EE538 DSPI

Module 9

Outline:

- Frequency domain analysis of
  - upsampling
  - downsampling
- Efficient downsampling
- Fractional Sampling Rate Conversion

P+M Text
10.2, 10.3, 10.9.8.

- Recall:

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{z[n]} \boxed{\begin{array}{c}\text{gain} = L \\ \text{LPF} \\ \omega_c \approx \frac{\pi}{L}\end{array}} \longrightarrow y[n]$$

$$x[n] = x_a\left(\frac{n}{F_s}\right) \qquad y[n] = x_a\left(\frac{n}{LF_s}\right)$$

$$\omega_p = 2\pi \frac{W}{LF_s} \Rightarrow \text{passband edge}$$

$$\omega_s = \frac{2\pi}{L} - 2\pi\frac{W}{LF_s} \Rightarrow \text{stopband edge}$$

$$z[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

- Frequency domain analysis of up-sampling

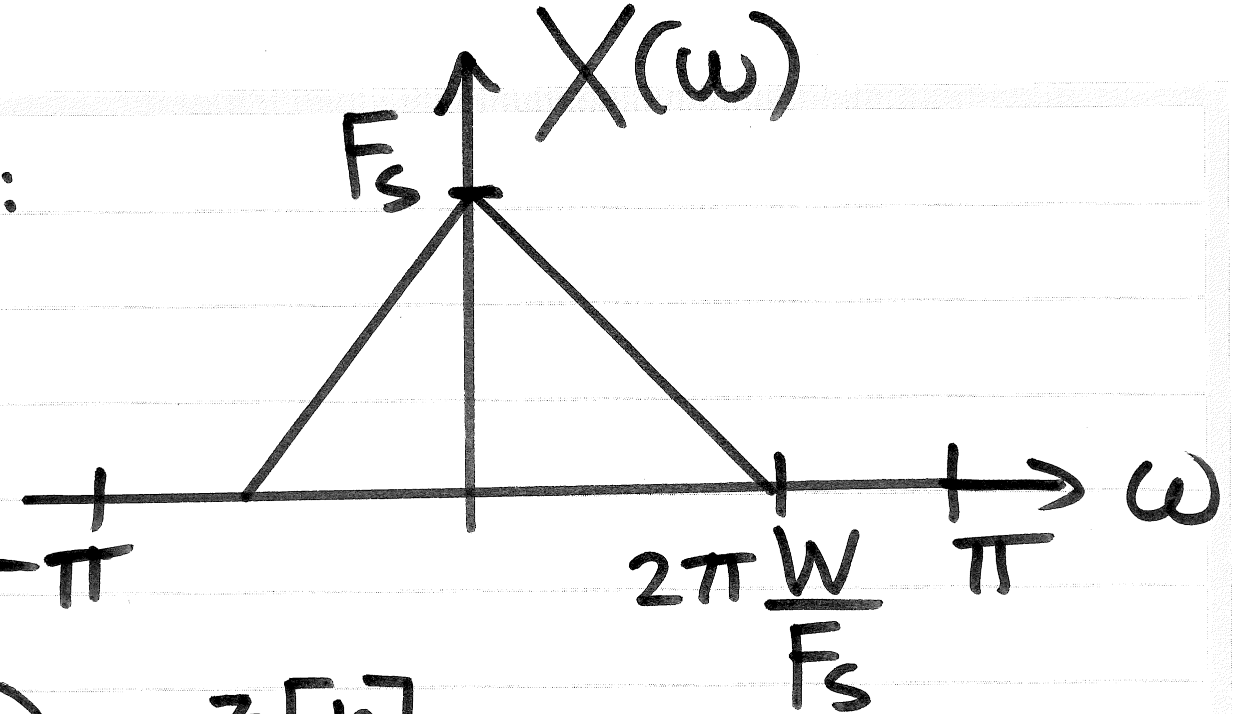
$$Z(\omega) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]\,e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} \delta[n-kL]\,e^{-j\omega n}$$

