EE648 (CC761-M) DSP II

Session 2 (Live: 1/14/99)

Outline :

- Review MMSE Criterion for Adaptive Filters – Sect. 12.2.1 1st Ed. of P+M
- Widrow LMS Algorithm – Sect. 12.2.2, 1st Ed. of P+M
- Convergence analysis of LMS
  - Sect. 12.2.3 , 1st Ed. of P+M

- recall: $E\{e^2[n]\}$

$$= E\{d^2[n]\} - \underline{r}_{dx}^T R_{xx}^{-1} \underline{r}_{dx}$$

$$+ (\underline{R}_{xx}\underline{h}_M - \underline{r}_{dx})^T \underline{R}_{xx}^{-1} (\underline{R}_{xx}\underline{h}_M - \underline{r}_{dx})$$

- optimum $\underline{h}_M$ satisfies

$$\underline{R}_{xx}\underline{h}_M - \underline{r}_{dx} = \underline{0} \Rightarrow \underline{h}_M^{opt} = \underline{R}_{xx}^{-1}\underline{r}_{dx}$$

- Minimum value of $E\{e^2[n]\}$:

$$= E\{d^2[n]\} - \underline{r}_{dx}^T (\underline{R}_{xx}^{-1}\underline{r}_{dx})$$

$$= E\{d^2[n]\} - \underline{r}_{dx}^T \underline{h}_M^{opt}$$

- in practice, don't know $\underline{R}_{xx}$ or $\underline{r}_{dx}$
- must estimate these quantities
- also: statistics of $x[n]$ and $d[n]$ may vary "slowly" with time

- adaptive approach:
- employ gradient-based search to iterate towards the filter coeff. vector minimizing $E\{e^2[n]\}$
- error surface is a hyperparaboloid in M-dimensional space

- search for "bottom of bowl"
- let gradient vector evolve with time in accordance with $\{x[n]\}$ and $\{d[n]\}$
- gradient operator:

$$\nabla_{\underline{h}_M} = \left[\frac{\partial}{\partial h[0]}, \frac{\partial}{\partial h[1]}, \cdots, \frac{\partial}{\partial h[M-1]} \right]^T$$

- $E\{e^2[n]\} = f(\underline{h}_M)$

$$= E\{d^2[n]\} - 2\,\underline{h}_M^T \underline{r}_{dx} + \underline{h}_M^T \underline{R}_{xx} \underline{h}_M$$

- easy to show:

$$\nabla_{\underline{h}_M}\left(\underline{h}_M^T \underline{r}_{dx}\right) = \underline{r}_{dx} \quad (M\times 1)$$

$$\nabla_{\underline{h}_M}\left(\underline{h}_M^T \underline{R}_{xx} \underline{h}_M\right) = 2\underbrace{\underline{R}_{xx}}_{M\times M}\underbrace{\underline{h}_M}_{M\times 1}$$

- thus:

$$\nabla_{\underline{h}_M} f(\underline{h}_M) = -2\,\underline{r}_{dx} + 2\,\underline{R}_{xx}\underline{h}_M$$

$$= \underline{0} \implies \underline{R}_{xx}\underline{h}_M = \underline{r}_{dx}$$

$$\implies \underline{h}_M^{opt} = \underline{R}_{xx}^{-1}\underline{r}_{dx}$$

• consider using method of "steepest descent" (classical gradient search)

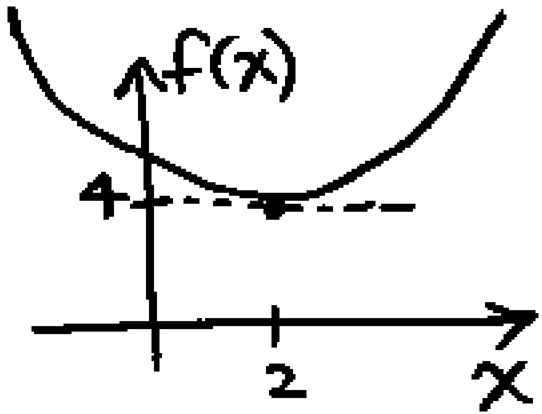
$$\underline{h}_M(\ell+1) = \underline{h}_M(\ell) - \frac{1}{2}\mu_\ell \nabla_{\underline{h}_M} f(\underline{h}_M^{(\ell)})$$

• each iteration: take "step" in direction of negative gradient

• μ_ℓ : step size $(0 < \mu_\ell < 1)$

• Simple example:

$$f(x) = x^2 - 4x + 8 = (x-2)^2 + 4$$



$$\text{Min } f(x) = 4 \text{ at } x = 2$$

$$\nabla_x f(x) = \frac{d}{dx} f(x) = 2(x-2)$$

- gradient search :

$$x(l+1) = x(l) - \frac{1}{2}\mu_l \, 2(x^{(l)}-2)$$

$$= x(l) - \mu_l (x^{(l)}-2)$$

- if $x(l) < 2 \Rightarrow -\mu_l (x(l)-2) > 0$

  $\Rightarrow$ take step in $+x$ direction

- if $x(l) > 2 \Rightarrow -\mu_l (x(l)-2) < 0$

  $\Rightarrow$ take step in $-x$ direction

- Widrow's LMS Algorithm
- Key features:
  - fix step size μ_ℓ
  - approximate $\underline{r}_{dx}$ and $\underline{R}_{xx}$ by instantaneous estimates

$$\hat{\underline{r}}_{dx}[n] = d[n]\,\underline{x}[n]$$

$$\hat{\underline{R}}_{xx}[n] = \underline{x}[n]\,\underline{x}^T[n]$$

- Substitute these into the expression for the gradient

$$\nabla_{\underline{h}_M} E\{e^2[n]\} = -2\,\underline{r}_{dx} + 2\,\underline{R}_{xx}\,\underline{h}_M$$

