EE648 (CC761-M) DSP II

Session 4 (Live: 1/21/99)

Outline:

- Development of Recursive Least Squares (RLS) Algorithm
  - Sect. 12.2.4 of 1st Ed. of P+M
- Adaptive Noise Cancellation Application Revisited

- Mathematical precursor: Matrix Inversion Lemma

$$\Big(w\underset{M\times M}{\underline{R}} + \underbrace{\underset{M\times 1}{\underline{x}}\;\underset{1\times M}{\underline{x}^T}}_{M\times M}\Big)^{-1}$$

$$= \frac{1}{w}\underline{R}^{-1} - \frac{1}{w}\,\frac{\underline{R}^{-1}\underline{x}\,\underline{x}^T\underline{R}^{-1}}{w + \underset{1\times M}{\underline{x}^T}\,\underset{M\times M}{\underline{R}^{-1}}\,\underset{M\times 1}{\underline{x}}}$$

$$\left(w\underline{R} + \underline{x}\,\underline{x}^T\right)\left(\frac{1}{w}\underline{R}^{-1} - \frac{1}{w}\,\frac{\underline{R}^{-1}\underline{x}\,\underline{x}^T\underline{R}^{-1}}{w + \underline{x}^T\underline{R}^{-1}\underline{x}}\right)$$

$$= \underline{I} + \frac{1}{w}\,\underline{x}\,\underline{x}^T R^{-1} - \frac{\underline{x}\,\underline{x}^T R^{-1}}{w + \underline{x}^T\underline{R}^{-1}\underline{x}} - \frac{1}{w}\,\frac{\underline{x}^T\underline{R}^{-1}\underline{x}\;\underline{x}\,\underline{x}^T R^{-1}}{w + \underline{x}^T\underline{R}^{-1}\underline{x}}$$

$$\frac{1}{w} - \frac{1}{w+\underline{x}^T \underline{R}^{-1} \underline{x}} - \frac{1}{w} \frac{\underline{x}^T \underline{R}^{-1} \underline{x}}{w+\underline{x}^T \underline{R}^{-1} \underline{x}}$$

$$\frac{w+\underline{x}^T \underline{R}^{-1} \underline{x} - w - \underline{x}^T \underline{R}^{-1} \underline{x}}{w\left(w+\underline{x}^T \underline{R}^{-1} \underline{x}\right)}$$

$$= 0$$

So the Matrix Inversion Lemma holds!

- Development of RLS Algorithm
- in RLS, $\underline{R}_{xx}$ and $\underline{r}_{dx}$ are estimated at time n as:

$$\hat{\underline{R}}_{xx}[n] = \sum_{l=0}^{n} \underline{x}[l]\,\underline{x}^T[l]\, w^{n-l}$$

$$\hat{\underline{r}}_{dx}[n] = \sum_{l=0}^{n} d[l]\,\underline{x}[l]\, w^{n-l}$$

- where $0 < w < 1$

- $w < 1$ is used in practice to weight past data samples less than the current data samples (to adapt to time-variations in the statistics of the underlying signal)

- RLS works to minimize the time-avg'd. error

$$\mathcal{E}[n] = \sum_{\ell=0}^{n} w^{n-\ell} \left\{ d[\ell] - \underline{h}_M^T[\ell]\, \underline{x}[\ell] \right\}^2$$

- taking gradient of $\mathcal{E}[n]$ wrt $\underline{h}_M[n]$ and setting $=0$ yields:

$$\hat{\underline{R}}_{xx}[n]\,\underline{h}_M[n] = \hat{\underline{r}}_{dx}[n]$$

- observe:

$$\hat{\underline{R}}_{xx}[n] = \sum_{\ell=0}^{n} w^{n-\ell}\,\underline{x}[\ell]\,\underline{x}^T[\ell]$$

$$= w \sum_{\ell=0}^{n-1} w^{n-1-\ell}\,\underline{x}[\ell]\underline{x}^T[\ell] + \underline{x}[n]\underline{x}^T[n]$$

$$= w\,\underline{R}_{xx}[n-1] + \underline{x}[n]\,\underline{x}^T[n]$$

• similarly :

$$\underline{\hat{r}}_{dx}[n] = w\,\underline{\hat{r}}_{dx}[n-1] + d[n]\,\underline{x}[n]$$

