Lec15-mwf

Saturday, February 3, 2018 10:53 AM

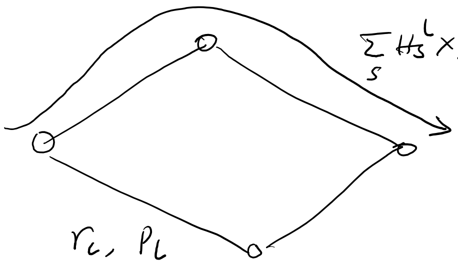
Optimal capacity - handout

Wednesday, February 19, 2020 8:55 AM

$H_s^l = 1$ if flow s uses link l

$\sum_s H_s^l x_s$: offered load to link l

$E[A_s(t)] = X_s$



$$\vec{r}(t) = g(\vec{p}(t), K(t))$$

g: rate-power function

$\vec{p}(t)$: global action vector

$K(t)$: global channel state

$\vec{p}(t) \in \textcircled{H}$

$P[K(t) = k] = \lambda_k$

Capacity Region

$$\left[\sum_s H_s^l X_s\right] \in \underbrace{\sum_k \lambda_k \,\text{Conv\_hull}\left\{ g(\vec{p}, k) \mid \vec{p} \in \textcircled{H} \right\}}_{\Lambda:\ \text{capacity region}}$$

Throughput-Optimal Scheduling

— $q^l(t+1) = [\, q^l(t) + \sum_s H_s^l A_s(t) - r^l(t) \,]^+$ $\qquad E[A_s(t)] = X_s$

$A_s(t)$: random arrivals

$[\cdot]^+$: projection to $[0, +\infty)$

— $\vec{p}(t) = \arg\max\limits_{\vec{p} \in \textcircled{H}} \sum q^l r^l$

$-\quad \vec{p}(t) = \underset{\vec{p} \in \textcircled{H}}{\arg\max} \sum \hat{\gamma}^l r^l$

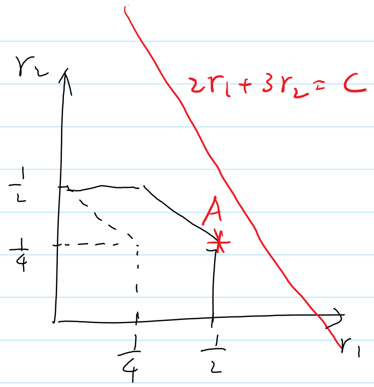
$\vec{r} = f(\vec{p}, K(t))$

gives larger weight to r^l if $\hat{\gamma}^l$ is large.

Maximizing a weighted-sum

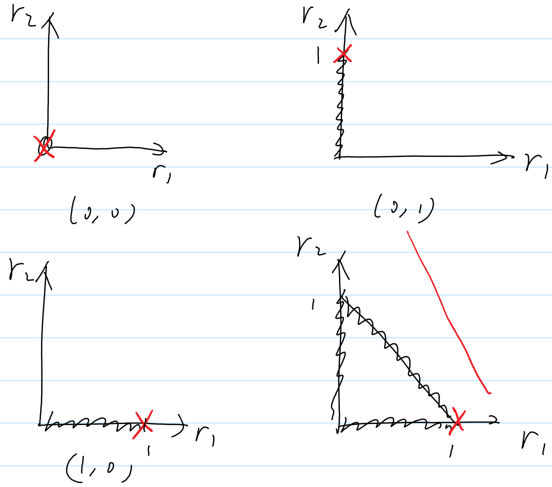
- Over this capacity region, suppose we want to maximize a weighted-sum

$$2r_1 + 3r_2$$



Three important facts

① The point A can be seen as the average of the maximizing rate vector at each channel state



$(0,0)\times\frac{1}{4} + (0,1)\times\frac{1}{4}$

$+ (1,0)\times\frac{1}{4} + (1,0)\times\frac{1}{4} = \left(\frac{1}{2}, \frac{1}{4}\right)$

