Problem 1. [35 points]

The analog signal \( x_a(t) \) with CTFT \( X_a(F) \) shown is input to the system above, where \( x(n) = x_a(n/F_s) \),

\[
w(n) = \begin{cases} 
  x\left(\frac{\omega}{2}\right), & n \text{ even} \\
  0, & n \text{ odd}
\end{cases}
\]

\[
h_{LP}(n) = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}, \quad -\infty < n < \infty,
\]

and \( \tilde{h}_{LP}(n) \) is the Discrete-Time Hilbert Transform of \( h(n) \). Note that \( y(n) = w(n) * h(n) \), where \( h(n) = h_{LP}(n) + j\tilde{h}_{LP}(n) \). (If necessary, see page 610 of the text for both a frequency and time-domain description of the DT Hilbert Transform.) Finally, \( z(n) = e^{j\frac{\pi}{2}n}y(n) \).

(25 pts) Plot the magnitude of the respective DTFT’s of \( x(n), w(n), y(n), h(n), \) and \( z(n) \) – five plots all together. In each case, plot the magnitude of the DTFT over \(-\pi < \omega < \pi\) (NOT just \( 0 < \omega < \pi \) since most of the sequences are complex-valued.) Show as much detail as possible.

(10 pts) Determine a closed-form expression (NO summation should appear in the final expression) for \( \tilde{h}_{LP}(n) \), the Hilbert Transform of \( h_{LP}(n) \), as a function of \( n \). Simplify as much as possible.
Problem 2. [35 points]

Consider the following two $N = 8$ point sequences:

\[ x(n) = \cos \left( \frac{\pi}{2} n \right) \{ u(n) - u(n - 8) \} \]

and

\[ h(n) = 0.5^n \{ u(n) - u(n - 8) \} , \]

where $u(n)$ is the DT unit step function. Let $X_8(k)$ and $H_8(k)$ denote the respective $N = 8$ point DFT's of $x(n)$ and $h(n)$.

The DFT of $x(n)$ is

\[ DFT \quad x(n) \rightarrow X_8(k) \]

and

\[ DFT \quad h(n) \rightarrow H_8(k) \]

(10 pts) Write a closed-form expression (NO summation in the final answer) for both $X_8(k)$ and $H_8(k)$, respectively (two answers), as a function of $k$. There’s no need to do a lot of work here if you make use of either standard DFT pairs or standard DTFT pairs.

(15 pts) Let $y_8(n)$ denote the $N = 8$ point inverse DFT of the product $Y_8(k) = X_8(k)H_8(k)$. Determine a closed-form expression (NO summation in final answer) for $y_8(n)$. Simplify as much as possible – there should be no $j = \sqrt{-1}$ in the final answer since $y_8(n)$ is real-valued for all $n$. (You can make the approximation $1 - \frac{1}{2^8} \approx 1$.)

(10 pts) Is there any value of $n$ for which $y_8(n) = y(n)$, where $y(n)$ is the linear convolution of $x(n)$ and $h(n)$ ($y(n) = x(n) * h(n)$)? If yes, state which values of $n$ does $y_8(n) = y(n)$. Explain your answer.

Problem 3. [10 points]

Consider reconstructing a signal from its samples by linearly interpolating between every two successive sample values (graphically, we draw a line between each pair of successive sample values.) From a theoretical point of view, the reconstruction process may be viewed in terms of the block diagram above. Plot the impulse response of the appropriate continuous time (CT) LTI filter, $h_a(t)$, in the diagram above that achieves this type of reconstruction.
Problem 4. [10 points]
A DT signal is obtained by sampling the CT signal \( x_a(t) = e^{-at}u(t) \) as \( x(n) = x_a(nT_s) = x_a(n/F_s) \). From class, we know that the DTFT of \( x(n) \), \( X(\omega) \), and the CTFT of \( x_a(t) \), \( X_a(F) \), are related as \( X(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a \left( \frac{F_s}{2\pi}(\omega + k2\pi) \right) \). Given the CTFT pair

\[
e^{-at}u(t) \quad CTFT \quad \frac{1}{a + j2\pi F}
\]

it follows that,

\[
X(\omega) = F_s \sum_{k=-\infty}^{\infty} \frac{1}{a + jF_s(\omega + k2\pi)} = ??
\]

Determine a closed-form expression (NO summation in final answer) for \( X(\omega) \) in terms of \( a \) and \( F_s \). (This problem requires very little work and was alluded to in class.)

Problem 5. [10 points]
Suppose we need to compute a \( N = 21 \) point DFT. The direct approach would require \( 21^2 = 441 \) complex multiplications. What would be the required number of complex multiplications if we employed a divide and conquer approach??