(1) 
a) 
\[ R_n(k) = \sum_{m=-\infty}^{\infty} x(m)w(n-m)x(m+k)w(n-k-m) \]
\[ R_n(-k) = \sum_{m=-\infty}^{\infty} x(m)w(n-m)x(m-k)w(n+k-m) \]

Let \( m = m' + k \):
\[ R_n(-k) = \sum_{m'=-\infty}^{\infty} x(m'+k)w(n-m'-k)x(m')w(n-m') \]

Rearranging and replacing \( m' \) by \( m \):
\[ R_n(-k) = \sum_{m=-\infty}^{\infty} x(m)w(n-m)x(m+k)w(n-k-m) \]
\[ R_n(-k) = R_n(k) \]
b) 
\[ R_n(k) = R_n(-k) = \sum_{m=-\infty}^{\infty} x(m)x(m-k)w(n-m)w(n+k-m) \]
\[ = \sum_{m=-\infty}^{\infty} x(m)x(m-k)w(n-m)w((n-m)+k) \]

Let \( h_k(n) = w(n)w(n+k) \)
Therefore \( h_k(n-m) = w(n-m)w(n-m+k) \)
and \( R_n(k) = \sum_{m=-\infty}^{\infty} x(m)x(m-k)h_k(n-m) \)
\[ x(m) = \sum_{r=-\infty}^{\infty} \delta(m - rP) \]

\[ w'(m) = \begin{cases} 
1 & ; \quad 0 \leq m \leq N - 1 \\
0 & ; \quad \text{otherwise} 
\end{cases} \]

where \( QP < N - 1 < (Q + 1)P \), for some integer \( Q \)

\[ R_n(k) = \sum_{m=0}^{N-1-k} [x(n+m)w'(m)][x(n+m+k)w'(k+m)] \]

The two sequences inside the windows will align exactly when \( n + m = n + m + k \geq k = 0 \). For other values of \( k \), the windows will not overlap completely. The other values of \( k \) which produce an alignment of the samples are \( k = iP \), where \( l \) is an integer in the range \(-Q \leq l \leq Q\). Then \( R_n(k) \) appears as shown.

Note that \( R_n(k) \) is independent of \( n \) since \( x(n) \) “looks” the same over any interval.
b) 

\[ \hat{R}_n(k) = \sum_{m=-\infty}^{\infty} s(n+m) \hat{w}_1(m) x(n+m+k) \hat{w}_2(m+k) \]

For \( 0 \leq m \leq N-1 \), \( x(n+m) \hat{w}_1(m) \) will be non-zero. For the same range of \( m \) and for \( 0 \leq k \leq K \), \( x(n+m+k) \hat{w}_2(m+k) \) will be non-zero. Therefore,

\[ \hat{R}_n(k) = \sum_{m=0}^{N-1} x(n+m)x(n+m+k) \]

\[ 0 \leq k \leq K \]

c) 

\[ \hat{R}_n(k) = \sum_{m=0}^{N-1} x(n+m)x(n+m+k) \]

Here we form the product over \( N-1 \) samples and sum the resulting samples. Each increment of \( k \) that shifts a sample of \( x(n+m+k) \) out of the interval \( 0 \leq m \leq N-1 \) also shifts a new but identical sample into the interval. As a result, \( \hat{R}_n(k) \) has the appearance

![Diagram]

\[ \hat{R}_n(k) \]

\[ k \]

-QP, -2P, -P, P, 2P, QP

\[ Q+1 \]

d) In the short-time autocorrelation function, for lags \( k \) that approach the window length (i.e., as \( k \) approaches \( N \)), the number of non-zero terms used in computing the short-time autocorrelation decreases, so that the autocorrelation function exhibits a significant roll-off. In the modified short-time autocorrelation function, \( \hat{w}_2 \) is chosen to include samples outside the interval of window \( \hat{w}_1 \), so that the same number of non-zero terms are used in computing \( \hat{R}_n(k) \) for each \( k \), \( 0 \leq k \leq K \). (Technically this makes \( \hat{R}_n \) a
cross-correlation function rather than an autocorrelation function. However, it does display peaks at multiples of the period of a periodic signal, as is desired for pitch analysis.)

\[(3) \quad D_n(k) = \frac{1}{N} \sum_{m=0}^{N-1} |x(n + m) - x(n + m - k)|\]

a) Given that

\[\frac{1}{N} \sum_{m=0}^{N-1} |x(m)| \leq \left[ \frac{1}{N} \sum_{m=0}^{N-1} |x(m)|^2 \right]^{1/2}\]