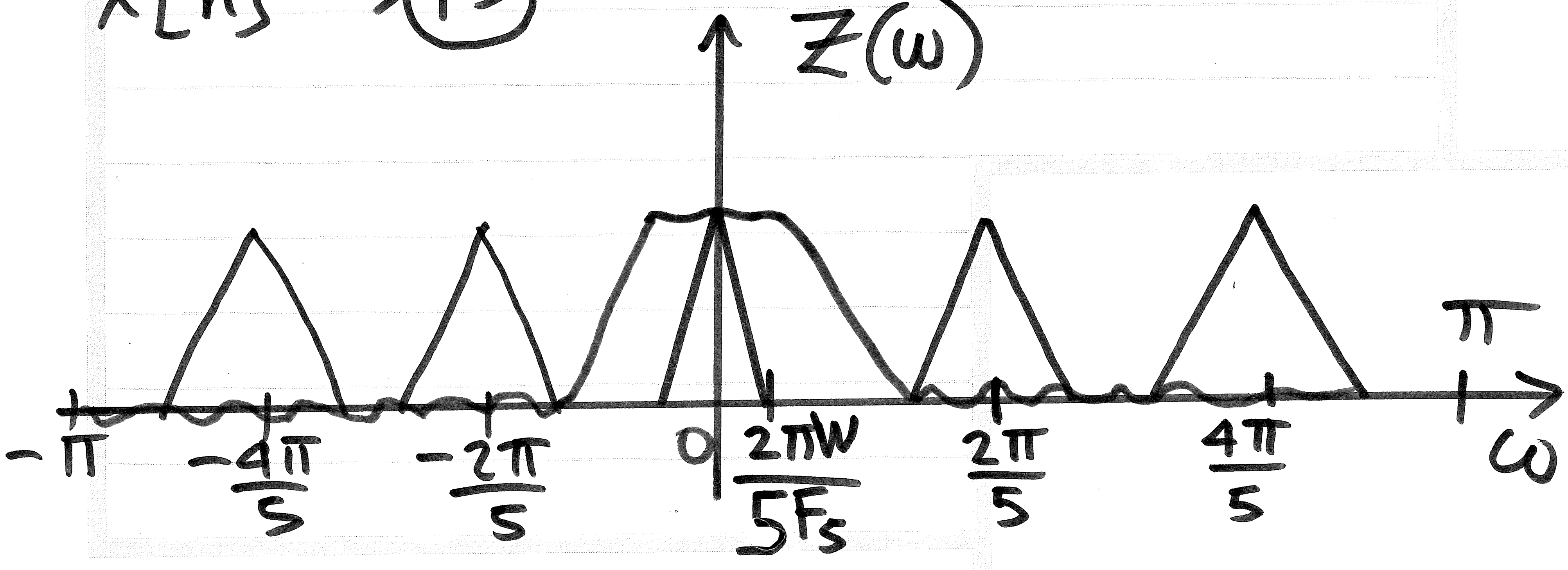
$$= \sum_{k=-\infty}^{\infty} x[k]\,e^{-j\omega k L}$$

$$= X(L\omega) \quad \left\}\begin{array}{l}\text{periodic } \omega \\ \text{period} = \dfrac{2\pi}{L}\end{array}\right.$$

- **Example:**



$x[n] \longrightarrow \uparrow 5 \longrightarrow z[n]$



- LPF removes the compressed replicas ("images") lying in $-\pi < \omega < \pi$ except that centered at $\omega = 0$
- referred to as anti-imaging filter
- if $F_s > 2W$, "don't care" region from $\frac{2\pi W}{L F_s}$ to $\frac{2\pi}{L} - \frac{2\pi W}{L F_s}$
- width $= \frac{2\pi}{L}\left\{1 - 2\frac{W}{F_s}\right\}$

On the Efficient Implementation of Upsampling

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-kL] \qquad L = \text{integer} > 1$$

