Lec19

Thursday, February 27, 2020 4:06 PM

Stochastic geometry - 10min

Friday, February 9, 2018 10:49 AM

- Focus on downlink. (Uplink can be treated in an analogous manner.)

- Consider a fixed mobile at the origin

- Key assumption #1: Base-stations (BS) are distributed according to a homogeneous Poisson Point Process (PPP) of intensity λ.

In optimal scheduling, we assume that node locations are given. ⇒ difficult to obtain closed form solutions

  - Roughly speaking, in any small area ΔA, the probability of having one BS in the area is $\lambda \Delta A$.

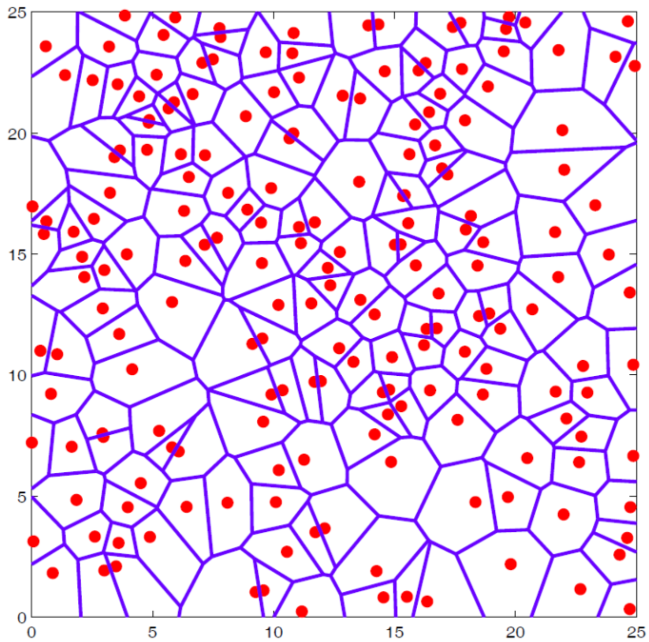
  - When ΔA is large, the # of BS is a Poisson random variable with mean $\lambda \Delta A$

$$P\{\#\text{ of BS} = k\} = e^{-\lambda \Delta A} \frac{(\lambda \Delta A)^k}{k}$$

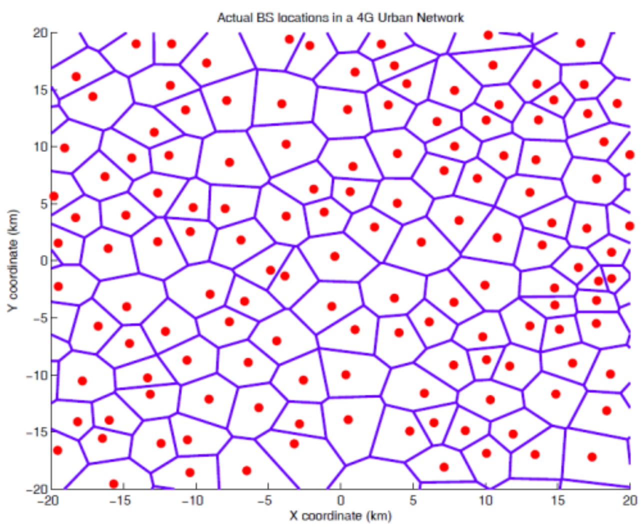
  - Independence:
    - For two disjoint areas, the # of BS in each area is independent of that of others.
    - Given that the # of BS in an area A is n, each BS is uniformly distributed in A, independently of others.

  - The independence assumption does not hold when BSs are arranged according to some patterns
    - may be more relevant for small cells in HetNets.

  - show Fig. 1 & Fig. 2 in Andrews et al.