Then by applying the inequality to the expression for \(\gamma_n(k)\), we obtain

\[D_n(k) \leq \left[ \frac{1}{N} \sum_{m=0}^{N-1} \{x(n + m) - x(n + m - k)\}^2 \right]^{1/2}\]

\[D_n(k) \leq \left[ \frac{1}{N} \sum_{m=0}^{N-1} \{x^2(n + m) - 2x(n + m)x(n + m - k) + x^2(n + m - k)\} \right]^{1/2}\]

Note that \(\sum_{m=0}^{N-1} x^2(n + m) = \hat{R}_n(0)\) for periodic \(x(n)\), we have:

\[\sum_{m=0}^{N-1} x^2(n + m - k) = \hat{R}_n(0)\]

\[\sum_{m=0}^{N-1} 2x(n + m)x(n + m - k) = 2\hat{R}_n(k)\]

Therefore, \(D_n(k) \leq \left[ \frac{1}{N} \{2\hat{R}_n(0) - 2\hat{R}_n(k)\} \right]^{1/2}\)
\( x(n) = \cos(\omega_0 n), y(n) = \begin{cases} 
1 & x(n) > C_L \\
0 & |x(n)| \leq C_L \\
-1 & x(n) < -C_L 
\end{cases} \)

\( C_L = 1.0, y(n) = 0 \quad \forall n \)
c) As $C_L \to 1$, the autocorrelation starts to look like an alternating pulse train (i.e. scaled unit-sample pulse train). However, as $C_L \to 1$ the amplitudes of the pulses $\to 0$. For a signal with time-varying amplitude, too “high” a clipping threshold may result in a loss of autocorrelation peaks, while too “low” a threshold causes the peaks to widen, which complicates pitch detection.

\[ C_L = 1.0, \phi_y(k) = 0 \quad \forall k \]

(5)

a) $\alpha_0 = -1; s'(n) = s(n)w(n)$

\[ e(n) = s'(n) - \sum_{i=1}^{p} \alpha_i s'(n-i) = -\sum_{i=0}^{p} \alpha_i s'(n-i) \]
\[ R_e(m) = \sum_{n=-\infty}^{\infty} e(n)e(n+m) \]
\[ = \sum_{n=-\infty}^{\infty} \sum_{i=0}^{p} \sum_{j=0}^{p} \alpha_i \alpha_j s'(n-i)s'(n+m-j) \]

Let \( n' = n + i \), then
\[ R_e(m) = \sum_{n'=-\infty}^{\infty} \sum_{i=0}^{p} \sum_{j=0}^{p} \alpha_i \alpha_j s'(n')s'(n' + m + i - j) \]

Let \( j = l + i \), then
\[ R_e(m) = \sum_{i=0}^{p} \sum_{l=-i}^{p-i} \alpha_i \alpha_{i+l} \sum_{n'=-\infty}^{\infty} s'(n')s'(n' + m - l) \]

Note:
\[ \alpha_{i+l} = 0 \begin{cases} & \text{for } i + l < 0 \geq l < -i \\ & \text{and } i + l > p \geq l > p - I \end{cases} \]

\[ R_e(m) = \sum_{l=-\infty}^{\infty} \left[ \sum_{i=0}^{p} \alpha_i \alpha_{i+l} \right] \left[ \sum_{n'=-\infty}^{\infty} s'(n')s'(n' + m - l) \right] \]
\[ = \sum_{l=-\infty}^{\infty} R_q(l)R_{q'}(m-l) \]

b) For a 10 kHz rate, a total of 121 values of \( m \) are needed to give \( R_e(m) \) from
\( m = 30 \) (3 msec) to \( m = 150 \) (15 msec). \( R_q(l) \) is a symmetric sequence of
length \( 2p+1 \) samples, and \( R_{q'}(l) \) is a symmetric sequence of length \( 2N-1 \)
samples. If we assume \( R_q(l) \) and \( R_{q'}(l) \) are available (i.e. no computation is
required), then for each value of \( m \), a total of \( 2p+1 \) multiplies and adds
are required to implement the convolution of \( R_q \) and \( R_{q'} \). Thus the total
computation for 121 values of \( m \) is:
\[ T_c = 121(2p+1) = 242p + 121 \]

For \( p = 10 \), a total of 2541 multiplies and adds are required to give \( R_e(n) \). If
\( R_q(l) \) and \( R_{q'}(l) \) are not available, the amount of computation grows linearly
with \( N \) (instead of \( p \)).