- Depends on how you "read out" the output values
- Read out or compute every L-th value starting at L different starting points:

$$n = \ell + n'L \qquad \ell = 0, 1, \ldots, L-1$$

$$y[\ell + n'L] = \sum_{k=-\infty}^{\infty} x[k]\, h[\ell + n'L - kL]$$

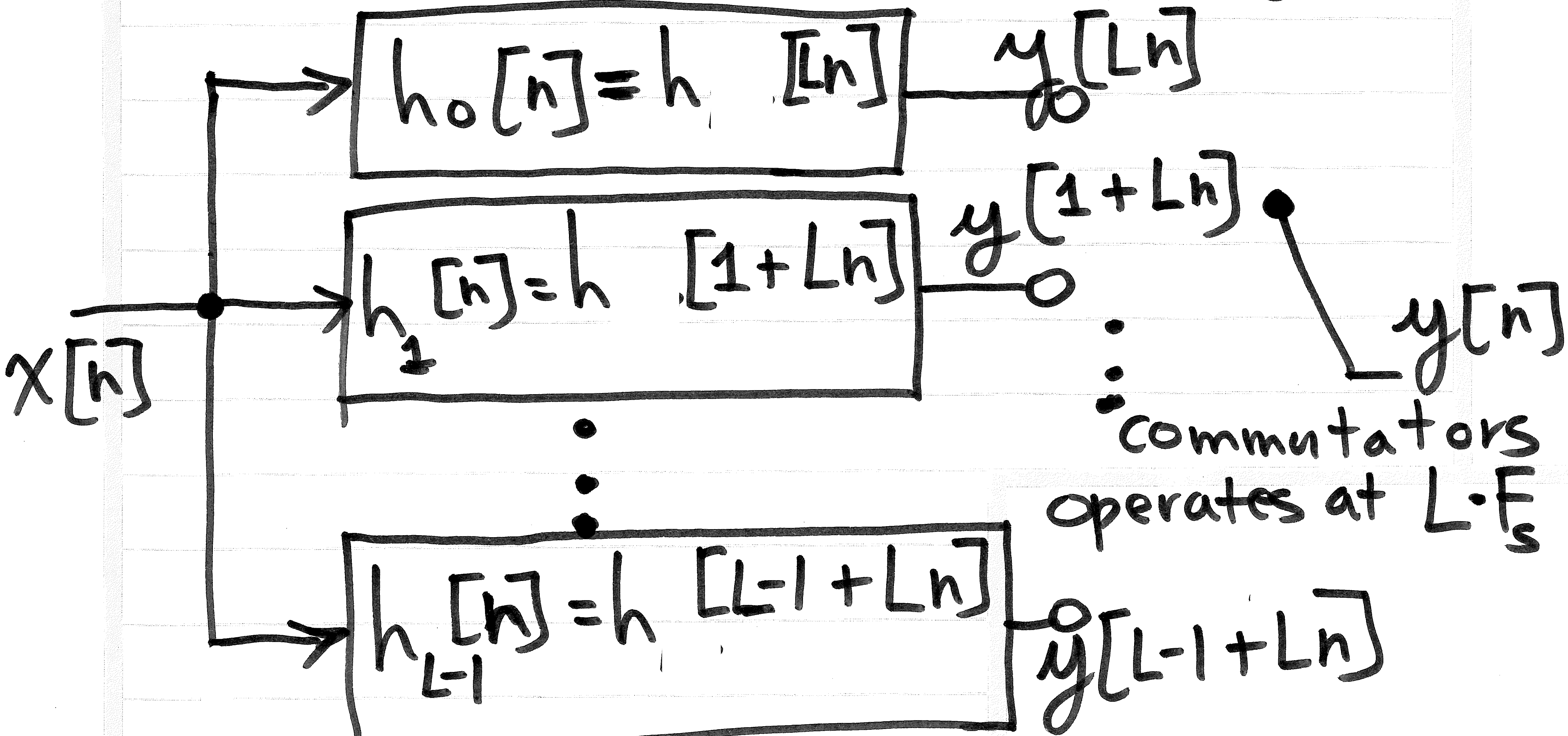
drop prime:

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[\ell + (n'-k)L]$$

$$\left.\begin{aligned} \Rightarrow y[\ell + nL] &= y_\ell[n] = x[n] * h_\ell[n] \\ \text{where: } h_\ell[n] &= h[\ell + nL] \end{aligned}\right\} \text{interleave outputs}$$

$$\ell = 0, 1, \ldots, L-1$$

- efficient implementation for general case of up-sampling by L



Note on Fractional Time-Shift

- Recall ideal Reconstruction Formula

$$x_a(t) = \sum_{k=-\infty}^{\infty} \underbrace{x_a(kT_s)}_{x[k]} \frac{\sin\left(\frac{\pi}{T_s}(t-kT_s)\right)}{\frac{\pi}{T_s}(t-kT_s)}$$

- Evaluate at : $t = \frac{l}{L}T_s + nT_s$ $\quad l \in \{0,1,\ldots,L-1\}$, L = integer

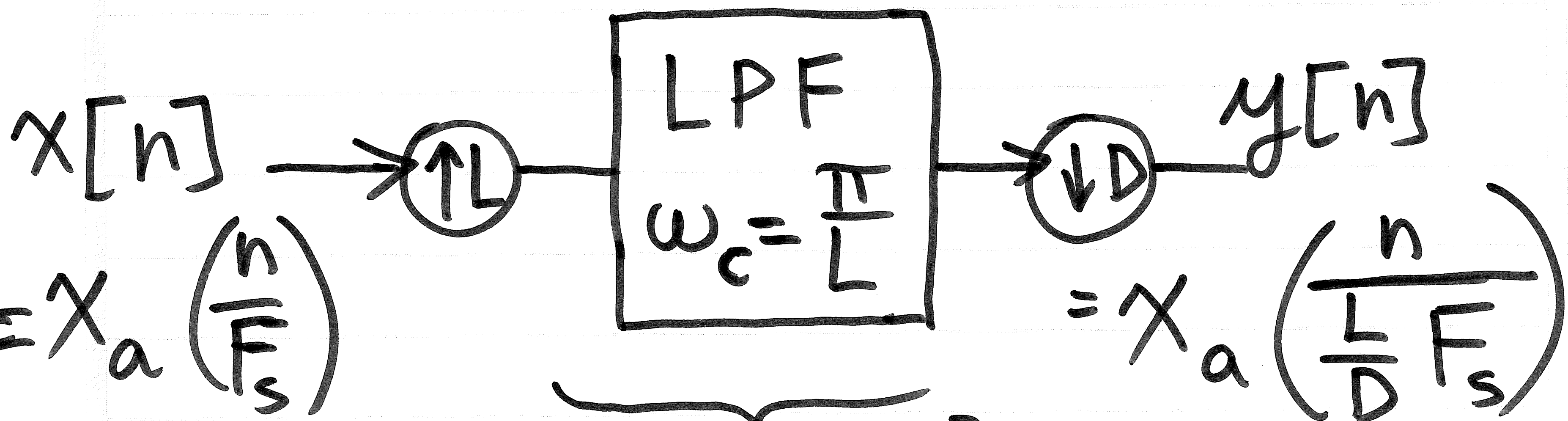
$$y[n] = x_a\left(\frac{l}{L}T_s + nT_s\right) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\frac{\pi}{T_s}\left(\frac{l}{L}T_s + nT_s - kT_s\right)\right)}{\frac{\pi}{T_s}\left(\frac{l}{L}T_s + nT_s - kT_s\right)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\pi\left(\frac{l}{L} + n - k\right)\right)}{\pi\left(\frac{l}{L} + n - k\right)}$$

$$= x[n] * h[n] \qquad h[n] = \frac{\sin\left(\pi\left(n + \frac{l}{L}\right)\right)}{\pi\left(n + \frac{l}{L}\right)}$$