PPP



Actual BS in 4G networks

Such a PPP assumption allows us to derive key quantities in closed form

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- The mobile is connected to the nearest BS at a distance r.
  b_0
- How far is r? For any value a

$$P[r > a] = P\{\# \text{ of } BS = 0 \text{ in area } \pi a^2\}$$

$$= e^{-\lambda \pi a^2}$$

$\Rightarrow$ The pdf of r is

$$\ldots e^{-\lambda \pi r^2}\Big| = 2\pi\lambda r \cdot e^{-\lambda \pi r^2}$$

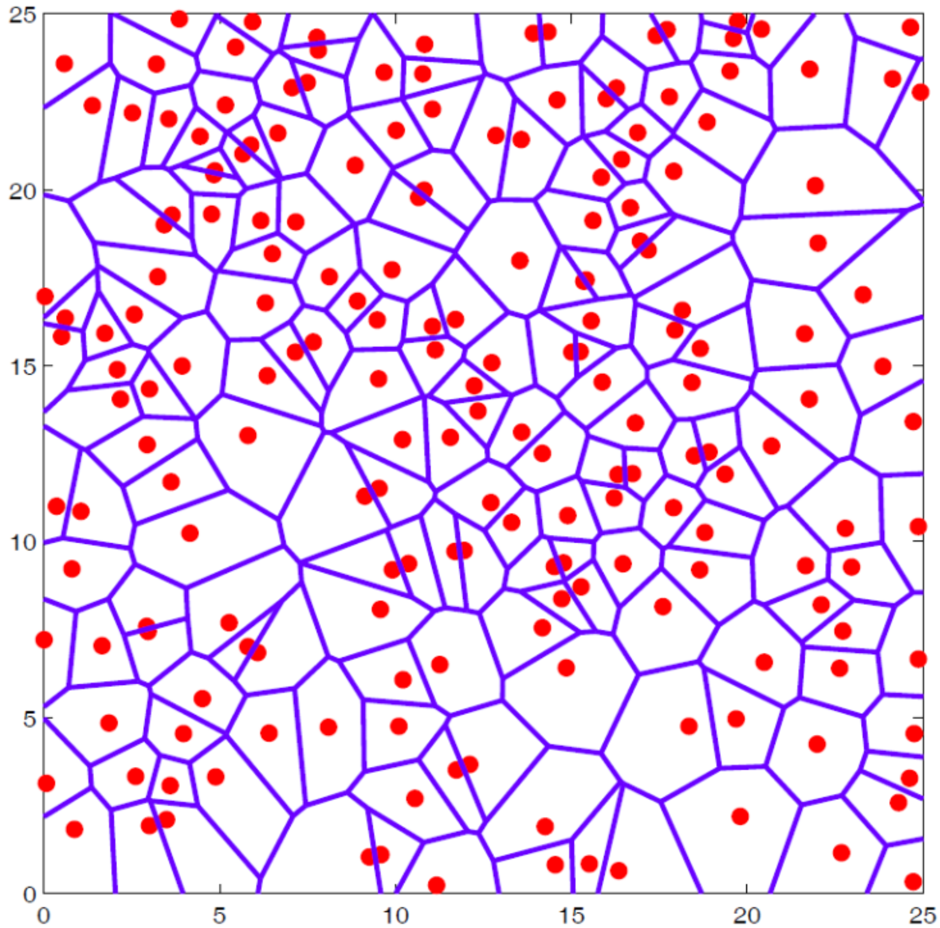
$\Rightarrow$ The pdf of r is

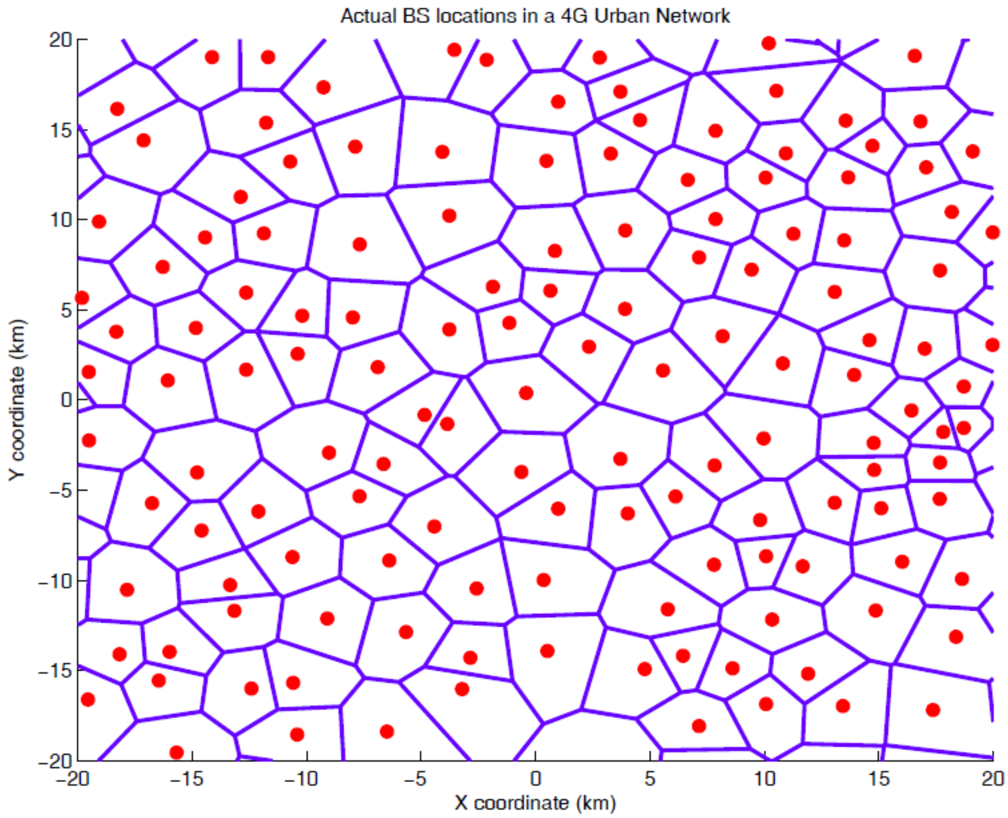
$$f(r) = \frac{d}{dr}\left[1 - e^{-\lambda\pi r^2}\right] = 2\pi\lambda r \cdot e^{-\lambda\pi r^2}$$

– We then then derive the distribution of received power

Tessellation

Tuesday, February 13, 2018 4:42 PM




Actual BS locations in a 4G Urban Network
Y coordinate (km)
X coordinate (km)
20
15
10
5
0
−5
−10
−15
−20
−20
−15
−10
−5
0
5
10
15
20

Key property - 10min

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- On the other hand, interference power is contributed as a sum over all over BSs. We need the following key property.

