\frac{2\log n}{n}}\right) \times O\left(\sqrt{\frac{2\log n}{n}}\right)$
- each cell can be activated once every $(4+2\Delta)^2$ slots
- with high prob, all cells have at least one node

Next, route packets first along the y-axis, then along the x-axis.

On average, each cell have $\frac{n}{m} = \sqrt{2n\log n}$ connections passing through it in each direction.

Sketch: Fix a particular cell.
The prob. that a connection goes through the cell
$= \frac{1}{m}$

Let X be the # of connections going through the cell.

$$E[X] = \frac{n}{m} = \sqrt{2n\log n}$$

However, some cell may have to support more that $E(X)$ connections

Claim: With high probability, all cell has at most $\frac{2n}{m} = \sqrt{8n\log n}$ connections passing through it in both directions.

We next show that $P\{X \geq 2\frac{n}{m}\}$ will be very small.

In particular, note that

$$E[e^{sX}] = \left(E\left[e^{s \cdot \mathbb{1}\{\text{A connection goes through the cell}\}}\right]\right)^n$$

$$= \left(1 - \frac{1}{m} + e^{s} \cdot \frac{1}{m}\right)^n$$

$$\leq \exp\left(\frac{n}{m}(e^s - 1)\right)$$

Hence, the prob. that $X \geq 2\frac{n}{m}$ is

$$\Pr\{X \geq \frac{2n}{m}\} \leq \frac{E[e^{sX}]}{e^{s \cdot \frac{2n}{m}}}$$

$$\leq \exp\left(\frac{n}{m}(e^s - 1 - 2s)\right)$$

Let $s = \log 2$

$$\Rightarrow e^s - 1 - 2s = 1 - 2\log 2 = -0.386 \leq -\frac{1}{4}$$

Since $\frac{n}{m} = \sqrt{2n\log n} \geq 8\log n$ for large n

$\Rightarrow \quad P\{\bar{X} \geq \frac{2n}{m}\} \leq \exp\{-2\log n\} = \frac{1}{n^2}$

$\Rightarrow \quad P\{$ any cell has more than $\frac{2n}{m}$ connections along the y-axis$\}$

$\leq \frac{1}{n^2} \cdot m^2 \to 0 \quad$ as $n \to +\infty$

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Rest of proof (lower bound) - 5min

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Since each cell allows W bits/sec passing through it, hence the end-to-end capacity for each connection is at least

$$\frac{1}{(4+2\Delta)^2} \cdot \frac{W}{2 \cdot \frac{2n}{m}} = \frac{W}{(4+2\Delta)^2 \cdot 4 \cdot \sqrt{2n\lg n}}$$

$$= \Theta\left(\frac{W}{\sqrt{n\lg n}}\right)$$

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Critique

Sunday, February 10, 2008 4:08 PM

1. multiple channels

2. Mobility

3. delay

4. Other interference models
   - physical model
   - fading

5. Multicast.