– In general,

$$\max \sum_l w_l r_l$$
$$\text{sub to } [r_l] \in \sum_k \lambda_k \, \text{Conv-hull}\{g(\vec{p},k) \mid \vec{p} \in \Phi\}$$

$$= \sum_k \lambda_k \cdot \left\{ \begin{array}{l} \max \sum_l w_l r_l \\ \text{sub to } [r_l] \in \text{Conv-hull}\{g(\vec{p},k) \mid \vec{p} \in \Phi\} \end{array} \right.$$

② The maximizer of a weighted-sum over a convex hull always occurs at the extreme points

$$\max \sum_l w_l r_l$$
$$\text{sub to } [r_l] \in \text{Conv-hull}\{g(\vec{p},k) \mid \vec{p} \in \Phi\}$$

$$= \max \sum_l w_l r_l$$
$$\text{sub to } [r_l] \in \{g(\vec{p},k) \mid \vec{p} \in \Phi\}$$

③ For any $[r_l^*] \in \Lambda$

$$\sum_l w_l r_l^* \leq \begin{array}{l} \max \sum_l w_l r_l \\ \text{sub to } [r_l] \in \Lambda \end{array}$$

Throughput-optimal scheduling - 10min

Sunday, January 27, 2008 11:29 AM

Consider the following dynamics.

Let q^l denote the queue length of link l.

Consider a slotted system.

- w.l.o.g. assume the length of a slot is 1.
- the channel state is fixed within a time slot. $K(t)$
- our scheme also uses a fixed action within a time slot $\vec{p}(t)$
- the capacity of each link is also fixed within a time slot

$$\vec{r}(t) = g(\vec{p}(t), K(t))$$

The queue-length then evolves as

$$q^l(t+1) = \left[q^l(t) + \sum_s H_s^l A_s(t) - r^l(t) \right]^+ \qquad E[A_s(t)] = X_s$$

random arrivals

projection to $[0, +\infty)$

<u>Key intuition:</u>

We need to make sure queues do not explode to infinity.

- To drain queues faster, we would like to choose $\vec{p}(t)$ such that $\underline{\underline{\vec{r}(t)}}$ is large

a vector

If the current rate of link l cannot support the offered load,

$\Rightarrow$ q^l will increase

$\Rightarrow$ should increase r^l, possibly at the cost of other links

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Consider the following policy:

Pick $\vec{p}(t)$ at each time such that

$$\vec{p}(t) = \underset{\substack{\vec{p} \in \textcircled{H} \\ \vec{r} = f(\vec{p}, K(t))}}{\arg\max} \sum q^l r^l$$

gives larger weight to r^l if q^l is large.

We will show that this scheduling policy will be able to stabilize all queues for any offered load $\vec{x} \in \Lambda$.

Note: The policy does not require knowledge of $\vec{x}$ or π

It only needs the current channel state $K(t)$

- queue-length based
- adaptive
- online solution.

Why some other policies will not work?

(1) choose the schedule that maximizes

$$\sum_l r_l$$

or $\sum_l w_l r_l$, where w_l is fixed.

(2) choose the schedule that maximizes

$$\sum_l r_l \mathbb{1}_{\{p_l > 0\}}$$

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Lyapunov stability - 5min

Sunday, January 27, 2008 11:38 AM

Main theoretical approach:

