EE648 (CC761-M) DSP II
Session 3 (hive: 1/19/99)

<u>Outline:</u>

- Further analysis of convergence of LMS
- Sect. 12.2.3 of 1st Ed. of P+M
- Application: Adaptive Noise Cancellation – Sect. 12.1 of 1st Ed. P+M

- recall LMS update eqn.:

$$\underline{h}_M[n+1] = \underline{h}_M[n] + \mu\, e[n]\, \underline{x}[n]$$

$$\underline{h}_M[n+1] - \underline{h}_M^{opt} = \underline{h}_M[n] - \underline{h}_M^{opt} + \mu\, e[n]\, x[n]$$

- take expected value of both sides:

$$\underline{c}[n+1] = \underline{c}[n] + \mu\, E\left\{ \left[d[n] - \underline{x}^T[n]\underline{h}_M[n] \right] \cdot \underline{x}[n] \right\}$$

- where: $\underline{c}[n] = E\left\{ \underline{h}_M[n] - \underline{h}_M^{opt} \right\}$

$$\underline{c}[n+1] = \underline{c}[n] + \mu\, \underline{r}_{dx}$$

$$\underbrace{- \mu\, E\left\{ \underline{x}[n]\, \underline{x}^T[n]\, \underline{h}_M[n] \right\}}$$

$$- \mu\, E\left\{ \underline{x}[n]\, \underline{x}^T[n] \left(\underline{h}_M[n] - \underline{h}_M^{opt} \right) \right\}$$

$$- \mu\, E\left\{ \underline{x}[n]\, \underline{x}^T[n] \right\} \underline{h}_M^{opt}$$

- See: "Adaptive Signal Processing" by Widrow & Stearns Prentice-Hall, 1985, Chap. 6

- Widrow proved error

$$\underline{h}_M[n] - \underline{h}_M^{opt}$$

is uncorrelated with data $\underline{x}[n]$. Thus:

$$E\left\{ \underline{x}[n]\,\underline{x}^T[n] \left(\underline{h}_M[n] - \underline{h}_M^{opt}\right) \right\}$$

$$= E\left\{ \underline{x}[n]\,\underline{x}^T[n] \right\} E\left\{ \underline{h}_M[n] - \underline{h}_M^{opt} \right\}$$

$$= \underline{R}_{xx}\, \underline{c}[n]$$

- in addition:

$$-\mu E\{\underline{x}[n]\underline{x}^T[n]\}\underline{h}_M^{opt}$$

$$= -\mu \underline{R}_{xx}(\underline{R}_{xx}^{-1}\underline{r}_{dx})$$

$$= -\mu \underline{r}_{dx}$$

- Ultimately, after substitution:

$$\underline{c}[n+1] = c[n] - \mu \underline{R}_{xx}\underline{c}[n]$$

$$= \{\underline{\underline{I}}_M - \mu \underline{R}_{xx}\}\underline{c}[n]$$

- consider eigenvalue decomposition of $\underline{R}_{xx}$: $\underline{R}_{xx} = \underline{U}\,\underline{\Lambda}\,\underline{U}^T$
- Since $\underline{R}_{xx}$ is symmetric

$$\underline{U}^T\underline{U} = \underline{I}_M = \underline{U}\,\underline{U}^T$$

$$\underline{c}[n+1] = \left\{\underline{U}\,\underline{U}^T - \mu\,\underline{U}\,\underline{\Lambda}\,\underline{U}^T\right\}\underline{c}[n]$$

$$\underline{U}^T\underline{c}[n+1] = \left\{\underline{I}_M - \mu\,\underline{\Lambda}\right\}\underline{U}^T\underline{c}[n]$$

- define: $\underline{c}^{\circ}[n] = \underline{U}^{T}\underline{c}[n]$

$$\underline{c}^{\circ}[n+1] = \left\{ \underline{\underline{I}}_{n} - \mu\,\underline{\underline{\Lambda}} \right\} \underline{c}^{\circ}[n]$$