Key property of PPP:

- For a given area ΔA, conditioned on the # of points of the PPP inside ΔA is k, the k points are uniformly distributed in ΔA, independently of each other.
- Suppose ~~that~~ is a "mark" function $f(x)$ for each location x.
- Let $x_1, x_2 \ldots$ be the PPP realization
- Consider $E\left[\prod_{i=1}^{+\infty} e^{-f(x_i)}\right] = E\left[e^{-\sum_{i=1}^{+\infty} f(x_i)}\right]$
    - Laplace functional of PPP
    - Think of $f(x_i)$ as the interference power of BS at x_i.

Then

$$E\left[\prod_{i=1}^{+\infty} e^{-f(x_i)}\right]$$

$$= \exp\left\{-\lambda \iint_{R^2} \left(1-e^{-f(x)}\right) dx\right\}$$

- Alternatively, if $f(x_i) > 0$,

$$E\left[\prod_{i=1}^{+\infty} f(x_i)\right] = \exp\left\{-\lambda \iint_{R^2} \left(1-f(x)\right) dx\right\}$$

- We will show a stronger version over an area A

$$E\Big[\prod_{\substack{i=1\\ x_i\in A}} e^{-f(x_i)}\Big]$$
$$= \exp\Big\{-\lambda \iint_A \left(1-e^{-f(x)}\right)dx\Big\}$$