- Lyapunov stability
- Find a Lyapunov function $V(\vec{q})$ such that

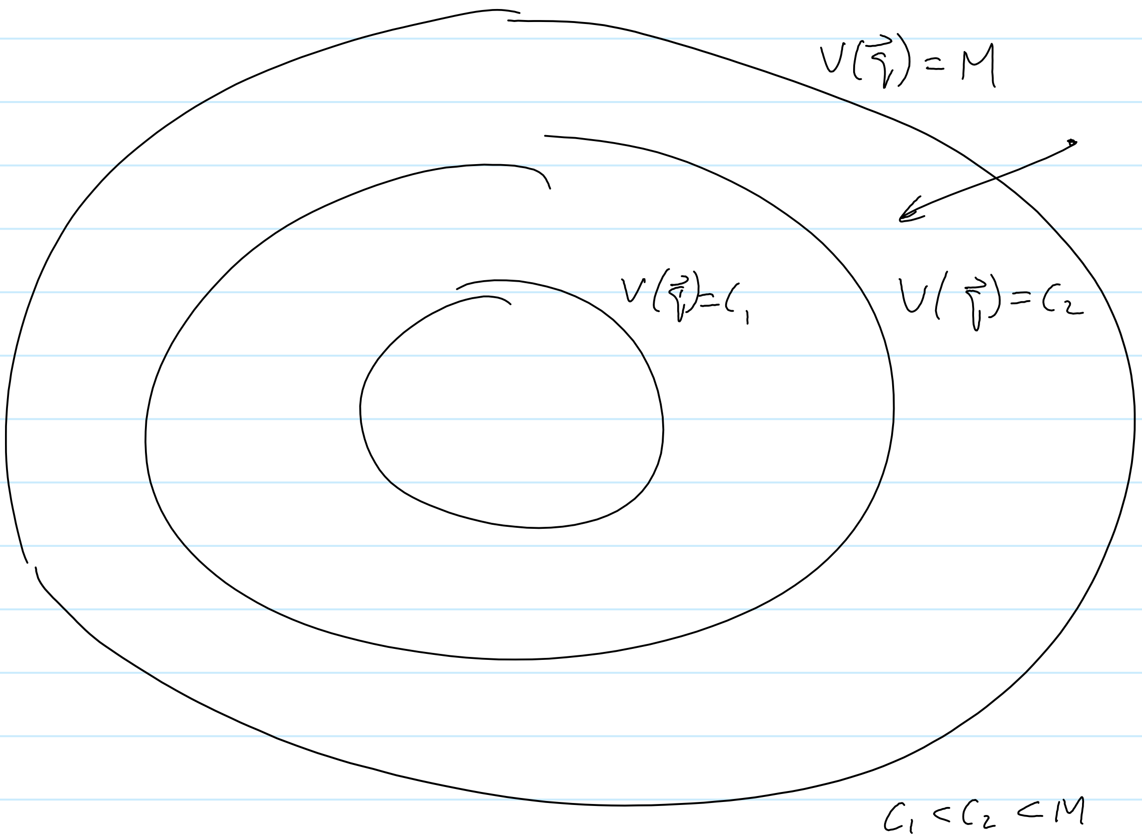
① $V(\vec{q}) \geq 0$ for all $\vec{q}$

$V(\vec{q}) \to +\infty$ as $\|\vec{q}\| \to +\infty$

② $$E\left[V(\vec{q}(t+1)) - V(\vec{q}(t)) \,\middle|\, \vec{q}(t) \right]$$

$$\leq -\varepsilon \|\vec{q}(t)\|$$

whenever $\|\vec{q}(t)\| \geq M$ for some constants ε & M



A negative drift for $V(\vec{q})$ whenever $V(\vec{q})$ is large.

$\Rightarrow$ $\vec{q}(t)$ can not explode to infinity

Note: the negative-drift condition can be relaxed to

$$E\left[V(\vec{q}(t+1)) - V(\vec{q}(t)) \,\middle|\, \vec{q}(t) \right]$$

$$\leq -\varepsilon \|\vec{q}(t)\| + M.$$

any constant
does not matter.

$\Rightarrow$

$$\leq -\frac{\varepsilon}{2}\|\vec{q}(t)\|$$

when $M \leq \frac{\varepsilon}{2}\|\vec{q}(t)\|.$

- When we derive the drift, we can ignore all terms that are bounded, and focus on those terms that grow with $\vec{q}$

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Proof of throughput-optimality - 10min