$$\hat{\nabla}_{\underline{h}_M[n]} = -2\,d[n]\,\underline{x}[n] + 2\,\underline{x}[n]\,\underline{x}^T[n]\,\underline{h}_M[n]$$

$$= -2\{d[n] - \underline{x}^T[n]\,\underline{h}_M[n]\}\,\underline{x}[n]$$

$$= -2\{d[n] - \hat{d}[n]\}\,\underline{x}[n]$$

$$= -2\,e[n]\,\underline{x}[n]$$

$$\underline{h}_M[n+1] = \underline{h}_M[n] - \frac{1}{2}\mu\,\hat{\nabla}_{\underline{h}_M} E\{e^2[n]\}$$

$$= \underline{h}_M[n] + \mu\,e[n]\,\underline{x}[n]$$

- Convergence Analysis of LMS
- show LMS udpate converges to $\underline{h}_M^{opt} = \underline{R}_{xx}^{-1} \underline{r}_{dx}$ in the mean provided $\{x[n]\}$ is stationary and

$$0 < \mu < \frac{2}{\lambda_{max}}$$

- where λ_{max} is largest eigenvalue of $\underline{R}_{xx}$

- Proof: define

$$\underline{c}[n] = \bar{\underline{h}}_M[n] - \underline{h}_M^{opt}$$

- where: $\bar{\underline{h}}_M[n] = E\{\underline{h}_M[n]\}$
- thus: $\underline{c}[n+1] = \bar{\underline{h}}_M[n+1] - \underline{h}_M^{opt}$
- take expected value of both sides of LMS update equation:

$$\bar{\underline{h}}_M[n+1] = \bar{\underline{h}}_M[n] + \mu E\{e[n]\underline{x}[n]\}$$

- Subtract $\underline{h}_M^{opt}$ from both sides

$$\underline{c}[n+1] = \underline{c}[n] + \mu E\{e[n]\underline{x}[n]\}$$

$$= \underline{c}[n] + \mu E\{d[n]\underline{x}[n]\} - \mu E\{\underline{x}[n]\underbrace{\underline{x}^T[n]\underline{h}_M[n]}_{\hat{d}[n]}\}$$

$$= c[n] + \mu \underline{r}_{dx} - \mu E\{\underline{x}[n]x^T[n]\underline{h}_M[n]\}$$

• asymptotically:

$$E\{\underline{x}[n]\underline{x}^T[n](\bar{\underline{h}}_M[n] + \Delta\underline{h}_M[n])\}$$

$$= \underline{R}_{xx}\bar{\underline{h}}_M[n] + E\{\underline{x}[n]x^T[n]\Delta\underline{h}_M[n]\}$$

• assuming $\Delta \underline{h}_M[n] << 0$,

$$\underline{c}[n+1] = \underline{c}[n] + \mu \underline{r}_{dx} - \mu \underline{R}_{xx} \bar{\underline{h}}_M[n]$$

$$= \underline{c}[n] + \mu \underline{R}_{xx} \underline{R}_{xx}^{-1} \underline{r}_{dx} - \mu \underline{R}_{xx} \bar{h}_M[n]$$

$$= \underline{c}[n] + \mu \underline{R}_{xx} \{ \underline{h}_M^{opt} - \bar{\underline{h}}_M[n] \}$$

$$= \underline{c}[n] - \mu R_{xx} \underline{c}[n]$$

$$\underline{c}[n+1] = \{ \underline{I} - \mu \underline{R}_{xx} \} \underline{c}[n]$$

• Consider eigenvalue decomposition of $\underline{R}_{xx}$

$$\underline{R}_{xx} = \underline{U}\,\underline{\Lambda}\,\underline{U}^T$$

• Since $\underline{R}_{xx}$ is positive-definite + symmetric

$$\underline{U}^T\underline{U} = \underline{I} = \underline{U}\,\underline{U}^T$$ } eigenvectors are orthonormal

$$\underline{c}[n+1] = \left\{\underline{U}\,\underline{U}^T - \mu\,\underline{U}\,\underline{\Lambda}\,\underline{U}^T\right\}\underline{c}[n]$$

$$\underline{U}^T\underline{c}[n+1] = \left\{\underline{I} - \mu\,\underline{\Lambda}\right\}\underline{U}^T\underline{c}[n]$$

• define $\underline{c}^{o}[n+1] = \underline{U}^T\underline{c}[n+1]$

$$\underline{c}^{o}[n+1] = \{\underline{I} - \mu \underline{\Lambda}\} \underline{c}^{o}[n]$$

- component-wise:

$$c^{o}[k; n+1] = (1 - \mu \lambda_k) c^{o}[k; n]$$

$$k = 1, \ldots, M$$

- recall:

$$h[n] = a\, h[n-1] \quad (h[n+1] = a\, h[n])$$

soln: $h[n] = a^n h[0]$

- thus: $c^{o}[k; n] = (1 - \mu \lambda_k)^n c^{o}[k; 0]$

$$k = 1, \ldots, M$$

- for convergence, require:

$$-1 < 1 - \mu \lambda_k < 1 \quad \text{for } k = 1, \dots, M$$

$$0 < \mu < \frac{2}{\lambda_k}$$

- to insure convergence:

$$0 < \mu < \frac{2}{\lambda_{max}}$$

- in practice:

$$\lambda_{max} < \sum_{k=1}^{M} \lambda_k = \text{trace}\{\underline{R}_{xx}\} = M r_{xx}[0]$$

- $$0 < \mu < \frac{2}{M r_{xx}[0]}$$