- To see this, for an area A, suppose the # of points in A is n.
  - Let $g(x) = f(x)\cdot 1_{\{x\in A\}}$

$$\Rightarrow E\Big[\prod_{i=1}^{+\infty} e^{-g(x_i)} \Big/ n\Big]$$
$$= E\Big[\prod_{i=1}^{n} e^{-g(x_i)}\Big]$$
$$= \prod_{i=1}^{n} \iint_A e^{-g(x)}\frac{1}{A}dx = \Big(\iint_A e^{-g(x)}\frac{1}{A}dx\Big)^n$$
$$\Rightarrow E\Big[\prod_{i=1}^{+\infty} e^{-g(x_i)}\Big]$$
$$= e^{-\lambda A}\sum_{n=0}^{+\infty}\frac{(\lambda A)^n}{n!}\cdot\Big(\iint_A e^{-g(x)}\frac{1}{A}dx\Big)^n$$
$$= e^{-\lambda A}\cdot e^{\lambda A\cdot \iint_A e^{-g(x)}\frac{1}{A}dx}$$
$$= e^{-\lambda \iint_A \left(1-e^{-g(x)}\right)dx}$$

- Taking $A\to \mathbb{R}^2$ gives the result.

Key property of PPP - handout

Friday, February 9, 2018 10:49 AM

$$E\left[\prod_{i=1}^{+\infty} e^{-f(x_i)}\right]$$

$$= \exp\left\{-\lambda \iint_{R^2} (1-e^{-f(x)})\, dx\right\}$$

- To see this, for an area A, suppose the # of points in A is n.
  - Then, one can show that, conditioned on n, each point x_i is uniform in A, independently of other points.
  - Let $g(x) = f(x) \cdot 1_{\{x \in A\}}$

$$\Rightarrow E\left[\prod_{i=1}^{+\infty} e^{-g(x_i)} \,\middle|\, n\right]$$

$$\Rightarrow E\left[\prod_{i=1}^{+\infty} e^{-g(x_i)}\right]$$

$$=$$

$$= e^{-\lambda A} \cdot e^{\lambda A \cdot \iint_A e^{-g(x)} \frac{1}{A} dx}$$

$$= e^{-\lambda \iint_A (1-e^{-g(x)})\, dx}$$

- Taking $A \to R^2$ gives the result.

Alternatively, if $f(x_i) > 0$,

$$E\left[\prod_{i=1}^{\tau} f(X_i)\right] = \exp\left\{-\lambda \iint_{R^2} (1 - f(x))\, dx\right\}$$

Total Interference Power - 15min

Friday, February 9, 2018 10:49 AM

- Assme unit transmission power at all BS
- The channel between the mobile and the nearest BS has two components
- The path-loss component is r^{-n}
- Key assumption #2: The multi-path fading component is Raleigh, i.e., α is distributed as

$$f(\alpha) = \frac{\alpha}{A^2} e^{-\frac{\alpha^2}{2A^2}}$$

  - This implies that α^2 is exponential with mean $\frac{1}{2A^2}$
- Hence, the overall received power at the mobile is

$$h \cdot r^{-n}$$

  - where h is exponential with mean $\frac{1}{\mu}$.

$$P[h \geq a] = e^{-\mu a}$$

- The SINR is

$$SINR = \frac{h r^{-n}}{I_r + \sigma^2}$$

  - where I_r is the total interference

$$I_r = \sum_{i \neq b_0} \underbrace{g_i}_{\text{fading}} \cdot \underbrace{R_i^{-n}}_{\text{path loss}} \cdot \underbrace{1}_{\text{unit power}}$$

- Given that h, g_i, r & R_i are all random, it seems very difficult to obtain a simple form for the SINR in the average sense.

- Below, we will see the Raleigh & PPP assumption simplifies the calculation.

- Our first goal is to derive a simple expression for the coverage probability, i.e., the probability that the SINR is above a threshold T.

- In particular, we will show that when $\sigma^2 = 0$

$$P(SINR > T) = \frac{1}{1 + \rho(T, \alpha)}$$

$\uparrow$ a number independent of λ & μ.

Coverage - 15min

Friday, February 9, 2018 11:10 AM

- Let us first study the coverage probability, i.e. the S2NR at the mobile is greater than a threshold T.