• $\underline{h}_M[n] = \underline{R}_{xx}^{-1}[n]\,\underline{\hat{r}}_{dx}[n]$ can be computed resursively from

$$\underline{h}_M[n-1] = \underline{\hat{R}}_{xx}^{-1}[n-1]\,\underline{\hat{r}}_{dx}[n-1]$$

using matrix inversion lemma

$$\underline{\hat{R}}_{xx}^{-1}[n] = \left\{ w\,\underline{R}_{xx}[n-1] + \underline{x}[n]\,\underline{x}^T[n] \right\}^{-1}$$

$$\hat{\underline{R}}^{-1}_{xx}[n] =$$

$$= \underbrace{\left\{ \frac{1}{w}\hat{\underline{R}}^{-1}_{xx}[n-1] - \frac{1}{w}\,\frac{\hat{\underline{R}}^{-1}_{xx}[n-1]\,\underline{x}[n]\,\underline{x}^T[n]\,\hat{\underline{R}}^{-1}_{xx}[n-1]}{w + \underline{x}^T[n]\,\hat{\underline{R}}^{-1}_{xx}[n-1]\,\underline{x}[n]} \right\}}$$

$$\underline{h}_M[n] = \underline{R}^{-1}_{xx}[n]\;\underbrace{\hat{\underline{r}}_{dx}[n]}_{\left\{ w\,\hat{\underline{r}}_{dx}[n-1] + d[n]\,\underline{x}[n] \right\}}$$

$$= \underline{h}_M[n-1] + 3 \text{ other terms}$$

• define: $\mu[n] = \underline{x}^T[n]\,\underline{R}^{-1}_{xx}[n-1]\,\underline{x}[n]$

$$+\frac{1}{w}\,\underline{R}_{xx}^{-1}[n-1]\,\underline{x}[n]\,d[n]$$

$$-\frac{1}{w+\mu[n]}\,\underline{R}_{xx}^{-1}[n-1]\,\underline{x}[n]\,\underline{x}^{T}[n]\,\underline{h}_{M}[n-1]$$

$$-\frac{1}{w}\,\frac{1}{w+\mu[n]}\,\underline{R}_{xx}^{-1}[n-1]\,\underline{x}[n]\,\mu[n]\,d[n]$$

$$=\frac{1}{w+\mu[n]}\,\underline{R}_{xx}^{-1}[n-1]\,\underline{x}[n]\cdot\left\{\frac{d[n]}{w}\big[w+\mu[n]-\mu[n]\big]-\underline{x}^{T}[n]\,\underline{h}_{M}[n-1]\right\}$$

$$\underline{h}_M[n] = h_M[n-1] +$$

$$+ \left\{\frac{1}{w + \mu[n]}\right\} \underline{\hat{R}}_{xx}^{-1}[n-1]\,\underline{x}[n] \left\{d[n] - \underline{h}_M^T[n-1]\,\underline{x}[n]\right\}$$

- where: $\mu[n] = \underline{x}^T[n]\,\underline{R}_{xx}^{-1}[n-1]\,\underline{x}[n]$

- define: $e[n, n-1] = d[n] - \underline{h}_M^T[n-1]\,\underline{x}[n]$

$$\left.\begin{aligned} &h_M[n] = \underline{h}_M[n-1] + \\ &\frac{e[n, n-1]}{w + \mu[n]}\,\underline{\hat{R}}_{xx}^{-1}[n-1]\,\underline{x}[n] \end{aligned}\right\} \text{RLS update}$$

Summary of RLS

0. Initialization: $\underline{h}_M[-1] = \underline{0}_M$, e.g.)

and $\underline{\hat{R}}^{-1}[-1] = \frac{1}{\sigma} \underline{I}_M$

1. $e[n, n-1] = d[n] - \underline{h}_M^T[n-1] \, \underline{x}[n]$

2. a. $\underline{f}[n] = \hat{R}_{xx}^{-1}[n-1] \, \underline{x}[n]$

   b. $u[n] = \underline{x}^T[n] \, \underline{f}[n]$

   c. $\underline{K}_M[n] = \underline{f}[n] \big/ (w + u[n])$

3. $\underline{h}_M[n] = \underline{h}_M[n-1] + e[n, n-1] \, \underline{K}_M[n]$

4. $\underline{R}_{xx}^{-1}[n] = \frac{1}{w} \left\{ \underline{R}_{xx}^{-1}[n-1] + \underline{K}_M[n] \, \underline{f}^T[n] \right\}$

Go to 1.