- component-wise:

$$c^{\circ}[k;n+1] = (1-\mu\lambda_k)\,c^{\circ}[k;n]$$

$$k = 1,\dots,M$$

- recall: $h[n] = a\,h[n-1]$

$\Rightarrow$ sol'n: $h[n] = a^{n}\,h[0]$

- thus: $c^{\circ}[k;n] = (1-\mu\lambda_k)^{n}\,c^{\circ}[k;0]$

- for convergence, require:

$$-1 < 1 - \mu\lambda_k < 1 \quad \text{for } k = 1, \ldots, M$$

$$1 > \mu\lambda_k - 1 > -1$$

$$-1 < \mu\lambda_k - 1 < 1$$

$$0 < \mu\lambda_k < 2$$

$$0 < \mu < \frac{2}{\lambda_k}$$

λ_k are strictly non-negative

- to insure convergence:

$$0 < \mu < \frac{2}{\lambda_{max}}$$

- to be conservative : $\mu < \frac{1}{\lambda_{max}}$
- in practice:

$$\lambda_{max} < \sum_{k=1}^{M} \lambda_k = \mathrm{trace}\{\underline{R}_{xx}\}$$

$$= M \, r_{xx}[0]$$

- thus:

$$0 < \mu < \frac{1}{M \, r_{xx}[0]}$$

$$\text{or } \left(\frac{2}{M \, r_{xx}[0]}\right)$$

- further analysis:
  - say, we choose: $\mu = \dfrac{1}{\lambda_{max}}$

$$c^{\circ}[k;n] = \left(1 - \frac{\lambda_k}{\lambda_{max}}\right)^n c^{\circ}[k;0]$$

$$k = 1, \ldots, M$$

- consider $k = M$, for which $\lambda_M = \lambda_{min}$

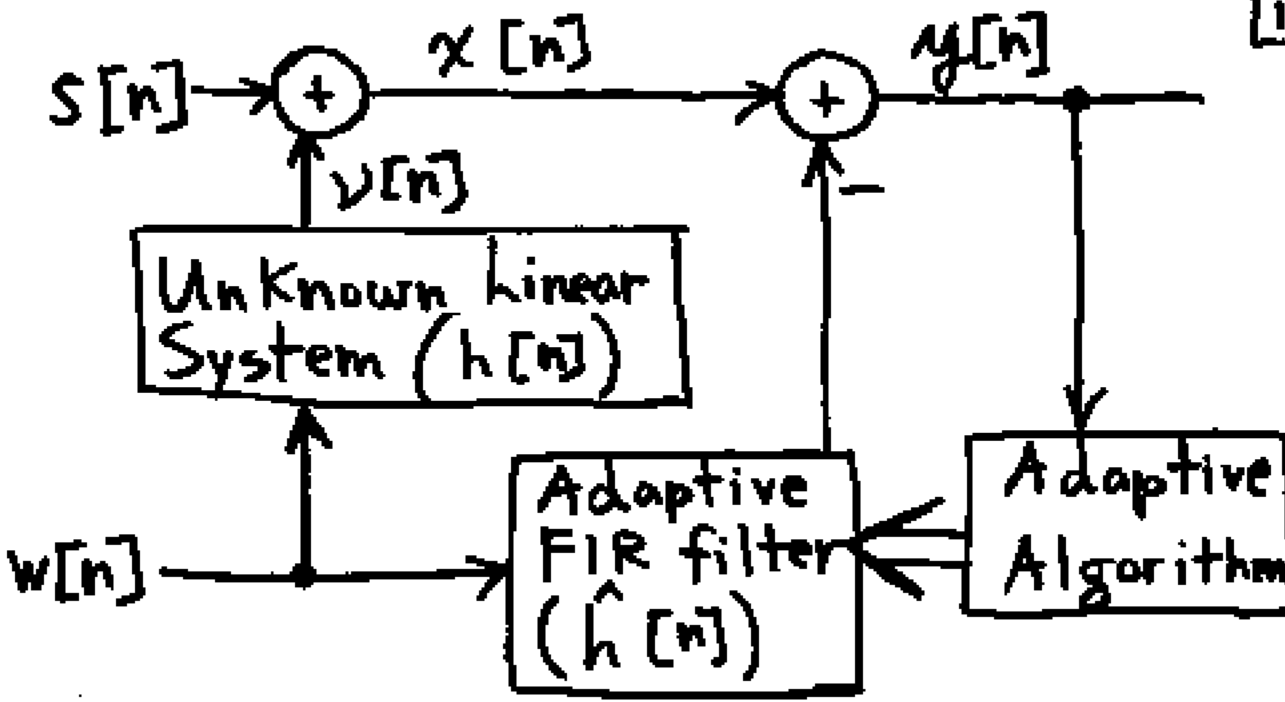
(assuming $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M$)