- First, condition on r (the distance to the serving BS).

$$
\begin{aligned}
& P[\,S2NR > T \mid r\,] \\
&= P\left[\frac{h r^{-n}}{I_r + \sigma^2} > T \,\Big|\, r\right] \\
&= E\left(P\left[\frac{h r^{-n}}{I_r + \sigma^2} > T \,\Big|\, I_r, r\right]\right) \\
&= E\left[P[\,h > T r^n (I_r + \sigma^2) \mid I_r, r\,]\right] \\
&= E\left[e^{-\mu T r^n (I_r + \sigma^2)} \,\Big|\, r\right] \\
&= e^{-\mu T r^n \cdot \sigma^2} \cdot \underbrace{E\left[e^{-\mu T r^n \cdot I_r} \,\Big|\, r\right]}_{\text{the m.g.f of } I_r}
\end{aligned}
$$

- Define $L_X(s) = E[e^{-sX}]$
- This term is $L_{I_r}(\mu T r^n)$

- In other words, if we know the Laplace transform of I_r, we can express the coverage probability in closed-form.

<u>Simplifying</u> the Laplace transform

- $I_r = \sum_{i \neq b_0} g_i R_i^{-n}$

[illegible] the location of the other BSs : Φ

— Condition on the location of the other BSs : Φ

$$E[e^{-sI_r} \mid \Phi]$$

$$= E[e^{-s \sum_{i \neq b_0} g_i R_i^{-n}} \mid \Phi]$$

$$= \prod_{i \neq b_0} E[e^{-s R_i^{-n} \cdot g_i} \mid \Phi]$$

— And then we need to take the integral over R_i

— When g_i is also exponential with mean $\frac{1}{\mu}$, the above expression can be greatly simplified.

$$E[e^{-sg}] = \int_0^{+\infty} e^{-s} \cdot e^{-\mu a} \mu \, da$$

$$= \frac{\mu}{s+\mu}.$$

$$\Rightarrow E[e^{-sI_r} \mid \Phi] = \prod_{i \neq b_0} \frac{\mu}{\mu + sR_i^{-n}}$$

Integrating over R_i's

— Each R_i is independent and is only outside r.

— $$L_{I_r}(s) = E\left[\prod_{i \neq b_0} \frac{\mu}{\mu + sR_i^{-n}}\right]$$

$$= \exp\left\{-\lambda \cdot \int_r^{+\infty} \left(1 - \frac{\mu}{\mu + sR^{-n}}\right) 2\pi R \, dR\right\}$$

$$\Rightarrow L_{I_r}(\mu T r^n) = \exp\left\{-\lambda \int_r^{+\infty} \left(1 - \frac{\mu}{\mu + \mu T \left(\frac{r}{R}\right)^n}\right) \pi \, dR^2\right\}$$

$$\Rightarrow L_{I_r}(\mu T r^n) = \exp\left\{ -\lambda \int_r \cdots \frac{\mu}{\mu + \mu T\left(\frac{r}{R}\right)^n} \cdots \right\}$$

$$= \exp\left\{ -\lambda \int_r^{+\infty} \frac{1}{\frac{1}{T}\left(\frac{R}{r}\right)^n + 1} \, \pi \, dR^2 \right\}$$

– The integral can be written as

$$\int_r^{+\infty} \frac{1}{\frac{1}{T}\left(\frac{R}{r}\right)^n + 1} \, d\left(\frac{R}{r}\right)^2 \cdot r^2$$

$$= \underbrace{\int_1^{+\infty} \frac{1}{\frac{1}{T}u^n + 1} \cdot du^2}_{\rho(T,n) \text{ a number}} \cdot r^2$$

Then $L_{I_r}(\mu T r^n) = e^{-\pi \lambda r^2 \rho(T,n)}$

– $P[\mathrm{SINR} > T)$

$$= \int_0^{+\infty} 2\pi\lambda r e^{-\lambda\pi r^2} dr \cdot e^{-\mu T r^n \cdot \sigma^2} \cdot e^{-\pi\lambda r^2 \rho(T,n)}$$

$\sigma^2 = 0$: noise is negligible compared to signals.

