Test Duration: 75 minutes.
Open Book but Closed Notes.
Calculators not allowed.
This test contains three problems.
All work should be done in the blue books provided.
Do not return this test sheet, just return the blue books.

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Problem 1. [30 points]

Consider a one-sided geometric sequence of the form

\[ x[n] = 31 \left( \frac{1}{2} \right)^n u[n] \]

where the constant 31 is just a multiplicative scalar that ultimately makes the numbers work out nice. The DTFT of \( x[n] \) is \( X(\omega) = \frac{31}{1 - 0.5e^{-j\omega}} \). Let \( X_N[k] \) be obtained as \( N \) samples of \( X(\omega) \) equi-spaced in frequency over \( 0 \leq \omega < 2\pi \). Further, let \( x_N[n] \) be obtained as the \( N \)-point Inverse DFT of \( X_N[k] = \frac{31}{1 - 0.5e^{-j\frac{2\pi}{N}}}. \) Determine \( x_N[n] \) when \( N = 5 \). That is, determine and explicitly list the numerical values of \( x_N[0], x_N[1], x_N[2], x_N[3], x_N[4] \), when \( N = 5 \). Actually computing an Inverse DFT is NOT the way to solve this problem.

Problem 2. [40 points] Consider the following window of length \( M - 1 \), where \( M \) is an even number.

\[ w[n] = e^{j\frac{2\pi}{M}n} \left\{ u[n] - u[n - \frac{M}{2}] \right\} \ast e^{-j\frac{2\pi}{M}n} \left\{ u[n] - u[n - \frac{M}{2}] \right\} \]

This “new” window is obtained as the convolution of one rectangular window of length \( \frac{M}{2} \) modulated by \( e^{j\frac{2\pi}{M}n} \) with another rectangular window of length \( \frac{M}{2} \) modulated by \( e^{-j\frac{2\pi}{M}n} \).

(a) Determine a closed-form expression for \( w[n] \) (that is, determine a simple analytical expression for the result obtained from performing the convolution.) Sketch \( w[n] \) for \( n = 0, 1, ..., M - 2 \).

(b) Is \( w[n] \) a symmetric or anti-symmetric window? Briefly justify your answer (that is, don’t just guess.)

(c) Let \( W(\omega) \) denote the DTFT of \( w[n] \). Determine a closed-form expression for \( W[\omega] \). Plot the magnitude \( |W(\omega)| \) over \( -\pi < \omega < \pi \) showing as much detail as possible. Explicitly point out the numerical values of the specific frequencies for which \( |W(\omega)| = 0 \).

(d) Analysis of mainlobe width of \( W(\omega) \): What is the null-to-null mainlobe width of \( W(\omega) \)? Is the mainlobe width of \( W(\omega) \) the same, larger, or smaller than the mainlobe width of the DTFT of a rectangular window of the same length, \( M - 1 \)? Briefly explain.

(e) Analysis of peak sidelobe of \( W(\omega) \): Is the peak sidelobe of \( W(\omega) \) the same, larger, or smaller than the peak sidelobe of the DTFT of a rectangular window of the same length, \( M - 1 \)? Briefly explain your answer.

(f) Analysis of sidelobes of \( W(\omega) \): What about the sidelobes other than the peak sidelobe? That is, excluding the mainlobe and the first peak sidelobe on either side of the mainlobe, are the sidelobes of \( W(\omega) \) the same, larger, or smaller than the sidelobes of the DTFT of a rectangular window of the same length, \( M - 1 \)? Briefly explain.
Problem 3. [30 points]

Let $x[n]$ be a discrete-time random process containing one real-valued sinewave plus noise as described by

$$x[n] = A \cos(\omega_0 n + \Theta) + \nu[n],$$

where the amplitude, $A$, and frequency, $\omega_0$, of the sinusoid are each deterministic but unknown constants and $\Theta$ is a random variable uniformly distributed over a $2\pi$ interval. $\nu[n]$ is a white noise process with autocorrelation $r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n-m]\} = \delta[m]$.

You are given the following three values of the true autocorrelation sequence $r_{xx}[m] = E\{x[n]x^*[n-m]\}$:

$$r_{xx}[0] = 3; \quad r_{xx}[1] = 1; \quad r_{xx}[2] = -1$$

(a) Knowing that $r_{xx}[m]$ satisfies $r_{xx}[m] = -a_1 r_{xx}[m-1] - a_2 r_{xx}[m-2] + \sigma^2 \delta[m]$, determine the numerical values of $a_1$ and $a_2$.

(b) Determine the numerical value of $r_{xx}[3]$.

(c) Plot the spectral density $S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} r_{xx}[m] e^{-jm\omega}$ over $-\pi \leq \omega \leq \pi$. 