- since: $1 - \dfrac{\lambda_{min}}{\lambda_{max}} > 1 - \dfrac{\lambda_k}{\lambda_{max}}$

- so the M-th term associated with the smallest eigenvalue, λ_{min}, takes the longest to converge
- $\dfrac{\lambda_{max}}{\lambda_{min}}$ determines the convergence rate of LMS
- the Recursive Least Squares (RLS) algorithm to be developed in Session 4 is not as sensitive to the eigenvalue spread of $\underline{R}_{xx}$

- Application: Adaptive Noise Cancellation
- specific problem:
- car phone imbedded in steering wheel (or wearing a head set)
- Speech is potentially masked by "car noise"
- <u>approach</u>: place a microphone at some point in car to pick up "car noise" only (negligible speech)
- exploit correlation between noise

picked up at "remote mike" and
noise contaminating speech
to form an estimate of the latter
and subtract it off



$x[n]$: plays the role of the "desired" signal, $d[n]$

$y[n]$: plays the role of the "error" signal, $e[n]$

- $s[n]$: speech-only (unobservable)
- $w[n]$: noise-only observed at "remote" sensor (mike)
- $\nu[n]$: filtered version of $w[n]$ that corrupts speech signal

$$\nu[n] = w[n] * h[n]$$

- $h[n]$: FIR filter of length M
- $x[n] = s[n] + \nu[n]$ $\Rightarrow$ picked up by transmitting mike

Assumption: $w[n]$ and $s[n]$ are independent random processes

- Choose $\hat{h}[n]$ to minimize

$$E\left\{\left[x[n] - \sum_{k=0}^{M-1} \hat{h}[n]\, w[n-k]\right]^2\right\}$$

$$= E\left\{\left[x[n] - \underline{\hat{h}}_M^T \underline{w}[n]\right]^2\right\}$$

- where:

$$\underline{\hat{h}}_M = \left[\hat{h}[0], \ldots, \hat{h}[M-1]\right]^T$$

$$\underline{w}[n] = \left[w[n], \ldots, w[n-(M-1)]\right]^T$$

$$\nabla_{\underline{\hat{h}}_M} E\{y^2[n]\} =$$

$$\nabla_{\underline{\hat{h}}_M} \Big\{ E\{x^2[n]\} - 2\,\underline{\hat{h}}_M^T E\{x[n]\,\underline{w}[n]\} + \underline{\hat{h}}_M^T E\{\underline{w}[n]\,\underline{w}^T[n]\}\underline{\hat{h}}_M \Big\}$$

$$= -2\,\underline{r}_{wx} + 2\,\underline{R}_{ww}\,\underline{\hat{h}}_M = \underline{0}$$

$$\underline{R}_{ww} = E\{\underline{w}[n]\,\underline{w}^T[n]\}$$

$$\underline{r}_{wx} = E\{x[n]\,\underline{w}[n]\}$$

- sol'n: $\underline{\hat{h}}_M^{opt} = \underline{R}_{ww}^{-1} \underline{r}_{wx}$
- show: $\underline{\hat{h}}_M^{opt} = \underline{h}$
- recall: $x[n] = s[n] + v[n]$

$$= s[n] + w[n] * h[n]$$

$$= s[n] + \underline{h}^T \underline{w}[n]$$

- $E\{y^2[n]\}$

$$= E\{[s[n] + \underline{h}^T \underline{w}[n] - \underline{\hat{h}}_M^T \underline{w}[n]]^2\}$$

$$= E\left\{\left[s[n] + (\underline{h} - \underline{\hat{h}}_M)^T \underline{w}[n]\right]^2\right\}$$

$$= E\{s^2[n]\} + 2(\underline{h} - \underline{\hat{h}}_M)^T \underbrace{E\{s[n]\,\underline{w}[n]\}}_{0}$$

$$+ (\underline{h} - \underline{\hat{h}}_M)^T E\{\underline{w}[n]\,\underline{w}^T[n]\}(\underline{h} - \underline{\hat{h}}_M)$$

$$= E\{s^2[n]\} + (\underline{h} - \underline{\hat{h}}_M)^T \underline{R}_{ww} (\underline{h} - \underline{\hat{h}}_M)$$

- thus: $\underline{\hat{h}}_M^{opt} = \underline{h}$ since $\underline{R}_{ww}$ is positive-definite