$P(\mathrm{SINR} > T)$

$$= \int_0^{+\infty} \pi\lambda \, e^{-\pi\lambda r^2 (1+\rho(T,n))} dr^2$$

$$= \frac{1}{1+\rho(T,n)}$$

Coverage - handout

Friday, February 9, 2018 11:10 AM

- Let us first study the coverage probability, i.e. the SINR at the mobile is greater than a threshold T.

- First, condition on r (the distance to the serving BS).

$$P[\,SINR > T \mid r\,]$$

$$= P\left[\frac{h r^{-n}}{I_r + \sigma^2} > T \,\Big|\, r\right]$$

$$= E\left(P\left[\frac{h r^{-n}}{I_r + \sigma^2} > T \,\Big|\, I_r, r\right]\right)$$

$$= E\left[\qquad\qquad \Big|\, I_r, r\right]$$

$$= E\left[\qquad\qquad \Big|\, r\right]$$

$$= e^{-\mu T r^n \cdot \sigma^2} \cdot \underbrace{E\left[e^{-\mu T r^n \cdot I_r} \,\Big|\, r\right]}_{\text{the m.g.f of } I_r}$$

- Define $L_X(s) = E[e^{-sX}]$
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- In other words, if we know the Laplace transform of I_r, we can express the coverage probability in closed-form.

<u>Simplifying</u> the Laplace transform

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- Condition on the location of the other BSs : Φ

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$= E[e^{-s\sum_{i\neq b_0} g_i R_i^{-n}} \mid \Phi]$

$=$

— And then we need to take the integral over R_i

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Integrating over R_i's

— Each R_i is independent and is only outside r.

— $L_{I_r}(s) = E\left[\prod_{i\neq b_0} \frac{\mu}{\mu + sR_i^{-n}}\right]$

$\Rightarrow L_{I_r}(\mu T r^n) = \exp\left\{-\lambda \int_r^{+\infty} \left(1 - \frac{\mu}{\mu + \mu T\left(\frac{r}{R}\right)^n}\right) \pi dR^2\right\}$

$$\Rightarrow L_{I_r}(\mu T r^n) = \exp\left\{ -\lambda \int_r \left(1 - \frac{\mu}{\mu + \mu T \left(\frac{r}{R}\right)^n}\right) \right\}$$

$$= \exp\left\{ -\lambda \int_r^{+\infty} \frac{\mu T \left(\frac{r}{R}\right)^n}{\mu + \mu T \left(\frac{r}{R}\right)^n} \pi \, dR^2 \right\}$$

$$= \exp\left\{ -\lambda \int_r^{+\infty} \frac{1}{\frac{1}{T}\left(\frac{R}{r}\right)^n + 1} \pi \, dR^2 \right\}$$

— The integral can be written as

$$\int_r^{+\infty} \frac{1}{\frac{1}{T}\left(\frac{R}{r}\right)^n + 1} \, d\left(\frac{R}{r}\right)^2 \cdot r^2$$

$$= \underbrace{\qquad\qquad\qquad\qquad}_{\rho(T,n) \text{ a number}}$$

Then $L_{I_r}(\mu T r^n) = e^{-\pi \lambda r^2 \rho(T,n)}$

— $P[\text{SINR} > T] =$

$$= \int_0^{+\infty} 2\pi\lambda r e^{-\lambda \pi r^2} dr.$$

$$e^{-\mu T r^n \cdot \sigma^2} \cdot e^{-\pi \lambda r^2 \rho(T,n)}$$

$\sigma^2 = 0$: noise is negligible compared to signals.

$P(\text{SINR} > T)$

$$= \int_0^{+\infty} \pi\lambda e^{-\pi\lambda r^2 (1 + \rho(T,n))} dr^2$$

$$= \frac{1}{1 + \rho(T,n)}$$

$$= \frac{1}{1+\rho(T, \lambda)}$$

— When fading is Raleigh and $\sigma^2 = 0$, the coverage probability is independent of λ

Implication

Sunday, February 11, 2018 11:39 AM

- When fading is Raleigh and $\sigma^2 = 0$, the coverage probability is independent of λ
- In other words, the network can be arbitrarily dense or sparse, but the coverage probability remains the same.
- Even if $\sigma^2 \neq 0$, likely hold when $\lambda \to +\infty$.