Tuesday, January 29, 2008 3:14 PM

We will show that the following function can serve as the Lyapunov function

$$V(\vec{q}) = \frac{1}{2} \sum_l (q^l)^2$$

Proof: Since

$$q^l(t+1) = \left[q^l(t) + \sum_s H_s^l A_s(t) - r^l(t) \right]^+$$

$$\Rightarrow \left(q^l(t+1)\right)^2 \le \left(q^l(t) + \sum_s H_s^l A_s(t) - r^l(t) \right)^2$$

$$= \left(q^l(t)\right)^2 + 2 q^l(t) \left[\sum_s H_s^l A_s(t) - r^l(t) \right]$$

$$+ \left[\sum_s H_s^l A_s(t) - r^l(t) \right]^2$$

If $A_s(t)$ is finite, and $r^l(t)$ is bounded, there exists a constant M_l such that

$$\left(q^l(t+1)\right)^2 \le \left(q^l(t)\right)^2 + 2 q^l(t) \left[\sum_s H_s^l A_s(t) + r^l(t) \right] + M_l$$

$$\Rightarrow V(\vec{q}(t+1)) \le V(\vec{q}(t)) + \sum_l q^l(t) \left[\sum_s H_s^l A_s(t) - r^l(t) \right] + \frac{1}{2} \sum M_l^2 .$$

$$\Rightarrow E\left(V(\vec{q}(t+1)) - V(\vec{q}(t)) \,\middle|\, \vec{q}(t) \right)$$

$$\leq \sum_l q^l(t) \sum_j H_j^l x_j - \sum_l q^l(t) E\left[r^l(t) \middle| q^l(t)\right]$$

$$+ \frac{1}{2} \sum M_l^2 .$$

Intuition:
The max-weight policy is chosen the minimize the negative drift!

For any $\vec{x}$ that lies strictly inside Ω

$$\Omega = \sum_K \lambda_K \text{ Conv-hull} \{ g(\vec{P}, K) \mid \vec{P} \in \Theta \}$$

$\Rightarrow$ $(1+\varepsilon)\vec{x} \in \Omega$ for some $\varepsilon > 0$

$\Rightarrow$ There exists α_K^m, $\overrightarrow{P_K^m} \in \Theta$, $\sum_m \alpha_K^m = 1$

such that

$$(1+\varepsilon) \sum_j H_j^l x_j \leq \sum_K \lambda_K \sum_m \alpha_K^m g_l(\overrightarrow{P_K^m}, k)$$

for all l.

Since $\sum_l q^l(t) r^l(t) = \max_{\vec{r} = g(\vec{P}, K(t))} \sum_l q^l(t) r^l$

$$\Rightarrow \sum q^l(t) g_l(\overrightarrow{P_K^m}, K) \mathbb{1}_{\{K(t)=K\}}$$

$$\leq \sum q^l(t) r^l(t) \mathbb{1}_{\{K(t)=K\}}$$

for all k, m

$$\Rightarrow \sum q^l(t)\, \lambda_k \sum_m \alpha_k^m\, g(\vec{p}_k^m, k)$$

$$\leq \sum q^l(t)\; E\left[r^l(t)\, 1_{\{K(t)=k\}} \,\middle|\, \vec{q}(t)\right]$$

$$\Rightarrow \sum_l q^l(t) \sum_k \lambda_k \sum_m \alpha_k^m\, g_l(\vec{p}_k^m, k)$$

$$\leq \sum_l q^l(t) \cdot E\left[r^l(t) \,\middle|\, \vec{q}(t)\right]$$

$$\Rightarrow (1+\varepsilon) \sum_l q^l(t) \sum_s H_s^l x_s \leq \sum_l q^l(t)\, E\left[r^l(t) \,\middle|\, q^l(t)\right]$$

Hence, we have

$$E\left[V(\vec{q}(t+1) - \vec{q}(t)) \,\middle|\, \vec{q}(t)\right]$$

$$\leq -\varepsilon \sum_l q^l(t) \sum_s H_s^l x_s + \frac{1}{2}\sum_l M_l^2$$

$$\leq -\varepsilon \cdot \left[\min_l \sum_s H_s^l x_s\right]\left(\sum_l q^l(t)\right) + \frac{1}{2}\sum_l M_l^2$$

$$\leq -\frac{\varepsilon}{2}\left[\min_l \sum_s H_s^l x_s\right]\left(\sum_l q^l(t)\right)$$

$$\text{if} \quad \sum_l q^l(t) \geq \frac{\frac{1}{2}\sum_l M_l^2}{\frac{\varepsilon}{2}\left[\min_l \sum_s H_s^l x_s\right]} \triangleq M$$