$$= x_a\left(\frac{l}{L}T_s + nT_s\right)$$

- Fractional Sampling Rate Conversion



$$\omega_c = \min\left\{\frac{\pi}{L}, \frac{\pi}{D}\right\}$$

- **Downsampling:**

$$x[n] \longrightarrow \boxed{\downarrow D} \longrightarrow y[n]$$

$$= x_a\left(\frac{n}{F_s}\right) \qquad\qquad = x_a\left(\frac{n}{F_s/D}\right)$$

$$\Downarrow$$

$$y[n] = x[Dn]$$

e.g.:

$$y[-1] = x[-D]$$

$$y[0] = x[0]$$

$$y[1] = x[D]$$

$$y[2] = x[2D] \quad \dots$$

- Math preamble:

$$\sum_{k=-\infty}^{\infty}\delta[n-kL] = \sum_{k=0}^{L-1}\frac{1}{L}e^{j\frac{2\pi kn}{L}}$$

$$= \frac{1}{L}\,\frac{1-e^{j2\pi n}}{1-e^{j2\pi n/L}}$$

$$= \begin{cases} 1, & n=\ell L,\ \ell \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

- Downsampling:

$$y[n] = x[Dn]$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} x[Dn]\, e^{-j\omega n}$$

$$= \sum_{n'=-\infty}^{\infty} x[n']\, e^{-j\frac{\omega n'}{D}}$$

$$\boxed{\begin{aligned} n' &= nD \\ n &= \frac{n'}{D} \end{aligned}}$$

$n' = \ell D$

$$= \sum_{n'=-\infty}^{\infty} x[n'] \frac{1}{D} \sum_{k=0}^{D-1} e^{j\frac{2\pi k n'}{D}}\, e^{-j\frac{\omega n'}{D}}$$

L3

$$Y(\omega) = \sum_{k=0}^{D-1} \frac{1}{D} \sum_{n=-\infty}^{\infty} x[n] e^{-j\left(\frac{\omega}{D} - \frac{2\pi k}{D}\right)n}$$

$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega - 2\pi k}{D}\right)$$

- note: if $F_{s_{new}} \equiv \frac{F_{s_{old}}}{D} > 2W$,

then $Y(\omega) = \frac{1}{D} X\left(\frac{\omega}{D}\right)$ for $|\omega| < \pi$

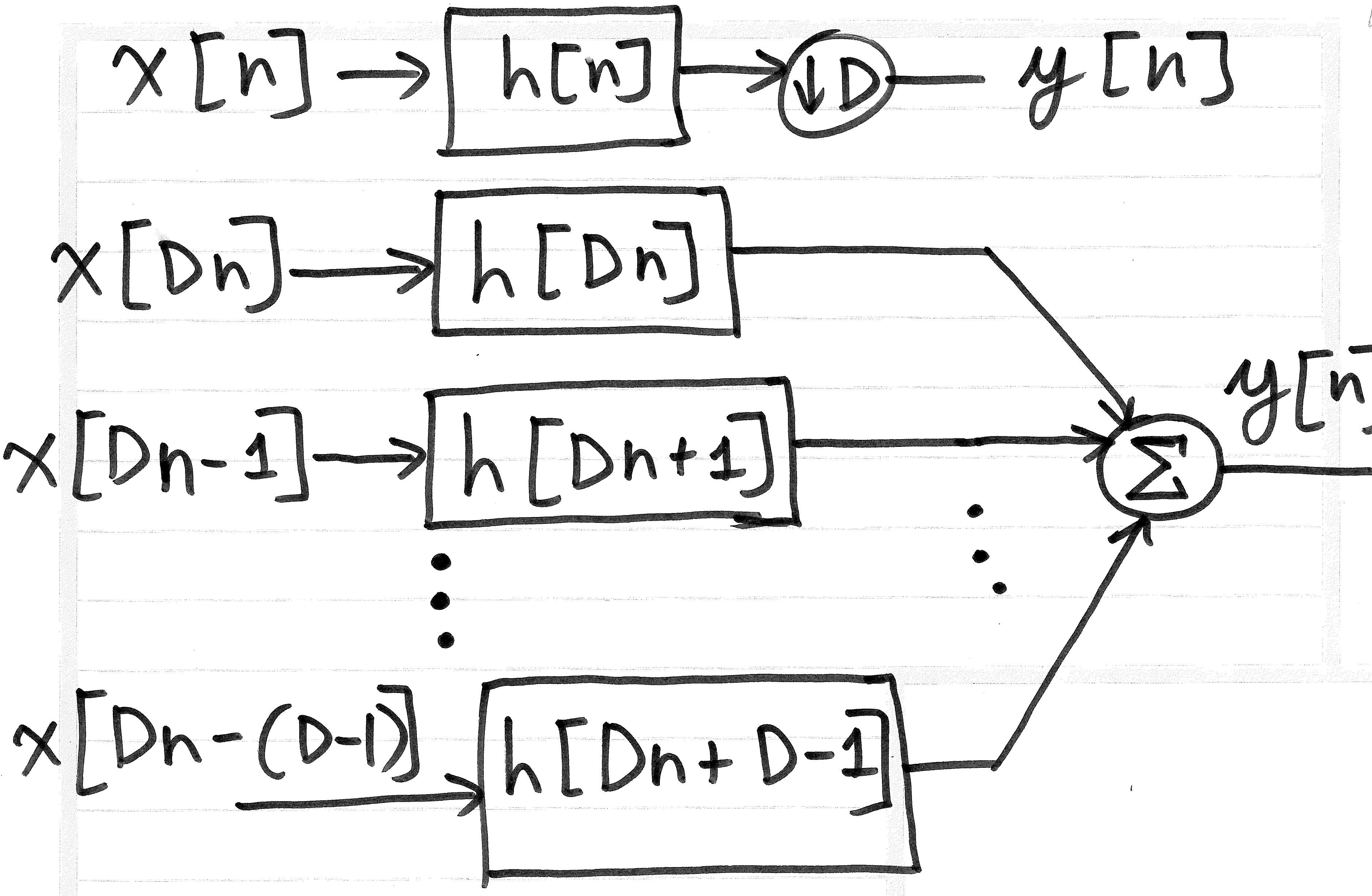
- Efficient Down-Sampling:

$$x[n] \longrightarrow \boxed{h[n]} \xrightarrow{w[n]} (D\downarrow) \longrightarrow y[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[Dn-k]$$

$$= \sum_{\ell=0}^{D-1} \sum_{k'=-\infty}^{\infty} h[k'D+\ell]\, x[Dn-(k'D+\ell)]$$

$$= \sum_{\ell=0}^{D-1} \sum_{k=-\infty}^{\infty} h[kD+\ell]\, x[D(n-k)-\ell]$$


x[n]
h[n]
↓D
y[n]
x[Dn]
h[Dn]
x[Dn-1]
h[Dn+1]
x[Dn-(D-1)]
h[Dn+D-1]
Σ
y[n]

Fractional Sampling Rate Alteration

- so far: $F_{s_{new}} = L F_{s_{old}}$

  L — integer

- desire:

  $F_{s_{new}} = \frac{L}{D} F_{s_{old}}$ where: D is integer

$$x[n] \rightarrow \boxed{\uparrow L} \xrightarrow{z[n]} \boxed{\text{LPF} \; \omega_c = \frac{\pi}{L}} \xrightarrow{y[n]} \boxed{\downarrow D} \xrightarrow{} w[n]$$

Decimator: $w[n] = y[Dn]$

Keep only every D-th sample

e.g. $w[0] = y[0]$

$w[1] = y[D]$

$w[2] = y[2D]$

Since: $y(n) = x_a\left(\frac{n}{LF_s}\right)$

$$w[n] = y[Dn] = x_a\left(\frac{Dn}{LF_s}\right) = x_a\left(\frac{n}{\frac{L}{D}F_s}\right)$$