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$$\Rightarrow \left(q^l(t+1)\right)^2 \le \left(q^l(t) + \sum_s H_s^l A_s(t) - r^l(t)\right)^2$$

If $A_s(t)$ is finite, and $r^l(t)$ is bounded, there exists a constant M_l such that

$$\left(q^l(t+1)\right)^2 \le \left(q^l(t)\right)^2 + 2q^l(t)\left[\sum_s H_s^l A_s(t) - r^l(t)\right] + M_l$$

$$\Rightarrow V(\vec{q}(t+1)) \le$$

$$\Rightarrow E\left[V(\vec{q}(t+1)) - V(\vec{q}(t)) \,\middle|\, \vec{q}(t) \right]$$

$$\leq$$

Intuition:

The max-weight policy is chosen the minimize the negative drift!

For any $\vec{x}$ that lies strictly inside Ω

$$\Omega = \sum_K \lambda_K \text{ Conv-hull} \{ g(\vec{p}, K) \mid \vec{p} \in \Theta \}$$

$$\Rightarrow \quad (1+\varepsilon)\vec{x} \in \Omega \quad \text{for some } \varepsilon > 0$$

$$\Rightarrow \quad \text{There exists } \alpha_K^m, \ \vec{p_K^m} \in \Theta, \ \sum_m \alpha_K^m = 1$$

such that

$$(1+\varepsilon) \sum_s H_s^l x_s \leq \sum_K \lambda_K \sum_m \alpha_K^m g_l(\vec{p_K^m}, k)$$

for all l.

$$\text{Since} \quad \sum_l q^l(t) r^l(t) = \max_{\vec{r} = g(\vec{p}, K(t))} \sum_l q^l(t) r^l$$

$$\Rightarrow \quad \sum q^l(t) g_l(\vec{p_K^m}, K) 1_{\{K(t)=k\}}$$

$$\leq \quad \sum q^l(t) r^l(t) 1_{\{K(t)=k\}}$$

for all k, m

$$\Rightarrow \quad \sum q^l(t)\, \lambda_k \sum_m \alpha_k^m\, g(\vec{p}_k^m, k) \le \sum q^l(t)\; E\left[r^l(t)\, 1_{\{k(t)=k\}} \,\middle|\, \vec{q}(t) \right]$$

$$\sum_l q^l(t) \sum_k \lambda_k \sum_m \alpha_k^m\, g_l(\vec{p}_k^m, k) \le \sum_l q^l(t) \cdot E\left[r^l(t) \,\middle|\, \vec{q}(t) \right]$$

$$\Rightarrow \quad (1+\varepsilon) \sum_l q^l(t) \sum_s H_s^l X_s \le \sum_l q^l(t)\, E\left[r^l(t) \,\middle|\, q^l(t) \right]$$

Hence, we have

$$E\left[V(\vec{q}(t+1) - \vec{q}(t)) \,\middle|\, \vec{q}(t) \right]$$

$$\le -\varepsilon \sum_l q^l(t) \sum_s H_s^l X_s + \frac{1}{2} \sum_l M_l^2$$

$$\le -\varepsilon \cdot \left[\min_l \sum_s H_s^l X_s \right] \left(\sum_l q^l(t) \right) + \frac{1}{2} \sum_l M_l^2$$

$$\le -\frac{\varepsilon}{2} \left[\min_l \sum_s H_s^l X_s \right] \left(\sum_l q^l(t) \right)$$

$$\text{if} \quad \sum_l q^l(t) \ge \frac{\frac{1}{2} \sum_l M_l^2}{\frac{\varepsilon}{2} \left[\min_l \sum_s H_s^l X_s \right]} \triangleq M$$

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